Ann. Inst. Statist. Math.
Vol. 45, No. 3, 445–452 (1993)

ON A SINGULARITY OCCURRING IN A SELF-CORRECTING POINT PROCESS MODEL*

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(Received September 24, 1991; revised October 26, 1992)

Abstract. In a self-correcting point process model a boundary point of the parameter set is shown to be singular. This means a local behavior of the model which is qualitatively different from the LAN (or LAMN) condition satisfied at the other parameter points. As a consequence we obtain a nonnormal limiting distribution of the ML-estimator normalized with the random Fisher information.

Key words and phrases: Self-correcting point process, locally asymptotically quadratic model, locally asymptotically Brownian functional, ML-estimator, nonnormal limiting distribution.

1. Introduction

We consider the simple self-correcting point process model of Inagaki and Hayashi (1990). Let $N = (N_t)_{t\geq 0}$ be a point process on $(\Omega, \mathcal{F})$ and let $(\mathcal{F}_t)_{t\geq 0}$ denote the filtration that it generates. For $\theta \in (0,\infty)^2$, let P_θ be a probability measure on $(\Omega, \mathcal{F})$ such that N has predictable intensity

$$\lambda(\theta, t) = \theta_1 1_{(-\infty,0]}(t - N_{t-}) + \theta_2 1_{(0,\infty)}(t - N_{t-}) \tag{1.1}$$

with respect to $(\mathcal{F}_t)$ under P_θ. The likelihood ratio process $L_t(\theta,\tau) = (dP_\tau \mid \mathcal{F}_t)/(dP_\theta \mid \mathcal{F}_t)$ is given by

$$\begin{aligned}\log L_t(\theta,\tau) &= \int_0^t \log\frac{\lambda(\tau,s)}{\lambda(\theta,s)} dN_s - \int_0^t (\lambda(\tau,s) - \lambda(\theta,s))ds \\ &= \log\frac{\tau_1}{\theta_1}\int_0^t 1_{(-\infty,0]}(s - N_{s-})dN_s \\ &\quad + \log\frac{\tau_2}{\theta_2}\int_0^t 1_{(0,\infty)}(s - N_{s-})dN_s\end{aligned} \tag{1.2}$$

* Work supported by a Heisenberg grant of the Deutsche Forschungsgemeinschaft.

$$
\begin{aligned}
&- (\tau_1 - \theta_1) \int_0^t 1_{(-\infty,0]}(s - N_s)ds \\
&- (\tau_2 - \theta_2) \int_0^t 1_{(0,\infty)}(s - N_s)ds.
\end{aligned}
$$

Let $\Theta = (0,1) \times (1,\infty)$. For $\theta \in \Theta$, the model has the standard properties of a locally asymptotically normal (LAN) model. In particular, the ML-estimator of θ is asymptotically efficient as $t \to \infty$ (see Inagaki and Hayashi (1990)).

At the boundary point $\theta_o = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ of Θ when the process N reduces to a Poisson process the log-likelihood ratio exhibits a qualitatively different behavior. It still admits a quadratic approximation such that the model is locally asymptotically quadratic (LAQ) but it is not locally asymptotically mixed normal (LAMN) at θ_o. In Luschgy (1992) such parameter points are termed singular. As a consequence we obtain a nonnormal limiting distribution for the ML-estimator normalized with the random Fisher information under the hypothesis $\theta = \theta_o$. This suggests a nonnormal approximation of the distribution of the ML-estimator under $\theta \in \Theta$ for moderate sample size t when the difference $\theta_2 - \theta_1$ of the intensity levels is small, more precisely, when $t^{1/2}(\theta_2 - \theta_1)$ is small.

This note provides further insight into the occurrence of singularities for stochastic process models. While quite often singularities occur under the hypothesis that the observation process is a martingale (see, for example, Luschgy (1992)), here the singularity occurs under the hypothesis that the release process $(t - N_t)_{t \geq 0}$ (see (1.1)) is a martingale.

2. LAQ condition

In this section we describe the local behavior of $\mathcal{E}_t = (P_\theta \mid \mathcal{F}_t : \theta \in \Theta_o)$, $\Theta_o = \Theta \cup \{\theta_o\}$, as $t \to \infty$. Let $\dot{L}_t(\theta)$ denote the first derivative of $L_t(\theta, \tau)$ with respect to τ at $\tau = \theta$, that is

$$
\dot{L}_t(\theta) = \begin{pmatrix} \dfrac{1}{\theta_1} \displaystyle\int_0^t 1_{(-\infty,0]}(s - N_{s-})dM_s(\theta) \\ \dfrac{1}{\theta_2} \displaystyle\int_0^t 1_{(0,\infty)}(s - N_{s-})dM_s(\theta) \end{pmatrix}; \tag{2.1}
$$

where

$$
M_t(\theta) = N_t - \int_0^t \lambda(\theta, s)ds.
$$

Note that $M(\theta)$ and $\dot{L}(\theta)$ are P_θ-martinales with $E_\theta|\dot{L}_t(\theta)|^2 < \infty$ and $E_\theta|M_t(\theta)|^2 < \infty$ for every $t \geq 0$. The Fisher information process is given by the brackett process

$$
\langle \dot{L}(\theta) \rangle_t = \begin{pmatrix} \dfrac{1}{\theta_1} \displaystyle\int_0^t 1_{(-\infty,0]}(s - N_s)ds & 0 \\ * & \dfrac{1}{\theta_2} \displaystyle\int_0^t 1_{(0,\infty)}(s - N_s)ds \end{pmatrix}. \tag{2.2}
$$

Let δ_t be a net of positive numbers with $\delta_t \to 0$. Then the net of experiments $\mathcal{E}_t$ is said to be LAQ at $\theta \in \Theta_o$ with localization rate δ_t as $t \to \infty$ if

$$\log L_t(\theta, \theta + \delta_t u_t) = u_t^T \delta_t \dot{L}_t(\theta) - \frac{1}{2} u_t^T \delta_t^2 \langle \dot{L}(\theta) \rangle_t u_t + o_{P_\theta}(1) \tag{2.3}$$

for every eventually bounded net (u_t) in $\mathbb{R}^2$ (T denotes transposition),

$$\mathcal{L}(\delta_t \dot{L}_t(\theta), \delta_t^2 \langle \dot{L}(\theta) \rangle_t \mid P_\theta) \xrightarrow{\mathcal{D}} (S(\theta), G(\theta)) \tag{2.4}$$

with a random symmetric 2×2 matrix $G(\theta)$ which is positive definite a.s. and

$$E\exp\left(u^T S(\theta) - \frac{1}{2} u^T G(\theta) u\right) = 1 \quad \text{for every } u \in \mathbb{R}^2 \tag{2.5}$$

(see Jeganathan (1988), LeCam and Yang (1990)). The LAMN case appears if $\mathcal{L}(S(\theta), G(\theta)) = \mathcal{L}(G(\theta)^{1/2} Z, G(\theta))$, where $\mathcal{L}(Z) = N(0, I_2)$ and Z and $G(\theta)$ are independent. The case where $G(\theta)$ is nonrandom is called LAN. In the LAMN case the condition (2.5) is automatically satisfied.

It has been shown by Inagaki and Hayashi (1990) that $\mathcal{E}_t$ satisfies LAN at every $\theta \in \Theta$ with rate $t^{-1/2}$ and

$$G(\theta) = (\theta_2 - \theta_1)^{-1} \begin{pmatrix} (\theta_2 - 1)/\theta_1 & 0 \\ * & (1 - \theta_1)/\theta_2 \end{pmatrix}.$$

We show that $\mathcal{E}_t$ satisfies LAQ at θ_o with the same rate but not LAMN.

THEOREM 2.1. *$\mathcal{E}_t$ satisfies LAQ at θ_o with rate $t^{-1/2}$ and*

$$S(\theta_o) = V_1, \qquad G(\theta_o) = \langle V \rangle_1, \tag{2.6}$$

where

$$V_s = \left(\int_0^s 1_{[0,\infty)}(W_r) dW_r, \int_0^s 1_{(-\infty,0)}(W_r) dW_r \right)^T,$$

$$\langle V \rangle_s = \begin{pmatrix} \int_0^s 1_{[0,\infty)}(W_r) dr & 0 \\ * & \int_0^s 1_{(-\infty,0)}(W_r) dr \end{pmatrix}, \qquad s \geq 0,$$

and W is a standard Wiener process.

The form of the limit $S(\theta_o) = V_1$ shows that $\mathcal{E}_t$ even has locally asymptotically Brownian functional (LABF) likelihood ratios at θ_o.

PROOF. For $t > 0$, let $Y_s^t = t^{-1/2} \dot{L}_{st}(\theta_o)$ and $M_s^t = t^{-1/2} M_{st}(\theta_o)$, $s \geq 0$. These processes are P_{θ_o}-martingales with respect to the filtration $(\mathcal{F}_{st})_{s \geq 0}$ with finite second moments and $Y_o^t = M_o^t = 0$. Note that

$$Y_s^t = \left(\int_0^s 1_{[0,\infty)}(M_{r-}^t) dM_r^t, \int_0^s 1_{(-\infty,0)}(M_{r-}^t) dM_r^t \right)^T$$

and

$$\langle Y^t \rangle_s = t^{-1} \langle \dot{L}(\theta_o) \rangle_{st} = \begin{pmatrix} \int_0^s 1_{[0,\infty)}(M_r^t)dr & 0 \\ * & \int_0^s 1_{(-\infty,0)}(M_r^t)dr \end{pmatrix}.$$

Let $Z^t = \begin{pmatrix} M^t \\ Y^t \end{pmatrix}$. We claim that

$$\mathcal{L}(Z^t \mid P_{\theta_o}) \xrightarrow{\mathcal{D}} \begin{pmatrix} W \\ V \end{pmatrix} \quad \text{as} \quad t \to \infty \tag{2.7}$$

(weak convergence in $D(\mathbb{R}_+, \mathbb{R}^3)$ equipped with the Skorokhod topology). Fix a sequence (t_n) going to infinity. Since

$$\langle Y_1^t \rangle_s + \langle Y_2^t \rangle_s = s \qquad \text{for every } t, s,$$

the sequence (Y^{t_n}) is tight in $D(\mathbb{R}_+, \mathbb{R}^2)$ (see Jacod and Shiryaev (1987), VI.4.13). It is well known that

$$\mathcal{L}(M^{t_n} \mid P_{\theta_o}) \xrightarrow{\mathcal{D}} W.$$

Therefore, (Z^{t_n}) is tight in $D(\mathbb{R}_+, \mathbb{R}^3)$. Let Q be a limit point of the sequence $(\mathcal{L}(Z^{t_n} \mid P_{\theta_o}))$ and let Z denote the coordinate process on $D(\mathbb{R}_+, \mathbb{R}^3)$. Clearly $Z_o = 0$ Q-a.s.

In order to identify the predictable characteristics (B, C, ν) of Z under Q we first compute the characteristics (B^t, C^t, ν^t) of Z^t under P_{θ_o} with respect to a continuous truncation function $h : \mathbb{R}^3 \to \mathbb{R}^3$. Since

$$\Delta Z_s^t = t^{-1/2}(\Delta N_{st}, 1_{[0,\infty)}(M_{st-}), 1_{(-\infty,0)}(M_{st-}))^T 1_{\{\Delta N_{st} \neq 0\}},$$

where $\Delta Z_s^t = Z_s^t - Z_{s-}^t$, one obtains for the random measure

$$\mu^t = \sum_{s \geq 0} \epsilon_{(s, \Delta Z_s^t)} 1_{\{\Delta Z_s^t \neq 0\}}$$

on $\mathbb{R}_+ \times \mathbb{R}^3$ associated with the jumps of Z^t

$$\mu^t([0,s] \times A) = \int_0^{st} 1_A(t^{-1/2}(1, 1_{[0,\infty)}(M_{r-}), 1_{(-\infty,0)}(M_{r-}))^T) dN_r$$

for every Borel set A in $\mathbb{R}^3$ with $0 \notin A$. Here ϵ_x denotes the point measure at x. Thus the compensator ν^t of μ^t is given by

$$\nu^t([0,s] \times A) = \int_0^{st} 1_A(t^{-1/2}(1, 1_{[0,\infty)}(M_r), 1_{(-\infty,0)}(M_r))^T) dr.$$

Since Z^t is a martingale,

$$B_s^t = \int_0^s \int (h(x) - x) \nu^t(dr, dx)$$

(see Jacod and Shiryaev (1987), II.2.29). In view of the fact that $h(x) = x$ in a neighborhood of 0 this yields $B_s^t = 0$ for every $s \geq 0$ and t large enough. Clearly $C^t = 0$ for every $t > 0$. The modified second characteristic $\tilde{C}^t$ therefore takes the form

$$\tilde{C}_s^t = \langle Z^t \rangle_s = \begin{pmatrix} s & \int_0^s 1_{[0,\infty)}(M_r^t)dr & \int_0^s 1_{(-\infty,0)}(M_r^t)dr \\ * & & \\ & \langle Y^t \rangle_s & \\ * & & \end{pmatrix}$$

for every $s \geq 0$ and t large enough. Now we apply the "limit identification theorem" (see Jacod and Shiryaev (1987), IX.2.11). Define the continuous process C as $\tilde{C}^t$ with M^t replaced by the first component Z_1 of Z. For every $s \geq 0$, the function

$$D(\mathbb{R}_+, \mathbb{R}^3) \to \mathbb{R}^{3\times 3}, \qquad \alpha \to C_s(\alpha)$$

is Q-a.s. Skorokhod continuous since $Q(Z_{1,s} = 0) = 0$ for every $s > 0$ (see Billingsley (1968), p. 232). All other conditions of the above mentioned theorem concerning the behavior of B^{t_n}, $\tilde{C}^{t_n}$, ν^{t_n} and C are easily seen to be satisfied with $B = 0$, $\nu = 0$ and $\tilde{C} = C$. The theorem gives that Z is a continuous local martingale under Q with characteristics $(0, C, 0)$ relative to the right continuous filtration that it generates.

It follows that

$$\mathcal{L}(Z \mid Q) = \mathcal{L}\left(\left(Z_1, \int_0^\cdot 1_{[0,\infty)}(Z_{1,r})dZ_{1,r}, \int_0^\cdot 1_{(-\infty,0)}(Z_{1,r})dZ_{1,r}\right)^T \middle| Q\right)$$

(see Jacod and Shiryaev (1987), IX.5.6). Since Z_1 is a standard Wiener process under Q, one obtains

$$Q = \mathcal{L}\begin{pmatrix} W \\ V \end{pmatrix}.$$

Hence $\mathcal{L}\begin{pmatrix} W \\ V \end{pmatrix}$ is the only limit point of the sequence (Z^{t_n}) which yields our claim (2.7).

The limit assertion of (2.4) is satisfied for the random variables given in (2.6). This is an immediate consequence of (2.7) and the continuous mapping theorem applied to the $\mathcal{L}\begin{pmatrix} W \\ V \end{pmatrix}$-a.s. continuous function

$$\begin{gathered} D(\mathbb{R}_+, \mathbb{R}^3) \to \mathbb{R}^2 \times \mathbb{R}^{2\times 2}, \\ \alpha \to \left(\begin{pmatrix} \alpha_2(1) \\ \alpha_3(1) \end{pmatrix}, D_1(\alpha)\right), \end{gathered}$$

where D_s is defined as $\langle Y^t \rangle_s$ with M^t replaced by Z_1. Furthermore, Lévy's arcsine law for the occupation time $\int_0^1 1_{[0,\infty)}(W_r)dr$ shows that $\langle V \rangle_1$ is positive definite a.s.

The condition (2.5) is satisfied by Novikov's criterion (see Revuz and Yor (1991), VIII.1.16). Using $\log(1+x) = x - (x^2/2) + o(x^2)$ as $x \to 0$, we find

$$\begin{aligned}\log L_t(\theta_o, \theta_o + t^{-1/2}u_t) &- t^{-1/2}u_t^T \dot{L}_t(\theta_o) + \frac{1}{2}t^{-1}u_t^T \langle \dot{L}(\theta_o)\rangle_t u_t \\ &= -\frac{1}{2}t^{-1}(u_{1,t}^2, u_{2,t}^2)^T \dot{L}_t(\theta_o) + o(t^{-1})N_t \\ &\to 0 \quad \text{in } P_{\theta_o}\text{-probability,}\end{aligned}$$

that is (2.3). The proof is complete. □

From this result and Girsanov's theorem one can deduce that $\mathcal{E}_t$ does not satisfy LAMN at θ_o (see Section 3). Thus θ_o is a singular parameter point.

3. Limiting distribution of the ML-estimator

As a consequence of the local properties of the model we derive the limiting distribution of the ML-estimator of θ. For $t > 0$, let

$$A_t = \{\langle \dot{L}(\theta)\rangle_t \text{ is positive definite}\}.$$

By (2.2), this set does not depend on θ and we have

$$A_t = \left\{\int_0^t 1_{(-\infty,0]}(s - N_s)ds > 0\right\} = \left\{\min_{s<t}(s - N_s) < 0\right\} \in \mathcal{F}_t.$$

Note that for every $\theta \in \Theta_o$, $P_\theta(A_t) < 1$ for every $t > 0$ and $P_\theta(A_t) \to 1$. By (1.2), $L_t(\theta_o, \theta)$ has a unique maximum on A_t. A ML-estimator $\hat{\theta}_t$ is given by

$$\begin{aligned}\hat{\theta}_{1,t} &= \frac{\int_0^t 1_{(-\infty,0]}(s - N_{s-})dN_s}{\int_0^t 1_{(-\infty,0]}(s - N_s)ds} \qquad \text{on } A_t, \\ \hat{\theta}_{2,t} &= \frac{\int_0^t 1_{(0,\infty)}(s - N_{s-})dN_s}{\int_0^t 1_{(0,\infty)}(s - N_s)ds}.\end{aligned} \tag{3.1}$$

For $\theta \in \Theta$,

$$\mathcal{L}(t^{1/2}(\hat{\theta}_t - \theta) \mid P_\theta) \xrightarrow{\mathcal{D}} N(0, G(\theta)^{-1}) \tag{3.2}$$

and under random norming

$$\mathcal{L}(\langle \dot{L}(\theta)\rangle_t^{1/2}(\theta_t - \theta) \mid P_\theta) \xrightarrow{\mathcal{D}} N(0, I_2) \tag{3.3}$$

(see Inagaki and Hayashi (1990)).

Under the Poisson hypothesis $\theta = \theta_o$ or in the nearly Poisson case we have:

THEOREM 3.1. *For every convergent net (u_t) in $\mathbb{R}^2$, $u_t \to u$,*

$$\mathcal{L}(t^{1/2}(\hat{\theta}_t - \theta_o - t^{-1/2}u_t) \mid P_{\theta_o + t^{1/2}u_t}) \xrightarrow{\mathcal{D}} \lambda(u), \tag{3.4}$$

$$\mathcal{L}(\langle \dot{L}(\theta_o)\rangle_t^{1/2}(\hat{\theta}_t - \theta_o - t^{-1/2}u_t) \mid P_{\theta_o + t^{1/2}u_t}) \xrightarrow{\mathcal{D}} \rho(u) \tag{3.5}$$

as $t \to \infty$ with

$$\lambda(u) \stackrel{\mathcal{D}}{=} \langle Y\rangle_1^{-1} Y_1, \qquad \rho(u) \stackrel{\mathcal{D}}{=} \langle Y\rangle_1^{-1/2} Y_1,$$

where

$$Y_s = \left(\int_0^s 1_{[0,\infty)}(Z_r)dB_r, \int_0^s 1_{(-\infty,0)}(Z_r)dB_r \right)^T,$$

B is a standard Wiener process and Z satisfies the stochastic differential equation

$$dZ_s = (u_1 1_{[0,\infty)}(Z_s) + u_2 1_{(-\infty,0)}(Z_s))ds + dB_s, \quad Z_o = 0, \quad 0 \le s \le 1. \tag{3.6}$$

PROOF. Let W be a standard Wiener process on a filtered probability space $(\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t\ge 0}, Q_o)$ and

$$dQ_u = \exp\left(u^T V_1 - \frac{1}{2} u^T \langle V\rangle_1 u \right) dQ_o,$$

where V is the Q_o-local martingale from Theorem 2.1. We have

$$\hat{\theta}_t - \theta = \langle \dot{L}(\theta)\rangle_t^{-1} \dot{L}_t(\theta) \qquad \text{on } A_t.$$

By Theorem 2.1 and "LeCam's first lemma", the nets $(P_{\theta_o + t^{-1/2}u_t} \mid \mathcal{F}_t)$ and $(P_{\theta_o} \mid \mathcal{F}_t)$ are mutually contiguous. Hence, by "LeCam's third lemma" and again Theorem 2.1, one obtains (3.4) and (3.5) with

$$\lambda(u) = \mathcal{L}(\langle V\rangle_1^{-1}(V_1 - \langle V\rangle_1 u) \mid Q_u),$$
$$\rho(u) = \mathcal{L}(\langle V\rangle_1^{-1/2}(V_1 - \langle V\rangle_1 u) \mid Q_u).$$

By Girsanov's theorem, the process $(B_s)_{0\le s\le 1}$ given by

$$\begin{aligned} B_s &= W_s - \langle W, u^T V\rangle_s \\ &= W_s - u_1 \int_0^s 1_{[0,\infty)}(W_r)dr - u_2 \int_0^s 1_{(-\infty,0)}(W_r)dr \end{aligned}$$

is a standard Wiener process under Q_u. We have

$$V_1 - \langle V\rangle_1 u = \left(\int_0^1 1_{[0,\infty)}(W_s)dB_s, \int_0^1 1_{(-\infty,0)}(W_s)dB_s \right)^T.$$

Setting $Z = W$ yields the assertions. □

The estimator $\hat{\theta}_t$ has the disadvantage to take values outside of Θ_o. Since $P_\theta(\hat{\theta}_t \in \Theta_o) \to 1$ for every $\theta \in \Theta$, one may choose a suitable modification of $\hat{\theta}_t$ without changing the limiting distributions under $\theta \in \Theta$. However, $P_{\theta_o}(\hat{\theta}_t \in \Theta_o) \to 1$ fails.

To see that $\mathcal{E}_t$ does not satisfy LAMN at θ_o, consider $R_u = \mathcal{L}(\int_0^1 1_{[0,\infty)}(W_s)ds \mid Q_u)$ with the notations of the proof of Theorem 3.1. For $u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, we get $R_u = \mathcal{L}(\int_0^1 1_{[0,\infty)}(s + W_s)ds \mid Q_o)$ and this distribution is different from R_o. Therefore, $\langle V \rangle_1$ is not ancillary for $(Q_u \mid u \in \mathbb{R}^2)$. The required property follows (see LeCam and Yang (1990), p. 81).

Remark. The phenomena discussed in this note is not completely new for point processes. Assuming (essentially) the exponential form intensity $\lambda(\theta, t) = \exp[\theta_1 + \theta_2(t - N_{t-})]$, Ogata and Vere-Jones (1984) showed that the ML-estimator of θ is asymptotically normal under the hypothesis $\theta \in \Theta = \mathbb{R} \times (0, \infty)$, while its limiting distribution at the boundary point $\theta_o = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is not even a mixture of normals. One can show that the model satisfies LABF (but not LAMN) at θ_o where here the condition (2.4) is much easier to verify than for the model (1.1) and it satisfies LAN at all other boundary points of Θ. Therefore, it is not the Poisson hypothesis $\theta_2 = 0$ for the process N but again the martingale hypothesis for the release process $(t - N_t)_{t \geq 0}$ under which a singularity occurs (in contrast to the statement of Ogata and Vere-Jones ((1984), p. 342)).

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