Ann. Inst. Statist. Math.
Vol. 45, No. 3, 579–597 (1993)

ON OPTIMUM INVARIANT TESTS OF EQUALITY OF INTRACLASS CORRELATION COEFFICIENTS

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(Received July 22, 1991; revised July 27, 1992)

Abstract. In this paper we address the problem of testing the equality of k intraclass correlation coefficients based on samples from independent p-variate normal populations, and explore various aspects of optimality through invariance. A UMPIU test is derived for $k = 2$, and LMMPIU test of SenGupta and Vermeire (1986) is indicated for $k > 2$. Several approximately optimum invariant tests are also proposed. The tests are compared with the approximate LR tests and Fisher's Z-tests derived in Konishi and Gupta (1987, 1989). As expected, the performance of the proposed tests turns out to be quite satisfactory and superior to the LR tests and Z-tests.

Key words and phrases: Intraclass correlation, invariance, locally most powerful invariant unbiased test, uniformly most powerful invariant unbiased test.

1. Introduction

The intraclass correlation coefficient ρ is frequently used to measure the degree of intrafamily resemblence with respect to characteristics such as blood pressure, weight, height, stature, lung capacity, etc. Statistical inference concerning ρ for a single sample problem based on a normal distribution has been studied by several authors (Scheffe (1959), Rao (1973), Rosner *et al.* (1977, 1979), Donner and Bull (1983), Srivastava (1984), Konishi (1985), Gokhale and SenGupta (1986), SenGupta (1988)). Surprisingly, however, its extension to multisample problems based on several multivariate normal distributions has received very little attention. While simultaneous estimation of several intraclass correlation coefficients can be handled without much difficulty, the problem of testing their equality can indeed be challenging. Below we review the literature on this latter problem.

For testing the equality of two intraclass correlation coefficients based on two independent multinormal samples, Donner and Bull (1983) discussed the likelihood ratio test. This, however, involves an iterative maximization of the likelihood function. Konishi and Gupta (1987) proposed a modified likelihood ratio test and derived its asymptotic null distribution. They also discussed another test procedure based on a modification of Fisher's Z-transformation, following Konishi (1985).

Most recently, Konishi and Gupta (1989) treated the problem of testing the equality of more than two intraclass correlation coefficients based on independent samples from several multinormal distributions. Noting that the implementation of the true likelihood ratio test can be very difficult, these authors proposed an approximate likelihood ratio (ALR) test, derived its asymptotic null and nonnull distributions, and also considered another test procedure based on a modification of Fisher's Z-transformation.

The existence of an "optimum" test, however, for the problem of testing the equality of k (two or more) intraclass correlation coefficients based on multinormal samples has not been attempted so far. *This is precisely the objective of the present investigation.* By employing the powerful tool of invariance, we investigate the existence of either a uniformly most powerful invariant unbiased (UMPIU) test or a locally best invariant unbiased (LBIU) test for the above problem. It turns out that, for $k = 2$, a UMPIU test exists quite generally. For $k > 2$, although a LBIU, more specifically locally most mean power unbiased (LMMPU) test of SenGupta and Vermeire (1986) can be easily described, its implementation seems to be quite difficult. Various approximately optimum invariant tests are suggested for $k > 2$, and compared with the ALR and Z-tests of Konishi and Gupta (1989). It turns out that the performance of our proposed tests is far superior to those of Konishi and Gupta (1989).

To describe the invariance approach, suppose that a random sample of size n_i is available from the p-variate normal population $N_p[\mu_i\mathbf{1}, \sigma_i^2\{(1-\rho_i)I_p + \rho_i\mathbf{1}\mathbf{1}'\}]$, $i = 1, 2, \ldots, k$, where μ_i, σ_i^2 and ρ_i are the common mean, common variance and common intraclass correlation in the i-th population. By using a standard canonical reduction (Rao (1973)), the underlying statistical model can be described as involving the variables $\{X_i, Y_i, Z_i, i = 1, \ldots, k\}$ distributed independently as

$$\begin{aligned} &X_i \sim N[\mu_i(pn_i)^{1/2}, \sigma_i^2(1+(p-1)\rho_i)], \quad i = 1, \ldots, k, \\ &Y_i \sim \sigma_i^2(1+(p-1)\rho_i)\chi^2_{\nu_i}, \quad i = 1, \ldots, k, \\ &Z_i \sim \sigma_i^2(1-\rho_i)\chi^2_{m_i}, \quad i = 1, \ldots, k \end{aligned} \tag{1.1}$$

where $\nu_i = n_i - 1$, $m_i = n_i(p-1)$, $i = 1, 2, \ldots, k$.

A natural group of transformations keeping the testing problem $H_0 : \rho_1 = \cdots = \rho_k$ *versus* H_1: *not all* ρ_i's *are equal* invariant is easily seen to be $\mathcal{G}$ whose typical element g can be expressed as $g = (\delta_i, \xi_i, i = 1, \ldots, k)$ where δ_i's are reals and $\xi_i > 0$, $i = 1, \ldots, k$. The group operation (action of g on $[(X_i, Y_i, Z_i), i = 1, \ldots, k]$) can be described as

$$g[(X_i, Y_i, Z_i), i = 1, \ldots, k] = [\xi_i X_i + \delta_i, \xi_i Y_i, \xi_i Z_i, i = 1, \ldots, k]. \tag{1.2}$$

A maximal invariant statistic under the above action is the vector of ratios $(Y_i/Z_i, i = 1, \ldots, k)$ and a maximal invariant parameter is given by $((1 + (p - 1)\rho_i)/(1-\rho_i), i = 1, \ldots, k)$. Writing $F_i = Y_i/Z_i$ and $\theta_i = (1+(p-1)/\rho_i)/(1-\rho_i)$, $i = 1, \ldots, k$ it then follows that an invariant test of H_0 versus H_1 under $\mathcal{G}$ must depend on the F_i's. In terms of the maximal invariant parameters θ_i's, the null hypotheses H_0 corresponds to their equality. By using the fact that the random variables F_i's are independently distributed as scaled F-variables with θ_i's as scale parameters, and employing yet another group of scale multiplication we derive below an optimum test of H_0 versus H_1. In Section 2, we consider the case $k = 2$ and derive a UMPIU test. Some asymptotic approximations are also suggested for large n_i's. Section 3 deals with the case $k > 2$. A LBIU, more specifically LMMPU test of SenGupta and Vermeire (1986) test is discussed and several asymptotically optimum invariant tests are proposed. It may be noted that Cohen *et al.* (1985) and most recently Kariya and Sinha (1991) discussed the LBI tests for homogeneity in multiparameter exponential families. However, this is not directly applicable in our problem since a scaled F-distribution does not belong to an exponential family.

It should also be noted that when the constants ν_i's and m_i's are equal (this happens when the sample sizes n_i's from the k populations are the same), there exists a permutation group $\mathcal{P}$ whose action keeps the underlying testing problem invariant. In both the cases of $k = 2$ and $k > 2$, we have used $\mathcal{P}$ to discuss relevant optimum invariant tests when the equality of n_i's holds.

We conclude this section with the important observation that, unlike the tests known so far which are valid only for large samples, our proposed optimum tests are valid invariant tests irrespective of the nature of sample sizes. This is a significant improvement for all k, and particularly for $k = 2$. Of course, as a referee pointed out, our invariant tests have the demerit that neither equal differences nor equal ratios of the intraclass correlations are detected with the same probability.

2. Test of $H_0 : \rho_1 = \rho_2$ versus $H_1 : \rho_1 \neq \rho_2$

For $k = 2$, the relevant invariant statistics are $F_i \sim \theta_i \chi^2_{\nu_i}/\chi^2_{m_i}$, $\theta_i = \{1 + (p-1)\rho_i\}/(1-\rho_i)$, $i = 1, 2$. Clearly the problem of testing $H_0 : \theta_1 = \theta_2$ versus $H_1 : \theta_1 \neq \theta_2$ remains invariant under a scalar multiplication by $c > 0$. A maximal invariant statistic is $U = F_1/F_2$ whose pdf depends only on $\delta = \theta_2/\theta_1$ and is given by

$$f_\delta(u) = \frac{u^{\nu_1/2-1}\delta^{\nu_1/2}}{B\left(\frac{\nu_1}{2}, \frac{m_1}{2}\right) B\left(\frac{\nu_2}{2}, \frac{m_2}{2}\right)} \cdot \left[\int_0^\infty \frac{x^{(\nu_1+\nu_2)/2-1}dx}{(1+x)^{(\nu_2+m_2)/2}(1+ux\delta)^{(\nu_1+m_1)/2}}\right]. \tag{2.1}$$

For testing $H_0 : \delta = 1$ versus $H_1 : \delta \neq 1$ based on the ultimate invariant statistic U, we proceed in the usual fashion (see Lehmann (1986), Chapter 4). It is argued below that the UMPU test $\phi(u)$ based on U is given by

$$\phi(u) = \begin{cases} 1 & \text{if } u < d_1 \text{ or } u > d_2 \\ 0 & \text{otherwise} \end{cases} \tag{2.2}$$

where d_1 and d_2 are obtained from the size and (local) unbiasedness conditions given respectively by

$$1-\alpha=\int_{d_1}^{d_2} f_{\delta=1}(u)du \tag{2.3}$$

and

$$0=\int \phi(u)f'_{\delta=1}(u)du \tag{2.4}$$

or, equivalently,

$$\frac{\nu_1}{\nu_1+m_1}(1-\alpha)=\int_{d_1}^{d_2}\frac{u^{\nu_1/2}}{B\left(\frac{\nu_1}{2},\frac{m_1}{2}\right)B\left(\frac{\nu_2}{2},\frac{m_2}{2}\right)} \cdot\left[\int_0^\infty \frac{x^{(\nu_1+\nu_2)/2}dx}{(1+x)^{(\nu_2+m_2)/2}(1+ux)^{(\nu_1+m_1)/2+1}}\right]du.$$

That the above test is globally unbiased follows from a comparison with the trivial test $\phi^*(u)\equiv\alpha$ (see Lehmann (1986), pp. 136–137). For the original model (1.1), this test is therefere UMPIU. Following Lehmann (1986), for any fixed $\delta\neq 1$, the rejection region R of a UMPIU test can be written as

$$R=\left\{u: f_\delta(u)\geq c_1^*\frac{\partial f_\delta(u)}{\partial\delta}\bigg|_{\delta=1}+c_2^* f_{\delta=1}(u)\right\}. \tag{2.5}$$

Using (2.1) and (2.4), R is equivalent to

$$\begin{aligned} R=\Bigg\{u: &\int_0^\infty \frac{x^{(\nu_1+m_1)/2-1}}{(1+x)^{(\nu_2+m_2)/2}(1+ux\delta)^{(\nu_1+m_1)/2}}dx \\ &\geq c_1\int_0^\infty \frac{ux^{(\nu_1+m_1)/2}dx}{(1+x)^{(\nu_2+m_2)/2}(1+ux)^{(\nu_1+m_1)/2+1}} \\ &+c_2\int_0^\infty \frac{x^{(\nu_1+m_1)/2-1}dx}{(1+x)^{(\nu_2+m_2)/2}(1+ux)^{(\nu_1+m_1)/2}}\Bigg\}. \end{aligned} \tag{2.6}$$

Making the transformation $v=ux$, R can be expressed as

$$\begin{aligned} R=\Bigg\{u: &\int_0^\infty \frac{v^{(\nu_1+m_1)/2-1}dv}{(1+v\delta)^{(\nu_1+m_1)/2}\left(1+\frac{v}{u}\right)^{(\nu_2+m_2)/2}} \\ &\geq c_1\int_0^\infty \frac{v^{(\nu_1+m_1)/2}dv}{\left(1+\frac{v}{u}\right)^{(\nu_2+m_2)/2}(1+v)^{(\nu_1+m_1)/2+1}} \\ &+c_2\int_0^\infty \frac{v^{(\nu_1+m_1)/2-1}dv}{\left(1+\frac{v}{u}\right)^{(\nu_2+m_2)/2}(1+v)^{(\nu_1+m_1)/2}}\Bigg\}. \end{aligned} \tag{2.7}$$

Define

$$(2.8)\qquad \psi(u) = \int_0^\infty \frac{v^{(\nu_1+m_1)/2-1}}{(1+v)^{(\nu_1+m_1)/2}\left(1+\frac{v}{u}\right)^{(\nu_2+m_2)/2}} \cdot \left\{ \left(\frac{1+v}{1+v\delta}\right)^{(\nu_1+m_1)/2} - c_1\frac{v}{1+v} - c_2 \right\} dv$$

so that $R = \{u : \psi(u) \geq 0\}$. It is proved in the Appendix that $\psi(u)$ is convex in u so that R is outside an interval, i.e., $R = \{u : u < d_1(\delta) \text{ or } u > d_2(\delta)\}$. Since $d_1(\delta)$ and $d_2(\delta)$ are obtained from (2.3) and (2.4), it follows that d_1 and d_2 are absolute constants, independent of δ, which justifies (2.2). The power of the test $\phi(u)$ for any $\delta \neq 1$ is obtained as

$$(2.9)\qquad \text{Power}_1(\delta) = 1 - \int_{d_1}^{d_2} f_\delta(u)du = 1 - \int_{d_1\delta}^{d_2\delta} f_{\delta=1}(u)du.$$

Computations to evaluate d_1, d_2 and hence power can be somewhat simplified using the following series expansion of the underlying integrands. Since, quite generally, it holds that

$$(2.10)\qquad \int_0^\infty \frac{x^{r-1}dx}{(1+x)^s(1+ux)^t} = \begin{cases} \sum_{j=0}^\infty \binom{t+j-1}{j}(1-u)^j B(r+j, s+t-r), & 0 < u \leq 1 \\ \sum_{j=0}^\infty \binom{t+j-1}{j}\frac{(u-1)^j}{u^{j+t}} B(r, s+t+j-r), & u > 1 \end{cases}$$

where $r = (\nu_1+\nu_2)/2$, $s = (\nu_2+m_2)/2$, $t = (\nu_1+m_1)/2$, we get

$$(2.11)\qquad f_\delta(u) = \frac{u^{\nu_1/2-1}\delta^{\nu_1/2}}{B\left(\frac{\nu_1}{2}, \frac{m_1}{2}\right)B\left(\frac{\nu_2}{2}, \frac{m_2}{2}\right)} \cdot \begin{cases} \sum_{j=0}^\infty \binom{t+j-1}{j}(1-u\delta)^j B(r+j, s+t-r), & 0 < u\delta \leq 1 \\ \sum_{j=0}^\infty \binom{t+j-1}{j}\frac{(u\delta-1)^j}{(u\delta)^{j+t}} B(r, s+t+j-r), & u\delta > 1, \end{cases}$$

so that

$$(2.12)\qquad f_{\delta=1}(u) = \frac{u^{\nu_1/2-1}}{B\left(\frac{\nu_1}{2}, \frac{m_1}{2}\right)B\left(\frac{\nu_2}{2}, \frac{m_2}{2}\right)} \cdot \begin{cases} \sum_{j=0}^\infty \binom{t+j-1}{j}(1-u)^j B(r+j, s+t-r), & 0 < u \leq 1 \\ \sum_{j=0}^\infty \binom{t+j-1}{j}\frac{(u-1)^j}{u^{j+t}} B(r, s+t+j-r), & u > 1, \end{cases}$$

and the integrand $f^*_{\delta=1}(u)$ in the RHS of the equation following (2.4) as

$$(2.13)\quad f^*_{\delta=1}(u) = \frac{u^{\nu_1/2}}{B\left(\frac{\nu_1}{2}, \frac{m_1}{2}\right) B\left(\frac{\nu_2}{2}, \frac{m_2}{2}\right)} \cdot \begin{cases} \sum_{j=0}^{\infty} \binom{t+j}{j} (1-u)^j B(r+1+j, s+t-r), & 0 < u \le 1 \\ \sum_{j=0}^{\infty} \binom{t+j}{j} \frac{(1-u)^j}{u^{j+t+1}} B(r+1, s+t+j-r), & u > 1. \end{cases}$$

We now proceed to provide two approximations of $\phi(u)$. Our one-step approximation of $\phi(u)$ is based upon replacing Z_i/m_i by its a.s. limit $\sigma_i^2(1-\rho_i)$, $i = 1, 2$. This is easily justified even for moderate sample size. Working with the statistics $F_i^* = Y_i m_i / Z_i \simeq \theta_i \chi^2_{\nu_i}$, $i = 1, 2$, derivation of an optimum invariant test based upon $U^* = F_1^*/F_2^*$, which is a maximal invariant under scale multiplication of F_1^* and F_2^*, is now straightforward. Since the pdf $f_\delta(u^*)$ of U^*, namely,

$$(2.14)\qquad f_\delta(u^*) = \frac{u^{*\nu_1/2-1} \delta^{\nu_1/2}}{B\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right) (1+\delta u^*)^{\frac{\nu_1+\nu_2}{2}}}, \qquad 0 < u^* < \infty,\ 0 < \delta < \infty$$

admits an MLR property, arguing as before, the UMPU test based on U^* is readily obtained as

$$(2.15)\qquad \phi(u^*) = \begin{cases} 1 & \text{if } u^* < e_1 \text{ or } u^* > e_2 \\ 0 & \text{otherwise} \end{cases}$$

where e_1 and e_2 satisfy the conditions:

$$(2.16)\qquad 1 - \alpha = \int_{e_1/(1+e_1)}^{e_2/(1+e_2)} \frac{x^{\nu_1/2-1}(1-x)^{\nu_2/2-1} dx}{B\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)}$$

and

$$(2.17)\qquad 1 - \alpha = \int_{e_1/(1+e_1)}^{e_2/(1+e_2)} \frac{x^{\nu_1/2}(1-x)^{\nu_2/2-1} dx}{B\left(\frac{\nu_1}{2}+1, \frac{\nu_2}{2}\right)}.$$

The power is then obtained as

$$(2.18)\qquad \begin{aligned} \text{Power}_2(\delta) &= 1 - \int_{e_1}^{e_2} f_\delta(u^*) du^* \\ &= 1 - \int_{e_1\delta}^{e_2\delta} f_{\delta=1}(u^*) du^* \\ &= 1 - \int_{e_1\delta/(1+e_1\delta)}^{e_2\delta/(1+e_2\delta)} \frac{x^{\nu_1/2-1}(1-x)^{\nu_2/2-1} dx}{B\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)}. \end{aligned}$$

The parameter consistency of the above test is obvious because for $\delta \to 0$ and $\delta \to \infty$, the power approaches 1.

Our two-step approximation of $\phi(u)$ is based upon the normal approximation of $\chi^2_{\nu_i}$ for large n_i's. It immediately follows from Lehmann ((1986), p. 376) that the optimum invariant test rejects H_0 whenever $\nu_1\nu_2(\nu_1+\nu_2)\{\ln(U^*\nu_2/\nu_1)\}^2/8 > \chi^2_{1;\alpha}$, and that the power of this test can be computed as

$$\text{Power}_3(\delta) = P\{\chi^2_1(\lambda) > \chi^2_{1;\alpha}\} \tag{2.19}$$

where the noncentrality parameter $\lambda = (\ln\delta)^2\nu_1\nu_2/2(\nu_1+\nu_2)$. It is clear that the above power tends to 1 as $\min(\nu_1,\nu_2)$ tends to ∞, thus guaranteeing the (sample size) consistency of this test. Moreover, since $\lambda = (\ln\delta^{-1})^2\nu_1\nu_2/2(\nu_1+\nu_2)$, it is obvious that $\text{Power}_3(\delta) = \text{Power}_3(\delta^{-1})$.

Remark 2.1. When $n_1 = n_2$, one obtains $\nu_1 = \nu_2$ and $m_1 = m_2$, implying that F_1 and F_2 are exchangeable statistics. The action of the multiplication group coupled with the permutation group $\mathcal{P}$ boils down to the consideration of a maximal invariant statistic $V = \max(U, 1/U)$ where $U = F_1/F_2$ as defined earlier, and the UMPI test rejects H_0 for large values of V i.e., for $V > c$. The constant c (obviously > 1) is obtained from the size condition (see (2.3) for a comparison),

$$\alpha = \int_c^\infty f_{\delta=1}(u)du + \int_0^{1/c} f_{\delta=1}(u)du. \tag{2.20}$$

The power of this test is given by

$$\begin{aligned}\text{Power}_4(\delta) &= \int_c^\infty f_\delta(u)du + \int_0^{1/c} f_\delta(u)du \\ &= \int_{c\delta}^\infty f_{\delta=1}(u)du + \int_0^{\delta/c} f_{\delta=1}(u)du.\end{aligned} \tag{2.21}$$

The series expansions given in (2.11) and (2.12) can be used to determine c and power rather easily. Obviously, as expected, $\text{Power}_4(\delta) = \text{Power}_4(\delta^{-1})$.

One-step approximation of the above test, which is analogous to (2.15), corresponds to the rejection of H_0 for $V^* \equiv \max(U^*, 1/U^*) > e$ where $e(> 1)$ satisfies:

$$\alpha = \int_{e/(1+e)}^1 \frac{x^{\nu/2-1}(1-x)^{\nu/2-1}dx}{B\left(\frac{\nu}{2},\frac{\nu}{2}\right)} + \int_0^{1/(1+e)} \frac{x^{\nu/2-1}(1-x)^{\nu/2-1}dx}{B\left(\frac{\nu}{2},\frac{\nu}{2}\right)}, \tag{2.22}$$

with its power given by

$$\begin{aligned}\text{Power}_5(\delta) = &\int_{e\delta/(1+e\delta)}^1 \frac{x^{\nu/2-1}(1-x)^{\nu/2-1}dx}{B\left(\frac{\nu}{2},\frac{\nu}{2}\right)} \\ &+ \int_0^{1/(1+e/\delta)} \frac{x^{\nu/2-1}(1-x)^{\nu/2-1}dx}{B\left(\frac{\nu}{2},\frac{\nu}{2}\right)}.\end{aligned} \tag{2.23}$$

Table 1(a). Exact cut-off points (c) of UMPIU test for $k = 2$, $\alpha = 0.05$.

n_1	n_2	$p/3$	4	5	6
5	5	13.513	12.013	11.345	10.968
10	10	5.236	4.804	4.600	4.480
15	15	3.709	3.456	3.333	3.261

Table 1(a). continued. Powers of UMPIU test for $k = 2$, $\alpha = 0.05$.

$n_1 = 5$, $n_2 = 5$

ρ_1 ρ_2	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9
					$p = 4$					$p = 6$
0.1	0.050	0.076	0.154	0.333	0.735	0.050	0.084	0.204	0.432	0.828
0.3	0.067	0.050	0.074	0.172	0.542	0.083	0.050	0.074	0.204	0.616
0.5	0.125	0.069	0.050	0.084	0.344	0.181	0.080	0.050	0.084	0.386
0.7	0.262	0.149	0.079	0.050	0.160	0.388	0.191	0.088	0.050	0.173
0.9	0.636	0.469	0.303	0.149	0.050	0.792	0.586	0.369	0.168	0.050
	$p = 3$					$p = 5$				

$n_1 = 10$, $n_2 = 10$

ρ_1 ρ_2	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9
					$p = 4$					$p = 6$
0.1	0.050	0.120	0.338	0.694	0.981	0.050	0.165	0.465	0.824	0.995
0.3	0.096	0.050	0.116	0.384	0.906	0.144	0.050	0.138	0.465	0.948
0.5	0.252	0.100	0.050	0.145	0.709	0.408	0.129	0.050	0.165	0.774
0.7	0.565	0.314	0.127	0.050	0.354	0.773	0.431	0.156	0.050	0.394
0.9	0.951	0.846	0.637	0.314	0.050	0.990	0.933	0.749	0.378	0.050
	$p = 3$					$p = 5$				

$n_1 = 15$, $n_2 = 15$

ρ_1 ρ_2	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9
					$p = 4$					$p = 6$
0.1	0.050	0.167	0.501	0.877	0.999	0.050	0.241	0.665	0.955	1.000
0.3	0.126	0.050	0.160	0.562	0.985	0.206	0.050	0.197	0.665	0.995
0.5	0.374	0.132	0.050	0.207	0.888	0.595	0.181	0.050	0.241	0.929
0.7	0.766	0.465	0.177	0.050	0.523	0.928	0.623	0.227	0.050	0.578
0.9	0.995	0.963	0.831	0.465	0.050	1.000	0.992	0.914	0.556	0.050
	$p = 3$					$p = 5$				

Again, as expected, $\text{Power}_5(\delta) = \text{Power}_5(\delta^{-1})$. In the case of the two-step approximation described earlier, the noncentrality parameter λ in (2.19) simplifies to $\nu(\ln \delta)^2/4$.

In Table 1(a) we have presented the exact cut-off points (c) and power of our UMPIU test (see (2.20) and (2.21)) for $n_1 = n_2 = 5, 10, 15$, $p = 3, 4, 5, 6$

Table 1(b). Simulated two-sided cut-off points (d_1, d_2) and powers of UMPIU test, $k = 2$, $\alpha = 0.05$.

	$n_1 = 5$, $n_2 = 10$					$n_1 = 10$, $n_2 = 15$				
ρ_1 ρ_2	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9
	$p = 3$, $d_1 = 0.088$, $d_2 = 6.656$					$p = 3$, $d_1 = 0.232$, $d_2 = 4.298$				
0.1	0.050	0.076	0.152	0.340	0.785	0.050	0.125	0.346	0.693	0.986
0.3	0.080	0.050	0.078	0.189	0.604	0.104	0.050	0.131	0.422	0.931
0.5	0.171	0.081	0.050	0.094	0.396	0.297	0.109	0.050	0.173	0.761
0.7	0.386	0.209	0.100	0.050	0.189	0.643	0.367	0.143	0.050	0.422
0.9	0.804	0.650	0.438	0.209	0.050	0.978	0.913	0.715	0.367	0.050
	$p = 4$, $d_1 = 0.103$, $d_2 = 6.123$					$p = 4$, $d_1 = 0.224$, $d_2 = 3.825$				
0.1	0.050	0.093	0.212	0.463	0.887	0.050	0.137	0.388	0.781	0.996
0.3	0.089	0.050	0.090	0.242	0.706	0.137	0.050	0.131	0.440	0.953
0.5	0.218	0.087	0.050	0.104	0.474	0.409	0.131	0.050	0.167	0.794
0.7	0.481	0.251	0.103	0.050	0.225	0.788	0.465	0.167	0.050	0.406
0.9	0.881	0.719	0.492	0.230	0.050	0.993	0.952	0.802	0.427	0.050
	$p = 5$, $d_1 = 0.097$, $d_2 = 5.399$					$p = 5$, $d_1 = 0.216$, $d_2 = 3.552$				
0.1	0.050	0.092	0.223	0.486	0.906	0.050	0.143	0.425	0.818	0.997
0.3	0.117	0.050	0.085	0.237	0.717	0.184	0.050	0.125	0.451	0.962
0.5	0.292	0.108	0.050	0.096	0.462	0.519	0.164	0.050	0.157	0.796
0.7	0.586	0.308	0.125	0.050	0.205	0.859	0.543	0.200	0.050	0.396
0.9	0.920	0.783	0.563	0.270	0.050	0.997	0.966	0.838	0.483	0.050

and $\alpha = 0.05$. In view of the symmetry of the power function with respect to ρ_1 and ρ_2, we have provided values of power for $\rho_1 \geq \rho_2$ when $p = 3, 5$, and for $\rho_1 \leq \rho_2$ when $p = 4, 6$. In Table 1(b) exact cut-off points (d_1, d_2) and power of the UMPIU test (see (2.2)) are shown for $(n_1, n_2) = (5, 10), (10, 15)$, $p = 3, 4, 5$ and $\alpha = 0.05$. Finally, in Table 1(c), powers of our proposed large sample tests (see (2.23) and (2.19)) and those of ALR and ZT of Konishi and Gupta (1989) are given for $n_1 = n_2 = 25$, $p = 3, 5$ and $\alpha = 0.05$. In Table 1(c), we have also included a negative value of ρ_2. The impressive superiority of our proposed tests and their simplicity are obvious.

3. Test of $H_0 : \rho_1 = \cdots = \rho_k$ versus $H_1 : \rho_i$'s unequal, $k > 2$

For $k > 2$, the relevant invariant statistics under the action of the group $\mathcal{G}$ are $(F_1, \ldots, F_k)$, where $F_i \sim \theta_i \chi^2_{\nu_i} / \chi^2_{m_i}$, and $\theta_i = \{1 + (p-1)\rho_i\}/(1 - \rho_i)$, $i = 1, \ldots, k$. As observed in Section 2, the equivalent problem of testing $H_0 : \theta_1 = \cdots = \theta_k$ versus $H_1 : \theta_i$'s unequal remains invariant under a scalar multiplication of each F_i by $c > 0$. A maximal invariant statistic is easily seen to be $\boldsymbol{U} = (U_1, \ldots, U_{k-1})'$,

Table 1(c). Powers of Proposed Tests* and ALR, ZT tests** for $k = 2$, $\alpha = 0.05$, $n_1 = 25$, $n_2 = 25$.

ρ_1 ρ_2		−0.2	0.1	0.3	0.5	0.7	0.9	−0.2	0.1	0.3	0.5	0.7	0.9
					$p = 3$						$p = 5$		
0.1	P_1	0.652	0.050	0.249	0.750	0.990	1.000	1.000	0.050	0.389	0.899	0.999	1.000
	P_2	0.671	0.050	0.262	0.768	0.992	1.000	1.000	0.050	0.406	0.911	0.999	1.000
	ALR	0.472	0.050	0.176	0.575	0.941	1.000	0.998	0.058	0.315	0.823	0.994	1.000
	ZT	0.472	0.050	0.175	0.572	0.939	1.000	0.998	0.050	0.314	0.820	0.994	1.000
0.3	P_1	0.954	0.249	0.050	0.265	0.852	1.000	1.000	0.389	0.050	0.338	0.917	1.000
	P_2	0.961	0.262	0.050	0.278	0.866	1.000	1.000	0.406	0.050	0.354	0.927	1.000
	ALR	0.861	0.184	0.050	0.201	0.708	0.999	1.000	0.323	0.050	0.278	0.894	1.000
	ZT	0.863	0.186	0.053	0.200	0.703	0.998	1.000	0.324	0.050	0.273	0.884	1.000
0.5	P_1	0.998	0.750	0.265	0.050	0.380	0.996	1.000	0.899	0.338	0.050	0.431	0.998
	P_2	0.999	0.768	0.278	0.050	0.397	0.997	1.000	0.911	0.354	0.050	0.448	0.999
	ALR	0.985	0.582	0.201	0.050	0.281	0.970	1.000	0.083	0.293	0.050	0.382	0.992
	ZT	0.986	0.587	0.203	0.050	0.277	0.969	1.000	0.834	0.295	0.050	0.375	0.991
0.7	P_1	1.000	0.990	0.852	0.380	0.050	0.852	1.000	0.999	0.917	0.431	0.050	0.872
	P_2	1.000	0.992	0.866	0.397	0.050	0.866	1.000	0.999	0.927	0.448	0.050	0.885
	ALR	1.000	0.936	0.679	0.258	0.050	0.693	1.000	0.994	0.851	0.358	0.050	0.806
	ZT	1.000	0.939	0.686	0.260	0.050	0.687	1.000	0.994	0.858	0.363	0.050	0.800
0.9	P_1	1.000	1.000	1.000	0.996	0.050	0.050	1.000	1.000	1.000	0.998	0.872	0.050
	P_2	1.000	1.000	1.000	0.997	0.866	0.050	1.000	1.000	1.000	0.999	0.885	0.050
	ALR	1.000	1.000	0.999	0.973	0.699	0.050	1.000	1.000	1.000	0.991	0.781	0.050
	ZT	1.000	1.000	0.999	0.973	0.702	0.050	1.000	1.000	1.000	0.992	0.786	0.050

*exact (numerical integration).
**simulated (Monote Carlo simulation) (see Konishi and Gupta (1989)).

where $U_i = F_i/F_k$, $i = 1, \ldots, k-1$, with its pdf $f_{\boldsymbol{\delta}}(\boldsymbol{u})$ given by

$$f_{\boldsymbol{\delta}}(\boldsymbol{u}) = \left[\frac{\prod_{i=1}^{k-1}(u_i^{\nu_i/2-1}\delta_i^{\nu_i/2})}{\prod_{i=1}^{k} B\left(\frac{\nu_i}{2}, \frac{m_i}{2}\right)}\right] \cdot \left[\int_0^\infty \frac{x^{\sum_1^k \nu_i/2-1}dx}{(1+x)^{(\nu_k+m_k)/2}\{\prod_{i=1}^{k-1}(1+xu_i\delta_i)^{(\nu_i+m_i)/2}\}}\right] \tag{3.1}$$

where $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_{k-1})'$, $\delta_i = \theta_k/\theta_i$, $i = 1, \ldots, k-1$. In terms of the final maximal invariant parameter $\boldsymbol{\delta}$, the problem reduces to testing the simple null hypothesis $H_0 : \boldsymbol{\delta} = \mathbf{1}$ versus $H_1 : \boldsymbol{\delta} \neq \mathbf{1}$. This is a genuine multiparameter problem for which usually no uniformly optimum unbiased test exists. It is also well known that even a locally most powerful unbiased test of type D (see Isaacson (1951)) is usually hard to construct. Below, following the ideas in SenGupta and Vermeire (1986), we describe a LMMPU test which maximizes 'average' local power among unbiased tests. The following expressions follow directly from (3.1).

$$\left.\frac{\partial f_{\boldsymbol{\delta}}(\boldsymbol{u})}{\partial \delta_i}\right|_{\boldsymbol{\delta}=\mathbf{1}} = \left[\frac{\nu_i}{2} - \frac{\nu_i+m_i}{2}\frac{\int_0^\infty \frac{u_ix}{1+u_ix}\psi_{\boldsymbol{u}}(x)dx}{\int_0^\infty \psi_{\boldsymbol{u}}(x)dx}\right] f_{\boldsymbol{\delta}=\mathbf{1}}(\boldsymbol{u}), \quad i = 1, 2, \ldots, k-1, \tag{3.2}$$

$$\begin{aligned}\left.\frac{\partial^2 f_{\boldsymbol{\delta}}(\boldsymbol{u})}{\partial \delta_i^2}\right|_{\boldsymbol{\delta}=\mathbf{1}} &= \left[\frac{\nu_i(\nu_i-2)}{4} - \frac{\nu_i(\nu_i+m_i)}{2}\frac{\int_0^\infty \frac{u_ix}{1+u_ix}\psi_{\boldsymbol{u}}(x)dx}{\int_0^\infty \psi_{\boldsymbol{u}}(x)dx}\right. \\ &\quad \left. + \frac{(\nu_i+m_i)(\nu_i+m_2+2)}{4}\frac{\int_0^\infty \left(\frac{u_ix}{1+u_ix}\right)^2\psi_{\boldsymbol{u}}(x)dx}{\int_0^\infty \psi_{\boldsymbol{u}}(x)dx}\right] f_{\boldsymbol{\delta}=\mathbf{1}}(\boldsymbol{u}), \\ &\qquad i = 1, \ldots, k-1,\end{aligned} \tag{3.3}$$

where

$$\psi_{\boldsymbol{u}}(x) = x^{\sum_1^k \nu_i/2-1} \Big/ \left\{(1+x)^{(\nu_k+m_k)/2}\prod_1^{k-1}(1+u_ix)^{(\nu_i+m_i)/2}\right\}. \tag{3.4}$$

The size and unbiasedness conditions of a test function $\phi(\boldsymbol{u})$ can be expressed as

$$\int \phi(\boldsymbol{u}) f_{\boldsymbol{\delta}=\mathbf{1}}(\boldsymbol{u}) d_{\boldsymbol{u}} = \alpha, \tag{3.5}$$

$$\int \frac{\phi_{\boldsymbol{u}}\prod_1^{k-1} u_i^{\nu_i/2-1}}{\prod_1^k B\left(\frac{\nu_i}{2}, \frac{m_i}{2}\right)}\left\{\int_0^\infty \frac{u_ix}{1+u_ix}\psi_{\boldsymbol{u}}(x)dx\right\} d\boldsymbol{u} = \frac{\nu_i}{\nu_i+m_i}\alpha, \quad i = 1, \ldots, k-1, \tag{3.6}$$

and following SenGupta and Vermeire (1986), the test function $\phi_0(\boldsymbol{u})$ of a LMMPU test is given by

$$\phi_0(\boldsymbol{u}) = \begin{cases} 1 & \text{if } \sum_1^{k-1}(\nu_i+m_i)(\nu_i+m_i+2)\int_0^\infty \left(\frac{u_i x}{1+u_i x}\right)^2 \psi_{\boldsymbol{u}}(x)dx \\ & \qquad \geq \sum_1^{k-1} c_i \int_0^\infty \frac{u_i x}{1+u_i x}\psi_{\boldsymbol{u}}(x)dx + c_0\int_0^\infty \psi_{\boldsymbol{u}}(x)dx \\ 0 & \text{otherwise} \end{cases} \tag{3.7}$$

where the constants $c_0, c_1, \ldots, c_{k-1}$ are chosen so that $\phi_0(\boldsymbol{u})$ satisfies (3.5) and (3.6). Unfortunately, although a formal description of a LMMPU test is available, its implementation turns out to be quite difficult. For $k = 3$, the integrals involved in (3.7) above can be simplified as indicated below. The simplification is essentially based on (2.10).

Write $r = \sum_1^k \nu_i/2$, $s = (\nu_k + m_k)/2$, $t_i = (\nu_i + m_i)/2$, $i = 1, 2$. Then,

$$\int_0^\infty \psi_{\boldsymbol{u}}(x)dx = \begin{cases} \sum_{j_1=0}^\infty \sum_{j_2=0}^\infty \binom{t_1+j_1-1}{j_1}\binom{t_2+j_2-1}{j_2} \\ \quad (1-u_1)^{j_1}(1-u_2)^{j_2} B(r+j_1+j_2, s+t_1+t_2-r), \\ \qquad \text{for } 0 < u_1, u_2 \leq 1 \\ \sum_{j_1=0}^\infty \sum_{j_2=0}^\infty \binom{t_1+j_1-1}{j_1}\binom{t_2+j_2-1}{j_2} \\ \quad (1-u_1)^{j_1}\frac{(u_2-1)^{j_2}}{u_2^{j_2+t_2}} B(r+j_1, s+t_1+t_2+j_2-r), \\ \qquad \text{for } 0 < u_1 < 1 < u_2 < \infty \\ \sum_{j_1=0}^\infty \sum_{j_2=0}^\infty \binom{t_1+j_1-1}{j_1}\binom{t_2+j_2-1}{j_2} \\ \quad \frac{(u_1-1)^{j_1}}{u_1^{j_1+t_1}}(1-u_2)^{j_2} B(r+j_2, s+t_1+t_2+j_1-r), \\ \qquad \text{for } 0 < u_2 < 1 < u_1 < \infty \\ \sum_{j_1=0}^\infty \sum_{j_2=0}^\infty \binom{t_1+j_1-1}{j_1}\binom{t_2+j_2-1}{j_2} \\ \quad \frac{(u_1-1)^{j_1}(u_2-1)^{j_2}}{u_1^{j_1+t_1}u_2^{j_2+t_2}} B(r, s+t_1+t_2+j_1+j_2-r), \\ \qquad \text{for } 1 < u_1, u_2 < \infty. \end{cases} \tag{3.8}$$

The other integrals

$$\int_0^\infty \frac{u_i x}{1+u_i x}\psi_{\boldsymbol{u}}(x)dx \quad \text{and} \quad \int_0^\infty \left(\frac{u_i x}{1+u_i x}\right)^2 \psi_{\boldsymbol{u}}(x)dx$$

can be analogously evaluated.

As in the case of $k = 2$, our one-step approximation of $\phi_0(\boldsymbol{u})$ is based upon replacing Z_i/m_i by its a.s. limit $\sigma_i^2(1-\rho_i)$, for $i = 1, \ldots, k$, which is equivalent to working with the statistics $F_i^* = Y_i m_i / Z_i \sim \theta_i \chi^2_{\nu_i}$, $i = 1, 2, \ldots, k$. For testing the hypothesis of homogeneity $H_0 : \theta_1 = \cdots = \theta_k$ versus $H_1 : \theta_i$'s unequal, we observe that under the action of a multiplicative group of transformations, the vector $\boldsymbol{U}^* = (U_1^*, \ldots, U_{k-1}^*)$, $U_i^* = F_i^*/F_k^*$, $i = 1, \ldots, k-1$, is a maximal invariant statistic with its pdf (depending only on a maximal invariant parameter $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_{k-1})$, $\delta_i = \theta_k/\theta_i, i = 1, \ldots, k-1$) given by

$$f_{\boldsymbol{\delta}}(\boldsymbol{u}^*) = \left\{ \frac{\Gamma\left(\frac{\sum_1^k \nu_i}{2}\right)}{\prod_1^k \Gamma\left(\frac{\nu_i}{2}\right)} \right\} \frac{(\prod_1^{k-1} u_i^{*\nu_i/2-1} \delta_i^{\nu_i/2})}{(1 + \sum_1^{k-1} u_i^* \delta_i)^{\sum_1^k \nu_i/2}}. \tag{3.9}$$

To derive the LMMPU test based on $\boldsymbol{u}^*$, we compute the following

$$\left.\frac{\partial f_{\boldsymbol{\delta}}(\boldsymbol{u}^*)}{\partial \delta_i}\right|_{\boldsymbol{\delta}=\mathbf{1}} = \left[\frac{\nu_i}{2} - \frac{(\sum_1^k \nu_i) u_i^*}{2(1 + \sum_1^{k-1} u_i^*)}\right] f_{\boldsymbol{\delta}=\mathbf{1}}(\boldsymbol{u}^*), \qquad i = 1, \ldots, k-1 \tag{3.10}$$

$$\begin{aligned} &\left.\frac{\partial^2 f_{\boldsymbol{\delta}}(\boldsymbol{u}^*)}{\partial \delta_i^2}\right|_{\boldsymbol{\delta}=\mathbf{1}} \\ &\quad = \left\{ \frac{-\nu_i}{2} + \frac{(\sum_1^k \nu_i) {u_i^*}^2}{2(1 + \sum_i^{k-1} u_i^*)^2} \right. \\ &\quad\quad \left. + \left[\frac{\nu_i}{2} - \frac{(\sum_1^k \nu_i) u_i^*}{2(1 + \sum_1^{k-1} u_i^*)}\right]^2 \right\} f_{\boldsymbol{\delta}=\mathbf{1}}(\boldsymbol{u}^*), \qquad i = 1, \ldots, k-1. \end{aligned} \tag{3.11}$$

The size and unbiasedness conditions of a test function $\phi(\boldsymbol{u}^*)$ can be expressed as

$$\int \phi(\boldsymbol{u}^*) f_{\boldsymbol{\delta}=\mathbf{1}}(\boldsymbol{u}^*) d\boldsymbol{u}^* = \alpha, \tag{3.12}$$

$$\int \phi(\boldsymbol{u}^*) \left(\frac{u_i^*}{1 + \sum_1^{k-1} u_i^*}\right) f_{\boldsymbol{\delta}=\mathbf{1}}(\boldsymbol{u}^*) d\boldsymbol{u}^* = \left(\frac{\nu_i}{\sum_1^k \nu_i}\right) \alpha, \qquad i = 1, 2, \ldots, k-1. \tag{3.13}$$

Since

$$\begin{aligned} &\sum_{i=1}^{k-1} \left.\frac{\partial^2 f_{\boldsymbol{\delta}}(\boldsymbol{u}^*)}{\partial \delta_i^2}\right|_{\boldsymbol{\delta}=\mathbf{1}} \\ &\quad = \left[\left(\frac{\sum \nu_i}{2}\right)\left(1 + \sum \frac{\nu_i}{2}\right)\left(\sum {u_i^*}^2\right) \Big/ \left(1 + \sum_1^{k-1} u_i^*\right)^2 \right. \\ &\quad\quad - \left(\frac{\sum \nu_i}{2}\right)\left(\sum_1^{k-1} \nu_i u_i^*\right) \Big/ \left(1 + \sum_1^{k-1} u_i^*\right) \\ &\quad\quad \left. + \frac{\nu_i}{2}\left(\frac{\nu_i}{2} - 1\right) \right] f_{\boldsymbol{\delta}=\mathbf{1}}(\boldsymbol{u}^*), \end{aligned} \tag{3.14}$$

we conclude from SenGupta and Vermeire (1986) that the LMMPU test function $\phi_1(\boldsymbol{u}^*)$ can be written as

$$\phi_1(\boldsymbol{u}^*) = \begin{cases} 1 & \text{if} \sum_{i=1}^{k-1} \left(\frac{u_i^*}{1+\sum_1^{k-1} u_i^*} - c_i \right)^2 \geq c_0 \\ 0 & \text{otherwise} \end{cases} \tag{3.15}$$

where the constants $c_0, c_1, \ldots, c_{k-1}$ are chosen so that $\phi_1(\boldsymbol{u}^*)$ satisfies (3.12) and (3.13). Making the transformation $v_i = u_i^*/(1+\sum_1^{k-1} u_i^*)$, $i = 1, \ldots, k-1$, the above can be simplified as

$$\phi_1(\boldsymbol{v}) = \begin{cases} 1 & \text{if} \sum_1^{k-1} (v_i - c_i)^2 \geq c_0 \\ 0 & \text{otherwise} \end{cases} \tag{3.16}$$

where the constants c_i's satisfy the conditions

$$\begin{aligned} &\int \phi_1(\boldsymbol{v}) f_0(\boldsymbol{v}) d\boldsymbol{v} = \alpha, \\ &\int \phi_1(\boldsymbol{v}) v_i f_0(\boldsymbol{v}) d\boldsymbol{v} = \frac{\nu_i}{\sum_1^k \nu_i} \alpha, \qquad i = 1, \ldots, k-1, \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} f_0(\boldsymbol{v}) = &\left[\Gamma\left(\frac{\sum_1^k \nu_i}{2} \right) \Big/ \prod_1^k \Gamma\left(\frac{\nu_i}{2}\right) \right] \left(\prod_1^{k-1} v_i^{\nu_i/2-1} \right) \\ &\cdot (1 - v_1 - \cdots - v_{k-1})^{\nu_k/2} \end{aligned} \tag{3.18}$$

represents a Dirichlet density. Choosing $c_i = E(v_i \mid H_0) = \nu_i / \sum_1^k \nu_i$, $i = 1, \ldots, k-1$, reduces $\phi_1(\boldsymbol{v})$ to

$$\tilde{\phi}_1(\boldsymbol{v}) = \begin{cases} 1 & \text{if} \sum_1^{k-1} \left(v_i - \frac{\nu_i}{\sum_1^k \nu_i} \right)^2 > c_0 \\ 0 & \text{otherwise} \end{cases} \tag{3.19}$$

where the sole constant c_0 satisfies the size condition. In general, the test based on $\tilde{\phi}_1(\boldsymbol{v})$ need not be unbiased. However, when $\nu_1 = \cdots = \nu_k$, in view of the symmetry of $f_0(\boldsymbol{v})$, this test is expected to be unbiased. We have computed the power of this test through simulation in Table 2(a), denoted as P_1, when $\nu_1 = \cdots = \nu_k$.

Our two-step approximation of $\phi_0(\boldsymbol{u})$ depends on the normal approximation of chi-square, and follows directly from Lehmann ((1986), p. 376). We define

$$Q^* = \sum_{i=1}^k \frac{1}{a_i^2} \left(Z_i - \frac{\sum_1^k Z_j / a_j^2}{\sum_1^k 1/a_j^2} \right)^2 \tag{3.20}$$

Table 2(a). Powers of Proposed Tests $(P_1, P_2)^*$ and ALR, ZT tests**, $k = 3$, $\alpha = 0.05$, $n_1 = n_2 = n_3 = 25$.

ρ_1	ρ_2	ρ_3	P_1	P_2	ALR	ZT	P_1	P_2	ALR	ZT
				$p=3$				$p=5$		
0	0.5	0.1	0.945	0.905	0.761	0.729	0.992	0.989	0.957	0.957
		0.2	0.924	0.874	0.721	0.685	0.988	0.982	0.948	0.941
		0.3	0.898	0.874	0.709	0.685	0.985	0.984	0.956	0.946
		0.4	0.888	0.905	0.754	0.729	0.988	0.991	0.970	0.965
		0.5	0.906	0.949	0.823	0.808	0.994	0.997	0.989	0.984
		0.6	0.949	0.984	0.914	0.899	0.998	1.000	0.996	0.996
		0.7	0.987	0.998	0.974	0.968	0.999	1.000	1.000	0.999
		0.8	0.999	1.000	0.997	0.997	1.000	1.000	1.000	1.000
		0.9	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

*exact (numerical integration).
**simulated (Monote Carlo simulation) (see Konishi and Gupta (1989)).

Table 2(b). Powers of Proposed Tests $(P_3)^*$ and ALR, ZT tests**, $k = 3$, $\alpha = 0.05$, $n_1 = n_2 = n_3 = 25$, $\delta_1 + \delta_2 = 2$, $(\delta_1, \delta_2) \neq (1, 1)$.

ρ_1	ρ_2	ρ_3	P_3	ALR	ZT	ρ_2	ρ_3	P_3	ALR	ZT
			$p=3$					$p=5$		
0	0.069	0.032	0.079	0.066	0.062	0.043	0.020	0.079	0.072	0.064
	0.143	0.063	0.182	0.120	0.101	0.091	0.038	0.182	0.119	0.112
	0.222	0.091	0.342	0.211	0.177	0.146	0.057	0.342	0.232	0.204
	0.308	0.118	0.572	0.331	0.301	0.211	0.074	0.572	0.378	0.355
	0.400	0.143	0.783	0.512	0.481	0.286	0.091	0.783	0.595	0.559
	0.500	0.167	0.935	0.731	0.695	0.375	0.107	0.935	0.799	0.778
	0.609	0.189	0.990	0.902	0.886	0.483	0.123	0.990	0.946	0.937
	0.727	0.211	1.000	0.989	0.985	0.615	0.138	1.000	0.995	0.995
	0.857	0.231	1.000	1.000	1.000	0.783	0.153	1.000	1.000	1.000

*exact (numerical integration).
**simulated (Monote Carlo simulation) (see Konishi and Gupta (1989)).

where

$$Z_i = \ln(F_i^*/\nu_i), \qquad a_i^2 = 2/\nu_i, \qquad i = 1, \ldots, k, \tag{3.21}$$

and reject the null hypothesis when $Q^* > \chi^2_{k-1;\alpha}$. It should be noted that this test is UMPI in the limiting normal distribution under actions of appropriate orthogonal transformations. The power of this test is given by

$$\text{Power}_2(\boldsymbol{\delta}) = P\{\chi^2_{k-1}(\lambda) > \chi^2_{k-1;\alpha}\} \tag{3.22}$$

where the noncentrality parameter λ is computed as

$$\lambda = \sum_{i=1}^{k} \frac{1}{a_i^2}\left(\xi_i - \frac{\sum_1^k \xi_i/a_j^2}{\sum_1^k 1/a_j^2}\right)^2 \tag{3.23}$$

$$= \sum_{i=1}^{k} \frac{(\ln\delta_i)^2}{a_i^2} - \frac{\left(\sum_1^k \frac{\ln\delta_j}{a_j^2}\right)^2}{\left(\sum_1^k \frac{1}{a_j^2}\right)}$$

since $\xi_i = \ln\delta_i$, $i = 1, \ldots, k$. Obviously, $\delta_k = 1$. The values of the above power for some selected combinations of design parameters appear in Table 2(a) (denoted as P_2). It may be noted that when the n_i's are equal, λ above reduces to $\lambda = (\nu/2)\sum_{i=1}^k\{\ln\delta_i - (1/k)(\sum_1^k \ln\delta_i)\}^2$. As before, the (sample size) consistency of this test is immediate.

Remark 3.1. When $n_1 = \cdots = n_k$, one obtains $\nu_1 = \cdots = \nu_k = \nu$ and $m_1 = \cdots = m_k = m$, implying thereby that the invariant statistics $F_1, \ldots, F_k$ under the action of the group $\mathcal{G}$ are exchangeable. Applying the multiplicative group on $F_1, \ldots, F_k$, this amounts to the exchangeability of $U_1, \ldots, U_{k-1}$ with the joint pdf given by (see (3.1))

$$f_{\boldsymbol{\delta}}(\boldsymbol{u}) = \frac{(\prod_1^{k-1} u_i)^{\nu/2-1}(\prod_1^{k-1}\delta_i)^{\nu/2}}{B^k\left(\frac{\nu}{2}, \frac{m}{2}\right)} \cdot \int_0^\infty \frac{x^{\nu k/2-1}dx}{(1+x)^{(\nu+m)/2}[\prod_1^{k-1}(1+xu_i\delta_i)]^{(\nu+m)/2}}. \tag{3.24}$$

We denote by $\mathcal{P}$ the permutation group of $(k-1)!$ elements and use $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_{k-1})$ to denote its typical element. It is then clear that the testing problem $H_0 : \boldsymbol{\delta} = \mathbf{1}$ versus $H_1 : \boldsymbol{\delta} \neq \mathbf{1}$ remains invariant under the action of $\mathcal{P}$, and the ratio of the nonnull to null distributions of a maximal invariant statistic $\boldsymbol{T}(\boldsymbol{U})$ is given by (see Wijsman (1967))

$$\frac{dP_{H_1}^{\boldsymbol{T}}(\boldsymbol{t})}{dP_{H_0}^{\boldsymbol{T}}(\boldsymbol{t})} = \left(\prod_1^{k-1}\delta_i\right)^{\nu/2} \cdot \left\{\frac{1}{(k-1)!}\sum_{\boldsymbol{\gamma}}\int_0^\infty \frac{x^{\nu k/2-1}}{(1+x)^{(\nu+m)/2}} \cdot \left[\prod_1^{k-1}(1+xu_{\gamma_i}\delta_i)\right]^{-(\nu+m)/2} dx \Big/ \int_0^\infty \frac{x^{\nu k/2-1}dx}{(1+x)^{(\nu+m)/2}[\prod_1^{k-1}(1+xu_i)]^{(\nu+m)/2}}\right\}. \tag{3.25}$$

Unfortunately, however, unlike in Cohen *et al.* (1985) and Kariya and Sinha (1991), inspite of the action of the group $\mathcal{P}$ (in addition to $\mathcal{G}$ and the multiplicative group), in general an LBIU test does *not* exist (see Kariya and Sinha ((1989), p. 29) for a definition of LBI test). This is solely due to the nonexponential nature of the underlying joint pdf.

On the other hand, derivation of an unbiased test function to maximize the local power in a specific direction is quite possible. For example, it can be shown that a test function in the direction of $\bar{\eta} = 0$ essentially coincides with the LMMPU test $\phi_0(\boldsymbol{u})$ displayed in (3.7) where the ν_i's and m_i's have to be taken equal to ν and m respectively, and $c_1 = \cdots = c_{k-1} = c^*$ and c_0 are obtained such that the size condition (3.5) and the appropriate unbiasedness condition are satisfied. Again, for $k = 3$, (3.8) can be used to simplify some computations.

A similar analysis based on the permutation group $\mathcal{P}$ can be easily done for our one-step approximate statistics $U_1^*, \ldots, U_{k-1}^*$, and it follows that, as in the previous case, in general a LBIU test does *not* exist. However, in the direction of $\bar{\eta} = 0$, which makes sense owing to the nature of the alternative $H_1 : \boldsymbol{\eta} \neq \mathbf{0}$, a restricted LBIU test $\phi^r(\boldsymbol{v})$ has the structure

$$\phi^r(\boldsymbol{v}) = \begin{cases} 1 & \text{if } \sum_1^{k-1}(v_i - \bar{v})^2 > c_0 \\ 0 & \text{otherwise} \end{cases} \tag{3.26}$$

where c_0 is chosen to satisfy the size condition. It should be noted that this test is different from the LMMPU test of SenGupta and Vermeire (1986) derived in (3.16) and also from its approximation $\tilde{\phi}_1(\boldsymbol{v})$ derived in (3.19). Of course, when the n_i's are all equal, the test function $\phi_1(\boldsymbol{v})$ in (3.16) reduces to

$$\phi_1(\boldsymbol{v}) = \begin{cases} 1 & \text{if } \sum_1^{k-1}(v_i - c_1)^2 \geq c_0 \\ 0 & \text{otherwise} \end{cases} \tag{3.27}$$

where c_0 and c_1 are chosen to satisfy the size condition and the appropriate unbiasedness condition.

In Table 2(a), we have presented the power of our proposed large sample tests (see (3.19) and (3.22)) as well as those of ALR and ZT tests of Konishi and Gupta (1989). Table 2(b) contains the power P_3 of our another test $\phi^r(\boldsymbol{v})$ (see (3.26)) and also those of ALR and ZT tests. As expected, in all the cases our proposed tests perform much better than the ALR and ZT tests.

Acknowledgements

Our sincere thanks are due to two referees for their critical comments and helpful suggestions which led to substantial improvements and a clearer presentation. The authors also thank Dr. Y. P. Chang for programming assistance.

Appendix

Here we provide a proof of the convexity of $\psi(u)$ defined in (2.8). Making the transformation: $w = v/(1+v)$, $\psi(u)$ is written as

$$\psi(u) = \int_0^1 \frac{w^{r-s-1}(1-w)^{s-1}}{\left(\frac{1}{u}+\frac{1}{w}-1\right)^s} f(w)dw \tag{A.1}$$

where

$$f(w) = \frac{1}{(1+\theta w)^r} - c_1 w - c_2 \tag{A.2}$$

and $r = (v_1 + m_1)/2$, $s = (v_2 + m_2)/2$, $\theta = \delta - 1$. We now use a powerful result of Karlin ((1968), p. 31, essentially Proposition 3.2). It is easy to verify that $f(w)$ is convex in w whenever $\delta \neq 1$, and hence that convexity of $\psi(u)$ follows once we establish that the function

$$K(w,u) = \frac{w^{r-s-1}(1-w)^{s-1}}{\left(\frac{1}{u}+\frac{1}{w}-1\right)^s} \tag{A.3}$$

is TP_3 (totally positive of order 3). Obviously it is enough to prove the TP_3 property of $K^*(w,u) = (1/u + 1/w - 1)^{-s}$. Now note the following representation of $K^*(w,u)$:

$$K^*(w,u) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-x/u} e^{-x/w} e^x x^{s-1} dx. \tag{A.4}$$

Since $e^{-x/u}$ and $e^{-x/w}$ are both TP_3 in the respective variables, applying the *Basic Composition Formula* of Karlin ((1968), p. 31), we conclude that $K^*(w,u)$ is TP_3. This implies the TP_3 property for $K(w,u)$ and hence establishes the convexity of $\psi(u)$.

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