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A CHARACTERIZATION OF DISCRETE UNIMODALITY WITH APPLICATIONS TO VARIANCE UPPER BOUNDS

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Abstract. Bertin and Theodorescu (1984, *Statist. Probab. Lett.*, **2**, 23–30) developed a characterization of discrete unimodality based on convexity properties of a discretization of distribution functions. We offer a new characterization of discrete unimodality based on convexity properties of a piecewise linear extension of distribution functions. This reliance on functional convexity, as in Khintchine's classic definition, leads to variance dilations and upper bounds on variance for a large class of discrete unimodal distributions. These bounds are compared to existing inequalities due to Muilwijk (1966, *Sankhyā, Ser. B*, **28**, p. 183), Moors and Muilwijk (1971, *Sankhyā, Ser. B*, **33**, 385–388), and Rayner (1975, *Sankhyā, Ser. B*, **37**, 135–138), and are found to be generally tighter, thus illustrating the power of unimodality assumptions.

Key words and phrases: Discrete distributions, unimodality, convexity, variance bounds.

1. Introduction

The concept of unimodality as it appears in most of the literature and textbooks extant is restricted to non-discrete distributions. Indeed, that is the case for the following definition, due to Khintchine, which is a widely used standard definition of unimodality.

DEFINITION 1.1. A cumulative distribution function F is said to be *unimodal* with *mode* m if F is convex on $(-\infty, m)$ and concave on (m, ∞). We say F is *unimodal about* m. Denote the class of unimodal distribution functions by $\mathcal{U}$.

First attributed to Khintchine by Gnedenko and Kolmogorov ((1954), p. 157), this definition, along with others, is discussed thoroughly in the treatise on the subject by Dharmadhikari and Joag-Dev ((1988), p. 2).

Unfortunately, the only discrete unimodal distributions under Khintchine's definition are degenerate. We are concerned with defining discrete unimodality in a manner analogous to that of Khintchine but without this drawback. Such a characterization affords, among other things, application of the many inequalities dependent on convexity. We shall demonstrate this utility by deriving variance upper bounds for a broad class of discrete unimodal distributions.

In Section 2 we present our characterization of discrete unimodality. We develop a variance dilation for discrete distributions in Section 3. In Section 4 we derive variance upper bounds for certain discrete unimodal distributions. Finally, in Section 5 we compare the new bounds with previously obtained bounds, finding that assumptions of discrete unimodality yield considerably tighter inequalities.

2. A new characterization of discrete unimodality

Let $\mathcal{F}_d$ be the collection of all discrete cumulative distribution functions (CDF's) with the support being a subset (not necessarily proper) of the positive integers. Let $F \in \mathcal{F}_d$ with support S. The density of F with respect to counting measure shall be called a *probability mass function* (PMF) with support S. We denote the mean and variance of F by μ_F and ν_F, respectively.

Characterizations of discrete unimodality have been considered by several authors. The subject has been reviewed through about 1987 in Dharmadhikari and Joag-Dev (1988). The following definition is commonly used and is due to Keilson and Gerber (1971).

DEFINITION 2.1. Let $F \in \mathcal{F}_d$ have support S and PMF f. Then f is said to be *discrete unimodal* about a *mode* $m \in S$ if $f(i) \geq f(i-1)$ for $i \leq m$ and $f(i) \leq f(i-1)$ for $i \geq m+1$, $i \in S$. The set of such points m is called the *mode set* and denoted by $\mathcal{M}$. We shall say that F is unimodal if f is unimodal. Denote the class of discrete unimodal CDF's by $\mathcal{U}_d$.

Characterizations of discrete unimodality based on convexity concepts have been advanced. Dharmadhikari and Jogdeo (1976) have made use of the convexity of the set of unimodal distributions on the integers $\mathbb{Z}$ with mode 0. Bertin and Theodorescu (1989) have noted that one of the oldest definitions of discrete unimodality makes use of convexity and is due to Mallows (1956). Mallows' approach, as well as that of Bertin and Theodorescu (1984), is based on a formal discretization of Definition 1.1. In particular, Bertin and Theodorescu have defined *discrete unimodality with mode m* as in Definition 2.1 but with the modification that $f(i) \leq f(i-1)$ for $i \geq m+2$. The motivation for the change is that the corresponding CDF restricted to the integers satisfies Definition 1.1. Dharmadhikari and Joag-Dev ((1988), p. 107) have noted advantages to this nonstandard definition, including the fact that the set of all unimodal distributions is the same under both definitions, with the indicated change made to Definition 2.1. A minor disadvantage is that $f(m) < f(m+1)$ so that m may not be a "mode" in the usual sense of that word.

In the main result of this section (Theorem 2.1) we offer a new characterization of discrete unimodality in terms of the convexity of a function of the CDF, namely the *extended* CDF. The characterization is intuitively appealing in that it is analogous to Khintchine's classic definition (Definition 1.1). We begin with several preliminary notions.

Let A be a subset of the real numbers $\mathbb{R}$ and define $A_d = A \cap \mathbb{Z}$. Following Bertin and Theodorescu (1984), if A is an interval, A_d is called a *discrete interval.* Let s and g be the smallest and greatest integers, respectively, in A. Then the *completion* of the discrete interval A_d in $\mathbb{R}$ is the interval $[s, g]$. In addition to the usual definition of convexity of a function, we shall need an analogous definition for functions on $\mathbb{Z}$. For a discrete interval I define $\mathrm{Int}(I) = \{i \in I : [i-1, i+1]_d \subset I\}$. A function $G : I \to \mathbb{R}$ is said to be *convex* at $i \in I$ if $i \notin \mathrm{Int}(I)$ or if $G(i) \leq [G(i+1) + G(i-1)]/2$. Concavity of functions on $\mathbb{Z}$ is defined analogously.

Let $\bar{I}$ be the completion of I. Denote by $\mathrm{Ext}(G)$ the piecewise linear extension to $\bar{I}$ of a function $G : I \to \mathbb{R}$. In our characterization of discrete unimodality we shall make use of the following lemma, due to Bertin and Theordorescu (1984).

LEMMA 2.1. *Let $G : I \to \mathbb{R}$ and suppose I is a discrete interval with completion $\bar{I}$. The following statements are equivalent.*

(i) *G is convex on I;*

(ii) *For $i \in \mathrm{Int}(I)$, $G(i) - G(i-1) \leq G(i+1) - G(i)$;*

(iii) *G is the restriction to I of a convex function $\mathrm{Ext}(G) : \bar{I} \to \mathbb{R}$;*

(iv) *$\mathrm{Ext}(G)$ is convex on $\bar{I}$.*

Given $F \in \mathcal{F}_d$ with support S, construct the *extended* CDF F^* by distributing the mass at $i \in S$ uniformly over $(i-1, i]$. Finally, define $\mathrm{Rest}(F)$ to be the restriction of a CDF F to $\mathbb{Z}$; that is, the function $\mathrm{Rest}(F)$ is defined by the ordered pairs $(z, F(z))$, $z \in \mathbb{Z}$. Note that $F^* = \mathrm{Ext}[\mathrm{Rest}(F)]$.

We now establish our characterization of discrete unimodality.

THEOREM 2.1. *Let $F \in \mathcal{F}_d$ have support S and extended CDF F^*. Let $\mathcal{M} \subset S$. Then $F \in \mathcal{U}_d$ with mode set $\mathcal{M}$ if and only if F^* is convex on $(-\infty, m]$ and concave on $[m-1, \infty)$ for all $m \in \mathcal{M}$.*

PROOF. Let $m \in \mathcal{M}$. Note that m is a jump point of F; that is, $F(m) > F(m-1)$. By Definition 2.1, $F \in \mathcal{U}_d$ if and only if $F(i)-F(i-1) \geq F(i-1)-F(i-2)$ when $i \leq m$ and $F(i) - F(i-1) \geq F(i+1) - F(i)$ when $i \geq m$, $i \in S$. Using (i) and (ii) of Lemma 2.1, this is equivalent to requiring that $\mathrm{Rest}(F)$ be convex on $(-\infty, m]_d$ and concave on $[m-1, \infty)_d$. By (iv) of the lemma, this is true if and only if F^* is convex on $(-\infty, m)$ and concave on $[m-1, \infty)$, in which case $F^* \in \mathcal{U}$. $\square$

Dharmadhikari and Joag-Dev ((1988), preface) characterize the role of convexity in unimodality theory as a "unifying thread"; they speak of a unified theory of unimodality. Characterizations of discrete unimodality which are based on convexity, such as Theorem 2.1 or Bertin and Theodorescu's (1984), are desirable because they fit easily into the unified theory and can be used to take advantage of the powerful tools of convex analysis. In the next section we shall see an example of that advantage.

3. Variance dilations

In this section, we develop a variance dilation for discrete distributions. A common method for comparing the variability of two distributions is to count the number of sign changes of the difference in their CDF's (see, for example, Whitt (1985), or Karlin ((1968), p. 20)). Let $h : \mathbb{R} \to \mathbb{R}$ and let $\mathcal{S}(h)$ be the number of sign changes of $h(w)$, where values of w such that $h(w) = 0$ are ignored. Let F and G be CDF's with PMF's f and g, respectively. If $\mathcal{S}(g - f) = 2$ with sign sequence $+, -, +$, then f and g are said to *satisfy the crossing condition* and we write $g \succ f$. If $\mathcal{S}(G - F) = 1$ with sign sequence $+, -$, we shall say that F and G *satisfy the crossing condition* and write $G \succ F$. It can be shown that $g \succ f$ implies $G \succ F$.

The following variance dilation, which utilizes the crossing condition, is due to Shaked (1980). Note that this result is not restricted to discrete distributions. For an overview of such results, see Shaked (1988).

THEOREM 3.1. *Let F and G be CDF's, and suppose $\mu_F = \mu_G$. If $G \succ F$, then $\nu_G \geq \nu_F$.*

In the next section we make use of this dilation by appealing to the properties of convex functions to establish the requisite crossings. We now present a discrete analogue to Theorem 3.1. The proof takes advantage of Bertin and Theodorescu's extension and restriction operators (Ext and Rest) discussed in the last section.

THEOREM 3.2. *Suppose F and G are discrete CDF's with common support S. Let F^* and G^* be the extended CDF's of F and G, respectively. Then $G^* \succ F^*$ if and only if $G \succ F$.*

PROOF. First, let u and v be nondecreasing functions on $\mathbb{Z}$. Then

$$u \succ v \text{ if and only if } \mathrm{Ext}(u) \succ \mathrm{Ext}(v). \tag{3.1}$$

This result is easily seen when we note the fact that $\mathrm{Ext}(u)$ and $\mathrm{Ext}(v)$ are piecewise linear functions on $\mathbb{Z}$. Now, since G is a discrete CDF, $G : S \to [0, 1]$. By definition, $\mathrm{Rest}(G) : S \to [0, 1]$ as well, its image being a subset of the range of G; that is,

$$\mathrm{Rest}(G)(z) = G(z) \forall z \in S.$$

It follows that $G \succ F$ if and only if $\mathrm{Rest}(G) \succ \mathrm{Rest}(F)$. From (3.1) we have that $\mathrm{Rest}(G) \succ \mathrm{Rest}(F)$ if and only if $\mathrm{Ext}[\mathrm{Rest}(G)] \succ \mathrm{Ext}[\mathrm{Rest}(F)]$. But $\mathrm{Ext}[\mathrm{Rest}(G)]$ and $\mathrm{Ext}[\mathrm{Rest}(F)]$ are just G^* and F^*, respectively, concluding the proof. □

COROLLARY 3.1. *Let F and G be discrete CDF's such that $\mu_F = \mu_G$. If $G^* \succ F^*$, then $\nu_G \geq \nu_F$.*

PROOF. The proof follows immediately from Theorems 3.1 and 3.2. □

The variance dilations established in Theorems 3.1 and 3.2, as well as in Corollary 3.1, will enable us to obtain variance upper bounds for discrete unimodal distributions in the next section.

4. Variance upper bounds for discrete unimodal distributions

Upper bounds on variance have been considered by several authors. For a concise overview, see Seaman and Odell (1988). Variance upper bounds for discrete distributions have been considered by Muilwijk (1966), Moors and Muilwijk (1971), Rayner (1975), and Klaassen (1985). Variance inequalities based on characterizations of distribution shape or location have been established for non-discrete distributions with bounded support by Gray and Odell (1967), Jacobson (1969), Seaman *et al.* (1987), and Dharmadhikari and Joag-Dev (1989).

We shall use the variance dilations developed in Section 3 to derive variance upper bounds for discrete unimodal distributions. For the sake of simplicity, we restrict attention to distributions with finite support $\{1, 2, \ldots, n\}$. Denote the class of all distribution functions with this support by $\mathcal{F}_n$. Let $\mathcal{U}_n$ be the class of discrete unimodal distribution functions with support $\{1, 2, \ldots, n\}$. All results may be proved for more general support. From Theorem 3.1 we immediately obtain the following inequality.

THEOREM 4.1. *Let $F \in \mathcal{U}_n$ have PMF f and suppose that $\mu_F = (n+1)/2$. If $f(1) < 1/n$ and $f(n) < 1/n$, then $\nu_F \leq (n^2-1)/12$.*

PROOF. The proof follows when we let g be the uniform PMF in Theorem 3.1. □

To progress beyond this result, we shall make use of the characterization of discrete unimodality presented in Section 2 and the consequent variance dilations. We begin by deriving variance upper bounds for certain discrete unimodal distributions for which both the mean and a mode are known.

Let $F \in \mathcal{U}_n$ with mean μ and mode $m = \min(\mathcal{M}) \neq 1$. Consider the PMF g, defined as follows:

$$g(x; \mu, m) = \begin{cases} \alpha_1 & x = 1, \ldots, m-1, \\ \alpha_2 & x = m, \ldots, n, \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha_1 = (n - 2\mu + m)/[n(m-1)]$, $\alpha_2 = (2\mu - m)/[n(n-m+1)]$, $\mu \in (1, n)$, and $m \in (\max\{2\mu - n, 2 - \delta\}, \min\{2\mu, n + \delta\}) \cap \mathbb{Z}$ with $\delta \in (0, 1)$. We shall refer to a PMF of the form g as a *step-mass function with step at* m. The quantities α_1 and α_2 will be used repeatedly in this section.

The step-mass function defined above has CDF G defined by

$$G(x; \mu, m) = \begin{cases} 0 & x < 1, \\ \alpha_1 t & t \leq x < t+1 \text{ and } 1 \leq t \leq m-1, \\ \alpha_1(m-1) + \alpha_2(t-m+1) & \\ & t \leq x < t+1 \text{ and } m \leq t \leq n-1, \\ 1 & x \geq n. \end{cases} \tag{4.1}$$

We shall need the extended CDF G^*, which is given by

$$G^*(x; \mu, m) = \begin{cases} 0 & x < 0, \\ \alpha_1 x & 0 \leq x < m-1, \\ 1 + \alpha_2(x-n) & m-1 \leq x < n, \\ 1 & x \geq n. \end{cases} \tag{4.2}$$

Note that G^* is a segmented linear function of x having slope α_1 between 0 and $m-1$ and slope α_2 between $m-1$ and n, and the two line segments coincide at $x = m-1$.

THEOREM 4.2. *Let G be defined as in (4.1). Let $F \in \mathcal{U}_n$ with mode $m = \min(\mathcal{M}) \neq 1$, mean μ, and PMF f. If $f(1) \leq \alpha_1 = [n - 2\mu + m]/[n(m-1)]$ and $f(n) \leq \alpha_2 = [2\mu - m]/[n(n-m+1)]$, then $\nu_F \leq \nu_G$; that is,*

$$\nu_F \leq \frac{m^2(n-2\mu+1) + m(\mu - 1 - 2n - n^2) + 2\mu n^2 + 3\mu n + \mu}{3(n-m+1)} - \mu^2 \equiv B_{\mu,m}.$$

PROOF. There are four cases. First, suppose $f(1) < \alpha_1$ and $f(n) < \alpha_2$. Define G and G^* as in (4.1) and (4.2), respectively. Then $\mu_G = \mu$ and $\nu_G = B_{\mu,m}$. Since F is unimodal with mode m, F^* is convex on $(-\infty, m-1]$ and concave on $[m-1, \infty)$. Also, since $f(1) < \alpha_1$ and $f(n) < \alpha_2$, F^* must cross G^* at least once; that is, $\mathcal{S}(F^* - G^*) \geq 1$. In fact, F^* and G^* satisfy the crossing condition (see Section 3) since they change from convex to concave at the same point. Thus, $G^* \succ F^*$ and by Corollary 3.1 $\nu_G \geq \nu_F$.

Now suppose $f(1) = \alpha_1$ and $f(n) < \alpha_2$. Then $F^*(1) = G^*(1)$, and, since F^* is convex on $[1, m-1]$, $F^*(x) \geq G^*(x)$ for $x \leq m-1$. Since $f(n) < \alpha_2$, $F^*(n-1) \geq G^*(n-1)$. Furthermore, since $F^*(x)$ is concave on $[m-1, n]$, $F^*(x) \geq G^*(x)$ on $[m-1, n]$. Thus, $F^*(x) \geq G^*(x)$ for all x so that the means cannot be equal. Since we have constructed G to have the same mean as F, we have a contradiction.

Next suppose $f(1) < \alpha_1$ and $f(n) = \alpha_2$. Then $F^*(x) \leq G^*(x)$ for all x so that, again, the means cannot be equal, and we have a contradiction. Finally, if $f(1) = \alpha_1$ and $f(n) = \alpha_2$, then $F^*(x) = G^*(x)$ for all x, and $\nu_F = \nu_G$. □

The variance upper bound in the previous theorem is a function of the mean and a mode. If the mean of a discrete unimodal distribution can be restricted in a certain way then we can obtain a variance upper bound which is a function of only the mode.

COROLLARY 4.1.1. *Suppose $F \in \mathcal{U}_n$ with PMF f and unknown mean $\mu \in (\max\{m/2, 1\}, \min\{(n+m)/2, n\})$, where $m = \min(\mathcal{M}) \neq 1$ and is known. If $f(1) \leq \alpha_1 = [n - 2\mu + m]/[n(m-1)]$ and $f(n) \leq \alpha_2 = [2\mu - m]/[n(n-m+1)]$, then*

$$\nu_F \leq \frac{(-2m^2 + m + 2n^2 + 3n + 1)^2 + 12m(n-m+1)(mn + m - 1 - 2n - n^2)}{36(n-m+1)^2}$$
$$\equiv B_m.$$

PROOF. The proof follows immediately from an application of the differential calculus. □

Although exact knowledge of μ is unnecessary to obtain B_m, we must have an upper and lower bound on μ in order to check that $\mu \in (\max\{m/2, 1\}, \min\{(n + m)/2, n\})$ and to verify that the conditions on $f(1)$ and $f(n)$ hold. We now obtain a result for the case when μ is known and m is unknown.

COROLLARY 4.1.2. *Let $F \in \mathcal{U}_n$ with PMF f, known mean μ, and unknown mode $m = \min(\mathcal{M})$. If $m \in (\max\{2\mu - n, 2(n-\mu)/(n-1)\}, \min\{2\mu, n+\delta\}) \cap \mathbb{Z}$ with $\delta \in (0,1)$ such that $f(1) \leq \alpha_1 = [n - 2\mu + m]/[n(m-1)]$ and $f(n) \leq \alpha_2 = [2\mu - m]/[n(n-m+1)]$, then $\nu_F \leq B_\mu$, where*

$$B_\mu = \begin{cases} \dfrac{(4n+1)\mu - n(n+1)}{3} - \mu^2 & \textit{if } 2\mu > n+1, \\ \dfrac{(n^2-1)}{12} & \textit{if } 2\mu = n+1, \\ \dfrac{(2n+5)\mu - 2(n+1)}{3} - \mu^2 & \textit{if } 2\mu < n+1. \end{cases}$$

PROOF. The proof is an application of the differential calculus. □

Example 4.1. Consider the three mass functions presented in Table 1. For these PMF's the bounds are closest to the variance for the "flattest" mass function, $f_3(x)$, and they are farthest from the variance for the mass function with the most prominent mode, $f_2(x)$.

Table 1. Discrete distributions, means, variances, and variance upper bounds for Example 4.1.

Distribution	x					Variance Bounds				
	1	2	3	4	5	μ_F	ν_F	$B_{\mu,m}$	B_m	B_μ
$f_1(x)$	.10	.40	.20	.20	.10	2.8	1.36	2.16	2.25	2.16
$f_2(x)$	.10	.20	.50	.10	.10	2.9	1.09	2.203	2.208	2.09
$f_3(x)$	.15	.20	.20	.30	.15	3.1	1.69	2.203	2.208	2.09

The bounds obtained in Theorem 4.2 and Corollaries 4.1.1 and 4.1.2 utilize a step PMF g with step at the mode m. Using similar methods of proof, one may obtain slightly different bounds by allowing g to step at $m+1$ or at $m-1$ although there is no uniform advantage in doing so. Given the mean, one can specify which of these various bounds (i.e., Theorem 4.2, Corollaries 4.1.1 and 4.1.2 using a step at m, $m+1$ or $m-1$) is tightest.

5. Comparison with previous variance upper bounds

We now compare the bounds derived in Section 4 with variance upper bounds for discrete distributions found in the literature. In particular, we shall compare the bounds derived in Theorem 4.2 ($B_{\mu,m}$), Corollary 4.1.1 (B_m), and Corollary 4.1.2 (B_μ) with those found in Muilwijk (1966), Moors and Muilwijk (1971), and Rayner (1975). The bounds we compare make various assumptions concerning what is known about the distribution in question. Thus, when deciding whether one bound is "better" than another, we must take care to consider the relative restriction of the assumptions employed.

We should note that Chernoff-type bounds have been developed for discrete distributions. Such bounds require much more extensive knowledge of the distribution than those bounds presented here. At the other extreme is a bound which makes use of *less* information than any considered here—the so-called "1/4-th bound". That is, for any distribution which has support which is a subset of the interval $[a, b]$, the variance of that distribution cannot exceed $(b-a)^2/4$. This well known result is of little interest here because, not surprisingly, it is inferior to all of the bounds considered in this paper. For a discussion of this variance upper bound see Seaman and Odell (1985, 1988) or Seaman *et al.* (1992). For an overview of Chernoff-type inequalities, including bounds for discrete distributions, see Klaassen (1985).

We begin by stating the bounds to be found in the literature. The most general has been given by Muilwijk (1966) and applies to any mass function having known finite support and expectation. We continue to state bounds in terms of support $\{1, 2, \ldots, n\}$. However, like the bounds developed in this paper, these inequalities hold for more general support. Again, denote by $\mathcal{F}_n$ and $\mathcal{U}_n$ the class of discrete distributions and discrete unimodal distributions, respectively, with support $\{1, 2, \ldots, n\}$.

THEOREM 5.1. (Muilwijk (1966)) *If* $F \in \mathcal{F}_n$, *then* $\nu_F \leq (n - \mu_F)(\mu_F - 1) \equiv M_\mu$.

Moors and Muilwijk (1971) have derived an upper bound which requires more knowledge about the probability distribution than the support and expected value. Specifically, the Moors and Muilwijk variance upper bound requires knowledge about the discrete probabilities themselves.

THEOREM 5.2. (Moors and Muilwijk (1971)) *Let* $F \in \mathcal{F}_n$ *have PMF* f *such that* $f(i) = r_i/r$, *where* r_i *and* r *are positive integral numbers and* $r = \sum r_i$. *Then*

$$\nu_F \leq (n - \mu_F)(\mu_F - 1) - (\gamma - \gamma^2)(n-1)^2/r \equiv M_{\mu,p},$$

where γ *denotes the fractional part of* $r(\mu_F - 1)/(n-1)$.

Rayner (1975) has obtained the following variance upper bound, which also assumes extensive knowledge of the probability distribution.

THEOREM 5.3. *Suppose* $F \in \mathcal{F}_n$ *with PMF* f. *Define* $t(x, y) = f(x)/[f(x) + f(y)]$. *Then*

$$\nu_F \leq \begin{cases} (n - \mu_F)(\mu_F - 1) - c(1, 2) & \text{if } f(1) \text{ and } f(2) \text{ are known,} \\ (n - \mu_F)(\mu_F - 1) + c(n-1, n) & \text{if } f(n-1) \text{ and } f(n) \text{ are known,} \end{cases}$$

where $c(x,y) = (y-x)[n-\mu_F - t(x,y)(n-1)]$. *If both of the pairs* $\{f(1), f(2)\}$ *and* $\{f(n-1), f(n)\}$ *are known, then the bound becomes*

$$\nu_F \le \min\{(n-\mu_F)(\mu_F - 1) - c(1,2), (n-\mu_F)(\mu_F - 1) + c(n-1,n)\} \equiv R_{\mu,p}.$$

We now compare the bounds $B_{\mu,m}$, B_μ, and B_m with M_μ, $M_{\mu,p}$, and $R_{\mu,p}$. The smallest support size for which $B_{\mu,m}$, B_μ, and B_m can be applied is $n = 3$. However, application of Rayner's bound for $n \le 4$ requires complete knowledge of the probability mass function f since μ_F must also be known. Thus, comparisons involving Rayner's bounds are of little interest for $n \le 4$.

We have found analytical comparisons to be intractable in all but two cases. We can show analytically that $B_\mu < M_\mu$ for all $n \ge 4$. Furthermore, $B_\mu < R_{\mu,p}$ when $n > 5$ and $(4n-2)/(n+1) < \mu_F < (n^2 - 2n + 3)/(n+1)$. Rayner (1975) has noted that $R_{\mu,p} \le M_\mu$ for all μ and p and has stated conditions under which $R_{\mu,p} \le M_{\mu,p}$.

We have empirically compared all of the bounds using several hundred differently shaped distributions and a variety of support sizes. Examples for comparison were easily obtained by discretizing unimodal members of the beta family. A few representative examples are shown in Table 2 and Figs. 1 and 2. These results are typical of what was observed in our numerical comparisons.

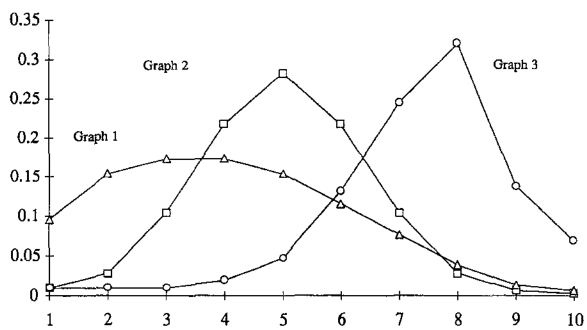

Fig. 1. Discrete probability distributions for Table 2.

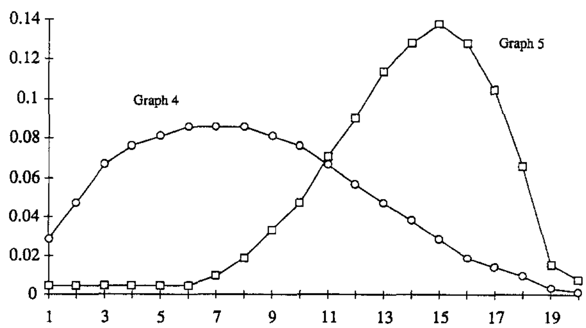

Fig. 2. Discrete probability distributions for Table 2.

Table 2. Values of the variance and the variance upper bounds for the distributions illustrated in Figs. 1 and 2. Numbers in italics refer to values of the support. The number of each row corresponds to the numbers on the graphs in Figs. 1 and 2.

	Probabilities										ν_F	$B_{\mu,m}$	B_m	B_μ	$R_{\mu,p}$	$M_{\mu,p}$	M_μ
1	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	2.10	8.33	8.36	9.33	17.26	19.91	20.00
	0.94	2.83	10.38	21.70	28.30	21.70	10.38	2.83	0.63	0.31							
2	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	2.57	7.22	8.69	9.73	13.33	16.62	16.71
	0.94	0.94	0.94	1.89	4.72	13.21	24.53	32.08	13.84	6.92							
3	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	4.12	9.09	9.25	10.02	15.86	18.20	18.30
	9.61	15.38	17.31	17.31	15.38	11.54	7.69	3.85	1.28	0.64							
4	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	10.34	31.33	35.03	42.18	73.22	79.26	79.64
	0.47	0.47	0.47	0.47	0.47	0.47	0.95	1.90	3.32	4.74							
	11	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19*	*20*							
	7.11	9.00	11.37	12.80	13.74	12.80	10.43	6.64	1.58	0.79							
5	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	16.94	35.54	36.03	41.85	79.92	84.45	84.66
	2.86	4.76	6.67	7.62	8.10	8.57	8.57	8.57	8.10	7.62							
	11	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19*	*20*							
	6.67	5.71	4.76	3.81	2.86	1.90	1.43	0.95	0.32	0.16							

For the examples in the table, the bounds M_μ and $M_{\mu,p}$ are uniformly larger than $B_{\mu,m}$, B_μ, and B_m. The differences increase with the size of the support. For $n = 5$ the performance of $B_{\mu,m}$ and $R_{\mu,p}$ is similar with neither clearly dominant. (Note that we have employed $R_{\mu,p}$, the minimum of the two applicable bounds given in Theorem 5.3. Of course, this choice is to the advantage of Rayner's bound.)

In our empirical studies, as n increased beyond 5, $B_{\mu,m}$ was found to be markedly superior to all of the other bounds considered here indicating the value of knowing that a distribution is unimodal.

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