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SIMULATION OF STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. A lot of discrete approximation schemes for stochastic differential equations with regard to mean-square sense were proposed. Numerical experiments for these schemes can be seen in some papers, but the efficiency of scheme with respect to its order has not been revealed. We will propose another type of error analysis. Also we will show results of simulation studies carried out for these schemes under our notion.

Key words and phrases: Numerical solution, stochastic differential equations, error analysis, order of convergence.

1. Introduction

We consider stochastic initial value problem (SIVP) for scalar autonomous Ito stochastic differential equation (SDE) given by

$$(1.1)\qquad \begin{aligned} &dX(t) = f(X)dt + g(X)dW(t), \qquad t \in [0, T], \\ &X(0) = x, \end{aligned}$$

where $W(t)$ represents the standard Wiener process and initial value x is a fixed value. In many literatures, whose partial list can be seen in the references of the present paper, numerical schemes for SDE (1.1) were proposed, which recursively compute sample paths (trajectories) of solution $X(t)$ at step-points. Numerical experiments for these schemes can be seen in some papers (Pardoux and Talay (1985), Liske and Platen (1987), Newton (1991)). However, the efficiency of numerical schemes has not been revealed in those works. We will propose error analysis separating error term into two parts (stochastic and deterministic parts). In this paper, we will show results of error behaviour in various numerical schemes on several SDEs under our notion. Then, in strong sense, the numerical features of schemes well reflect in the deterministic part of error.

In the next section, we describe several numerical schemes in mean-square sense. Section 3 discusses our notion of error analysis in detail with five examples and Section 4 shows the numerical result of order of convergence of scheme for these examples. Finally, Section 5 is devoted to describe our conclusion on error analysis and several future aspects.

2. Numerical schemes for SDEs

In the following, we present numerical schemes. They adopt an equidistant discretization of the time interval $[0, T]$ with stepsize

$$h = \frac{T}{N} \qquad \text{for fixed natural number } N.$$

Furthermore,

$$t_n = nh, \qquad n \in \{1, 2, \dots, N\}$$

denotes the n-th step-point. We abbreviate

$$\bar{X}_n = \bar{X}(t_n) \qquad \text{and} \qquad \Phi_n = \Phi(\bar{X}_n),$$

for all $n \in \{0, \dots, N\}$ and functions $\Phi : \boldsymbol{R} \mapsto \boldsymbol{R}$.

When $X(t)$ and $\bar{X}_n$ stand for the exact and the numerical solutions of SIVP (1.1), respectively, the local error from $t = t_{n-1}$ to $t = t_n$ and the global error from $t = t_0$ to $t = T = t_N$ are defined by the following:

$$\boldsymbol{E}(|X(t_n) - \bar{X}_n|^2 \mid X(t_{n-1}) = \bar{X}_{n-1} = \bar{x}_{n-1}),$$
$$\boldsymbol{E}(|X(T) - \bar{X}_N|^2 \mid X_0 = \bar{X}_0 = \bar{x}_0),$$

where $\bar{x}_{n-1}$, $\bar{x}_0$ are arbitrary real values. Then, the local and global orders are defined as follows.

DEFINITION 1. The numerical scheme $\bar{X}_n$ is of local order γ, of global order β iff

$$\boldsymbol{E}(|X(t_n) - \bar{X}_n|^2 \mid X(t_{n-1}) = \bar{X}_{n-1} = \bar{x}_{n-1}) = O(h^{\gamma+1}) \qquad (h \downarrow 0),$$
$$\boldsymbol{E}(|X(T) - \bar{X}_N|^2 \mid X_0 = \bar{X}_0 = \bar{x}_0) = O(h^{\beta}) \qquad (h \downarrow 0),$$

respectively.

Remark. While the equation $\gamma = \beta$ holds in numerical methods for ODE under a mild assumption, it isn't satisfied for SDE (see Saito and Mitsui (1992)). Also, another definition of order of convergence may be given by

$$\boldsymbol{E}(|X(T) - \bar{X}_N| \mid X_0 = \bar{X}_0 = \bar{x}_0) = O(h^{\beta'}) \qquad (h \downarrow 0)$$

to be consistent with the deterministic order of convergence (Kloeden and Platen (1989)). But we use the order concept in Definition 1 to make it easy to investigate the global error. Thus, the reader might read as $\beta = 2\beta'$.

The following three random variables will be used in the $(n+1)$-st time step of the schemes:

$$\Delta W_n = W(t_{n+1}) - W(t_n),$$
$$\Delta Z_n = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} dW(r) ds,$$
$$\Delta \bar{Z}_n = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} dr dW(s).$$

They are obtained as sample values of normal random variables using the transformation

$$\Delta W_n = \xi_{n,1} h^{1/2},$$
$$\Delta Z_n \overset{d}{=} \frac{1}{2}\left(\xi_{n,1} + \frac{\xi_{n,2}}{\sqrt{3}}\right) h^{3/2},$$
$$\Delta \bar{Z}_n \overset{d}{=} \frac{1}{2}\left(\xi_{n,1} - \frac{\xi_{n,2}}{\sqrt{3}}\right) h^{3/2}$$

and, together with them, we further use $\Delta\tilde{W}_n = \xi_{n,2}h^{1/2}$, where $\xi_{n,1}$, $\xi_{n,2}$ are mutually independent $N(0,1)$ random variables.

Remark. In the mean-square sense ΔZ_n and $\Delta \bar{Z}_n$ cannot be expressed in terms of the random variables $\xi_{n,1}, \xi_{n,2}, \ldots, \xi_{n,m}$ which are mutually independent $N(0,1)$ ones. Thus, any numerical scheme cannot attain order 3 (Rümelin (1982), Pardoux and Talay (1985), Newton (1991)). However, we derived the above expressions for ΔZ_n and $\Delta \bar{Z}_n$ in the weak sense. In the simulation on digital computer with pseudo-random numbers we might expect these random variables behave well for the approximate solution.

Numerical schemes.

1. $\gamma = 1$, $\beta = 1$.

Euler-Maruyama scheme (Maruyama (1955)):

$$\bar{X}_{n+1} = \bar{X}_n + f_n h + g_n \Delta W_n. \tag{2.1}$$

2. $\gamma = 2$, $\beta = 2$.

Heun scheme (McShane (1974)):

$$\bar{X}_{n+1} = \bar{X}_n + \frac{1}{2}[F_1 + F_2]h + \frac{1}{2}[G_1 + G_2]\Delta W_n, \tag{2.2}$$

where

$$F_1 = F(\bar{X}_n),$$
$$G_1 = g(\bar{X}_n),$$
$$F_2 = F(\bar{X}_n + F_1 h + G_1 \Delta W_n),$$
$$G_2 = g(\bar{X}_n + F_1 h + G_1 \Delta W_n),$$
$$F(x) = \left[f - \frac{1}{2}g'g\right](x).$$

Taylor scheme (Mil'shtein (1974)):

$$\bar{X}_{n+1} = \bar{X}_n + f_n h + g_n \Delta W_n + \frac{1}{2}[g'g]_n((\Delta W_n)^2 - h). \tag{2.3}$$

Derivative-free scheme (Kloeden and Platen (1989)):

$$\bar{X}_{n+1} = \bar{X}_n + F_1 h + G_1 \Delta W_n + [G_2 - G_1]h^{-1/2}\frac{(\Delta W_n)^2 - h}{2}, \tag{2.4}$$

where

$$F_1 = f(\bar{X}_n),$$
$$G_1 = g(\bar{X}_n),$$
$$G_2 = g(\bar{X}_n + G_1 h^{1/2}).$$

FRKI scheme (Newton (1991)):

$$\bar{X}_{n+1} = \bar{X}_n + F_1 h + G_2 \Delta W_n + [G_2 - G_1] h^{1/2}, \tag{2.5}$$

where

$$F_1 = f(\bar{X}_n),$$
$$G_1 = g(\bar{X}_n),$$
$$G_2 = g(\bar{X}_n + G_1(\Delta W_n - h^{1/2})/2).$$

3. $\gamma = 3$, $\beta = 2$.
Improved 3-stage RK scheme (Saito and Mitsui (1992)):

$$\bar{X}_{n+1} = \bar{X}_n + \frac{1}{4}[F_1 + 3F_3]h + \frac{1}{4}[G_1 + 3G_3]\Delta W_n + \frac{1}{2\sqrt{3}}\left[f'g - g'f - \frac{1}{2}g''g^2\right]_n h\Delta\tilde{W}_n, \tag{2.6}$$

where

$$F_1 = F(\bar{X}_n),$$
$$G_1 = g(\bar{X}_n),$$
$$F_2 = F\left(\bar{X}_n + \frac{1}{3}F_1 h + \frac{1}{3}G_1 \Delta W_n\right),$$
$$G_2 = g\left(\bar{X}_n + \frac{1}{3}F_1 h + \frac{1}{3}G_1 \Delta W_n\right),$$
$$F_3 = F\left(\bar{X}_n + \frac{2}{3}F_2 h + \frac{2}{3}G_2 \Delta W_n\right),$$
$$G_3 = g\left(\bar{X}_n + \frac{2}{3}F_2 h + \frac{2}{3}G_2 \Delta W_n\right),$$
$$F(x) = \left[f - \frac{1}{2}g'g\right](x).$$

Taylor scheme (Mil'shtein (1974)):

$$\bar{X}_{n+1} = \bar{X}_n + f_n h + g_n \Delta W_n + \frac{1}{2}[g'g]_n((\Delta W_n)^2 - h) + [f'g]_n \Delta Z_n + \left[g'f + \frac{1}{2}g''g^2\right]_n \Delta\bar{Z}_n + \frac{1}{6}[g'^2 g + g''g^2]_n((\Delta W_n)^3 - 3h\Delta W_n). \tag{2.7}$$

4. $\gamma = 3$, $\beta = 3$.
Taylor scheme (Kloeden and Platen (1989)):

$$\begin{aligned}(2.8)\quad \bar{X}_{n+1} &= \bar{X}_n + f_n h + g_n \Delta W_n + \frac{1}{2}[g'g]_n((\Delta W_n)^2 - h) \\ &+ [f'g]_n \Delta Z_n + \left[g'f + \frac{1}{2}g''g^2\right]_n \Delta \bar{Z}_n \\ &+ \frac{1}{6}[g'^2 g + g''g^2]_n((\Delta W_n)^3 - 3h\Delta W_n) + \frac{1}{2}\left[f'f + \frac{1}{2}f''g^2\right]_n h^2.\end{aligned}$$

5. $\gamma = 2$, $\beta = 2$ but $\gamma = 3$, $\beta = 3$ for linear equation.
ERKI scheme (Newton (1991)):

$$\begin{aligned}(2.9)\qquad \bar{X}_{n+1} &= \bar{X}_n + \frac{1}{2}[F_1 + F_2]h + \frac{1}{40}[37G_1 + 30G_3 - 27G_4]\Delta W_n \\ &+ \frac{1}{16}[8G_1 + G_2 - 9G_3]\sqrt{3h},\end{aligned}$$

where

$$\begin{aligned} F_1 &= f(\bar{X}_n), \\ G_1 &= g(\bar{X}_n), \\ F_2 &= f(\bar{X}_n + F_1 h + G_1 \Delta W_n), \\ G_2 &= g\left(\bar{X}_n - \frac{2}{3}G_1(\Delta W_n + \sqrt{3h})\right), \\ G_3 &= g\left(\bar{X}_n + \frac{2}{9}G_1(3\Delta W_n + \sqrt{3h})\right), \\ G_4 &= g\left(\bar{X}_n - \frac{20}{27}F_1 h + \frac{10}{27}(G_2 - G_1)\Delta W_n - \frac{10}{27}G_2\sqrt{3h}\right). \end{aligned}$$

3. Stochastic and deterministic parts in the global error

We apply the numerical schemes described in the last section to five examples. For obtaining the mean-square error of the approximation, it is important that the exact value of the realizations of the solution $X(T)$ can be determined. As mentioned in Introduction, the error analysis has not been carried out successfully. In our view, it could be solved by separating mean-square error into two parts (stochastic and deterministic), namely

$$(3.1)\qquad \|X(T) - \bar{X}_N\| \le \|X(T) - \hat{X}_N\| + \|\hat{X}_N - \bar{X}_N\|,$$

where $\|\cdot\|$ denotes

$$\|X\| = \{\boldsymbol{E}(|X|^2)\}^{1/2},$$

and $\hat{X}_N$ the discretized exact solution realized by using pseudo-random numbers generated in digital computer (see the following Examples). From this reason, we will call $\hat{X}_N$ the realized exact solution, and the first term in the right-hand side as stochastic part, the second term as deterministic part. We anticipate that the effect of order of numerical schemes appears in deterministic part. A similar notion in weak sense is seen in Talay (1984). However, we don't expect the stochastic part could be minimized by means of taking sufficiently large sample number. The discussion can be seen in Janssen (1984) and Kanagawa (1989) from mathematical point of view. However, stochastic part will have to be investigated by some stochastic tests.

For example, if the solution $X(t)$ of SDE (1.1) can be expressed

$$X(t) = F\left(t; W(t), \int_0^t W(s)ds, \int_0^t sdW(s), \dots\right), \tag{3.2}$$

for $F \in C^\infty$, then the realized exact solution is

$$\hat{X}_n = F\left(t; \sum_{i=0}^{n-1} \Delta W_i, \sum_{i=0}^{n-1} \Delta Z_i, \sum_{i=0}^{n-1} \Delta \bar{Z}_i, \dots\right). \tag{3.3}$$

Therefore, we can easily calculate the deterministic part, whereas stochastic part will be calculated with the distances between $\sum_{i=0}^{n-1} \Delta W_i$ and $W(t)$, $\sum_{i=0}^{n-1} \Delta Z_i$ and $\int_0^t W(s)ds$, and so on at each step points ($t = ih$; $i = 0, \dots, n-1$) in distribution sense.

Note that our error analysis is slightly different from one carried out by Liske and Platen (1987) or Newton (1991). They estimated directly the error $\|X(T) - \bar{X}_N\|^2$. Here our realized exact solution corresponds to the truncated one for the exact solution they thought of. In the sequel, we study only the deterministic part.

Example 1. Linear case (Martingale).

$$dX = XdW(t), \qquad t \in [0, T], \;\; X_0 = 1. \tag{3.4}$$

The exact solution of (3.4) is

$$X(t) = \exp\left\{-\frac{1}{2}t + W(t)\right\}, \tag{3.5}$$

while the realized exact solution $\hat{X}_n$ is

$$\hat{X}_n = \exp\left\{-\frac{1}{2}t_n + \hat{W}_n\right\}. \tag{3.6}$$

Example 2. Linear case (Submartingale).

$$dX = Xdt + XdW(t), \qquad t \in [0, T], \;\; X_0 = 1. \tag{3.7}$$

The exact solution of (3.7) is

$$X(t) = \exp\left\{\frac{1}{2}t + W(t)\right\}, \tag{3.8}$$

while the realized exact solution $\hat{X}_n$ is

$$\hat{X}_n = \exp\left\{\frac{1}{2}t_n + \hat{W}_n\right\}. \tag{3.9}$$

Example 3. Linear case (Supermartingale).

$$dX = -Xdt + XdW(t), \qquad t \in [0,T], \;\; X_0 = 1. \tag{3.10}$$

The exact solution of (3.10) is

$$X(t) = \exp\left\{-\frac{3}{2}t + W(t)\right\}, \tag{3.11}$$

while the realized exact solution $\hat{X}_n$ is

$$\hat{X}_n = \exp\left\{-\frac{3}{2}t_n + \hat{W}_n\right\}. \tag{3.12}$$

In Examples 1 to 3 $\hat{W}_n$ is given by

$$\hat{W}_n = \sum_{i=1}^{n} \xi_{i,1} h^{1/2}.$$

Example 4. Non-linear case (Liske and Platen (1987)).

$$dX = -\left(\sin 2X + \frac{1}{4}\sin 4X\right)dt + \sqrt{2}(\cos X)^2 dW(t), \qquad t \in [0,T], \;\; X_0 = 1. \tag{3.13}$$

The exact solution of (3.13) is

$$X(t) = \arctan(V(t)) \tag{3.14}$$

with

$$V(t) = V_0 e^{-t} + \sqrt{2}\int_0^t e^{s-t} dW(s), \qquad V_0 = \tan(1). \tag{3.15}$$

The realized exact solution $\hat{X}_n$ is $\hat{X}_n = \arctan(\hat{V}_n)$ where $\hat{V}_n$ is computed by the following three ways according to the order of numerical scheme;

Case 1. $\gamma = 1, 2$.

$$\hat{V}_n = e^{-t}\left(V_0 + \sqrt{2}\sum_{i=1}^{n} e^{t_{i-1}} \Delta W_i\right).$$

Case 2. $\gamma = 3$, use $\xi_{n,1}$ and $\xi_{n,2}$.

$$\hat{V}_n = e^{-t}\left(V_0 + \sqrt{2}\sum_{i=1}^{n} e^{t_{i-1}}(\Delta W_i + \Delta \bar{Z}_i)\right).$$

Case 3. $\gamma = 3$, use only $\xi_{n,1}$ but no $\xi_{n,2}$ and ERKI scheme.

$$\hat{V}_n = e^{-t}\left(V_0 + \sqrt{2}\sum_{i=1}^{n} e^{t_{i-1}}(\Delta W_i + \Delta U_i)\right)$$

where

$$\Delta U_i = \frac{1}{2}\xi_{i,1}h^{3/2}.$$

In this case, we simplify the expression for the realized exact solution in the schemes (2.7) and (2.8) by replacing the terms ΔZ_n and $\Delta \bar{Z}_n$ with ΔU_n or in the scheme (2.6) by removing the correction term $(1/2\sqrt{3})[f'g - g'f - (1/2)g''g^2]_n h \cdot \Delta \tilde{W}_n$.

Example 5. Non-linear case (Gard (1988)).

$$dX = X(1-X)dt + XdW(t), \qquad t \in [0,T], \; X_0 = 0.5. \tag{3.16}$$

The exact solution of (3.16) is

$$X(t) = \frac{\exp(0.5t + W(t))}{2 + \int_0^t \exp(0.5s + W(s))ds} \tag{3.17}$$

On the other hand, the realized exact solution $\hat{X}_n$ is computed by the following three ways, similar in Example 4.

Case 1. $\gamma = 1, 2$.

$$\hat{X}_n = \frac{\exp(0.5t_n + \hat{W}_n)}{2 + \sum_{i=1}^{n} \exp(0.5t_{i-1} + \hat{W}_{i-1})h}.$$

Case 2. $\gamma = 3$, use $\xi_{n,1}$ and $\xi_{n,2}$.

$$\hat{X}_n = \frac{\exp(0.5t_n + \hat{W}_n)}{2 + \sum_{i=1}^{n} \exp(0.5t_{i-1} + \hat{W}_{i-1})(h + \Delta Z_i + h^2/2)}.$$

Case 3. $\gamma = 3$, use only $\xi_{n,1}$ but no $\xi_{n,2}$ and ERKI scheme.

$$\hat{X}_n = \frac{\exp(0.5t_n + \hat{W}_n)}{2 + \sum_{i=1}^{n} \exp(0.5t_{i-1} + \hat{W}_{i-1})(h + \Delta U_i + h^2/2)},$$

where

$$\hat{W}_n = \sum_{i=1}^{n} \xi_{i,1}h^{1/2}, \qquad \Delta U_i = \frac{1}{2}\xi_{i,1}h^{3/2}.$$

The realized exact solution is expressed as in similar way for Example 4.

4. Numerical results

We chose $T = 0.5$, the sample number $= 10000$, and the stepsizes $h = 2^{-4}, 2^{-5}, 2^{-6}$. To each schemes, we calculate the error:

$$e = \frac{1}{10000} \sum_{k=1}^{10000} (\hat{X}_N^k - \bar{X}_N^k)^2, \tag{4.1}$$

where superscript k means the k-th trajectory of each solutions. This quantity tends to the square of the deterministic part in (3.1). We show the computed mean-square error (4.1) for Example 1 in Table 1 or Figs. 1 and 2; for Example 2 in Table 2 or Figs. 3 and 4; for Example 3 in Table 3 or Figs. 5 and 6; for Cases 1, 2 and 3 of Example 4 in Table 4 or Figs. 7 to 9; for Cases 1, 2 and 3 of Example 5 in Table 5 or Figs. 10 to 12, respectively.

Note that scheme (2.4) is just same as scheme (2.3) in Examples 1, 2, 3 and 5 or scheme (2.8) is (2.7) in Example 1.

Table 1. Example 1.

$h\backslash Scheme$	(2.1)	(2.2)	(2.3) & (2.4) & (2.5)
2^{-4}	3.25×10^{-2}	2.33×10^{-3}	7.36×10^{-4}
2^{-5}	1.29×10^{-2}	4.23×10^{-4}	1.44×10^{-4}
2^{-6}	6.29×10^{-3}	8.90×10^{-5}	3.32×10^{-5}

$h\backslash Scheme$	(2.6)	(2.7) & (2.8) & (2.9)
2^{-4}	9.49×10^{-5}	1.92×10^{-5}
2^{-5}	1.16×10^{-5}	1.53×10^{-6}
2^{-6}	2.06×10^{-6}	1.59×10^{-7}

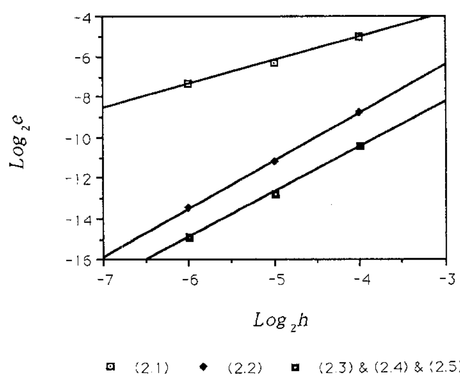

Fig. 1. Schemes (2.1), (2.2), (2.3), (2.4) and (2.5) for Example 1.

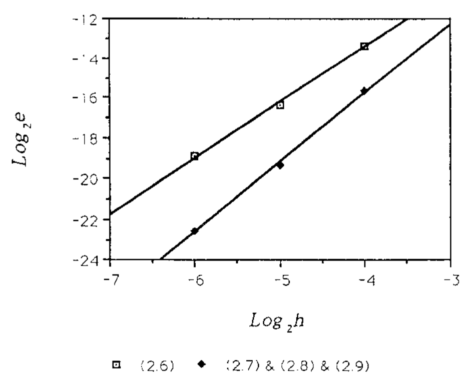

Fig. 2. Schemes (2.6), (2.7), (2.8) and (2.9) for Example 1.

Table 2. Example 2.

$h\backslash Scheme$	(2.1)	(2.2)	(2.3) & (2.4) & (2.5)
2^{-4}	1.17×10^{-1}	1.00×10^{-2}	2.37×10^{-2}
2^{-5}	4.04×10^{-2}	1.87×10^{-3}	5.15×10^{-3}
2^{-6}	1.78×10^{-2}	3.98×10^{-4}	1.13×10^{-3}

$h\backslash Scheme$	(2.6)	(2.7)	(2.8) & (2.9)
2^{-4}	3.53×10^{-4}	1.79×10^{-3}	5.35×10^{-4}
2^{-5}	3.75×10^{-5}	3.28×10^{-4}	4.16×10^{-5}
2^{-6}	6.08×10^{-6}	7.22×10^{-5}	4.54×10^{-6}

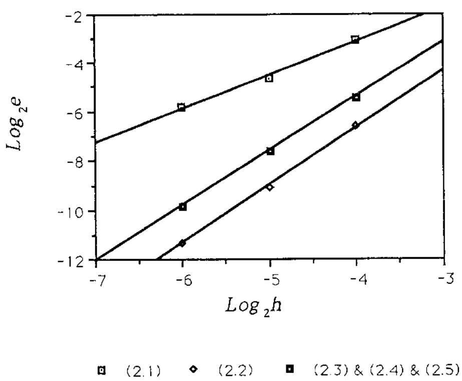

Fig. 3. Schemes (2.1), (2.2), (2.3), (2.4) and (2.5) for Example 2.

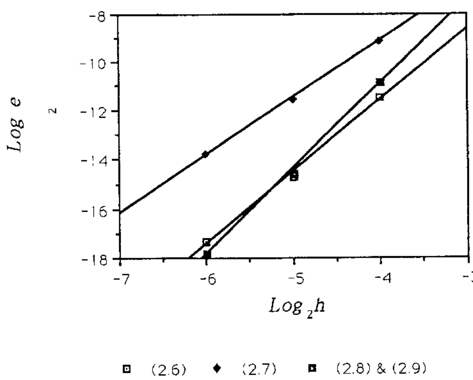

Fig. 4. Schemes (2.6), (2.7), (2.8) and (2.9) for Example 2.

Table 3. Example 3.

$h\backslash Scheme$	(2.1)	(2.2)	(2.3) & (2.4) & (2.5)
2^{-4}	1.20×10^{-2}	8.48×10^{-4}	1.93×10^{-3}
2^{-5}	5.06×10^{-3}	1.52×10^{-4}	4.77×10^{-4}
2^{-6}	2.44×10^{-3}	3.19×10^{-5}	9.84×10^{-5}

$h\backslash Scheme$	(2.6)	(2.7)	(2.8) & (2.9)
2^{-4}	3.68×10^{-5}	1.84×10^{-4}	4.54×10^{-5}
2^{-5}	4.52×10^{-6}	4.25×10^{-5}	5.11×10^{-6}
2^{-6}	7.86×10^{-7}	9.83×10^{-6}	6.44×10^{-7}

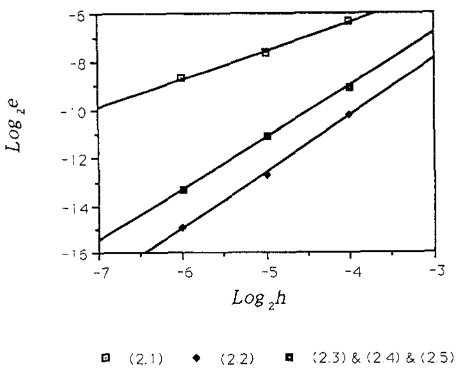

Fig. 5. Schemes (2.1), (2.2), (2.3), (2.4) and (2.5) for Example 3.

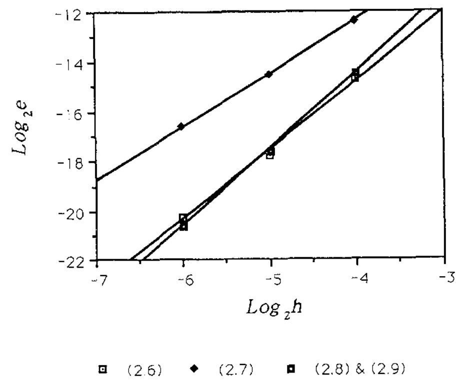

Fig. 6. Schemes (2.6), (2.7), (2.8) and (2.9) for Example 3.

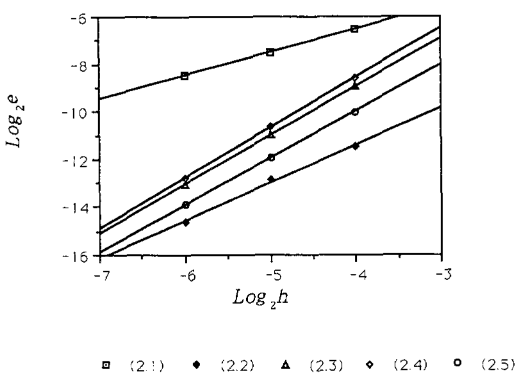

Fig. 7. Schemes (2.1), (2.2), (2.3), (2.4) and (2.5) for Case 1 of Example 4.

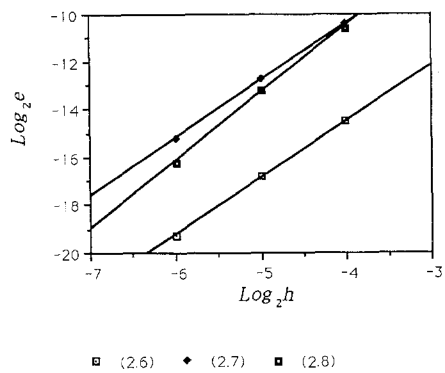

Fig. 8. Schemes (2.6), (2.7) and (2.8) for Case 2 of Example 4.

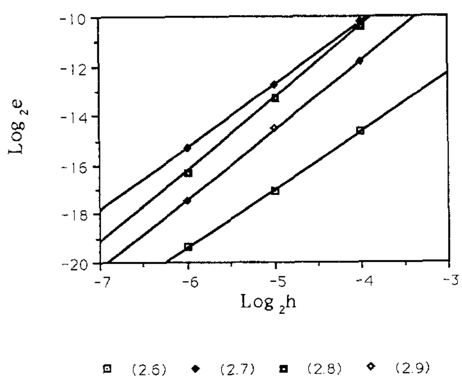

Fig. 9. Schemes (2.6), (2.7), (2.8) and (2.9) for Case 3 of Example 4.

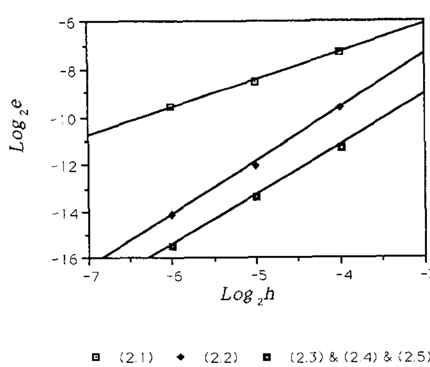

Fig. 10. Schemes (2.1), (2.2), (2.3), (2.4) and (2.5) for Case 1 of Example 5.

Table 4. Example 4.

Case 1

$h\backslash Scheme$	(2.1)	(2.2)	(2.3)
2^{-4}	1.08×10^{-2}	3.55×10^{-4}	2.00×10^{-3}
2^{-5}	5.62×10^{-3}	1.36×10^{-4}	4.94×10^{-4}
2^{-6}	2.77×10^{-3}	3.96×10^{-5}	1.17×10^{-4}

$h\backslash Scheme$	(2.4)	(2.5)
2^{-4}	2.54×10^{-3}	9.57×10^{-4}
2^{-5}	6.16×10^{-4}	2.53×10^{-4}
2^{-6}	1.39×10^{-4}	6.41×10^{-5}

Case 2

$h\backslash Scheme$	(2.6)	(2.7)	(2.8)
2^{-4}	4.31×10^{-5}	7.56×10^{-4}	6.59×10^{-4}
2^{-5}	8.78×10^{-6}	1.45×10^{-4}	1.06×10^{-4}
2^{-6}	1.60×10^{-6}	2.66×10^{-5}	1.31×10^{-5}

Case 3

$h\backslash Scheme$	(2.6)	(2.7)	(2.8)	(2.9)
2^{-4}	3.94×10^{-5}	8.68×10^{-4}	7.41×10^{-4}	2.73×10^{-4}
2^{-5}	7.30×10^{-6}	1.44×10^{-4}	1.02×10^{-4}	4.22×10^{-5}
2^{-6}	1.49×10^{-6}	2.50×10^{-5}	1.27×10^{-5}	5.54×10^{-6}

Table 5. Example 5.

Case 1

$h\backslash Scheme$	(2.1)	(2.2)	(2.3) & (2.4) & (2.5)
2^{-4}	6.34×10^{-3}	1.25×10^{-3}	4.15×10^{-4}
2^{-5}	2.78×10^{-3}	2.41×10^{-4}	9.57×10^{-5}
2^{-6}	1.32×10^{-3}	5.63×10^{-5}	2.24×10^{-5}

Case 2

$h\backslash Scheme$	(2.6)	(2.7)	(2.8)
2^{-4}	1.74×10^{-5}	5.20×10^{-5}	2.27×10^{-5}
2^{-5}	2.46×10^{-6}	1.14×10^{-5}	3.08×10^{-6}
2^{-6}	3.38×10^{-7}	2.33×10^{-6}	3.88×10^{-7}

Case 3

$h\backslash Scheme$	(2.6)	(2.7)	(2.8)	(2.9)
2^{-4}	2.21×10^{-5}	9.98×10^{-5}	2.19×10^{-5}	8.44×10^{-6}
2^{-5}	2.19×10^{-6}	1.46×10^{-5}	2.96×10^{-6}	1.03×10^{-6}
2^{-6}	3.06×10^{-7}	2.56×10^{-6}	3.61×10^{-7}	1.15×10^{-7}

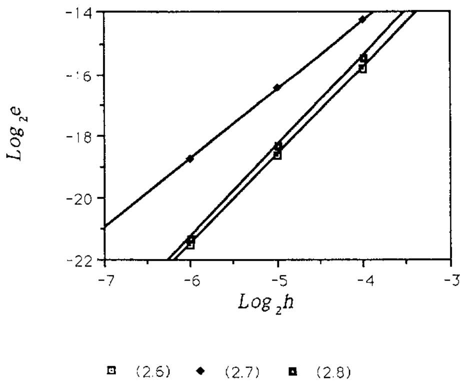

Fig. 11. Schemes (2.6), (2.7) and (2.8) for Case 2 of Example 5.

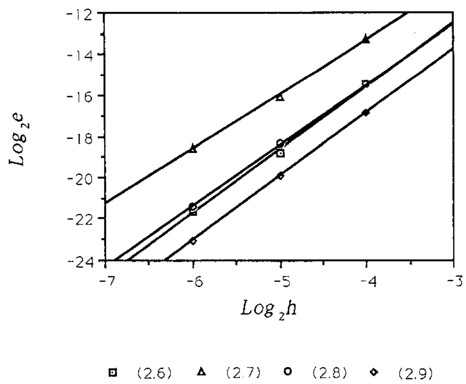

Fig. 12. Schemes (2.6), (2.7), (2.8) and (2.9) for Case 3 of Example 5.

Table 6. Fitted slopes for error-stepsize curve.

Example*Scheme*	(2.1)	(2.2)	(2.3)	(2.4)	(2.5)
1	1.2	2.4	2.2	2.2	2.2
2	1.4	2.3	2.2	2.2	2.2
3	1.1	2.4	2.1	2.1	2.1
4-case 1	1.0	1.6	2.0	2.1	1.9
5-case 1	1.1	2.2	2.1	2.1	2.1

Example*Scheme*	(2.6)	(2.7)	(2.8)	(2.9)
1	2.8	3.5	3.5	3.5
2	2.9	2.3	3.4	3.4
3	2.8	2.1	3.1	3.1
4-case 2	2.4	2.4	2.8	—
4-case 3	2.4	2.6	2.9	2.8
5-case 2	2.8	2.2	2.9	—
5-case 3	3.1	2.6	3.0	3.1

In all figures, the least-square linear fitting was incorporated for the error-stepsize curve. The fitted slopes are shown in Table 6. From the results we can conclude that the third order Taylor scheme (2.8) is superior to other schemes from our viewpoint of global error. Also, when the scheme carried out by using only one pseudo-random number at each step, ERKI method (2.9) may be preferred to the third order Taylor scheme (2.8), because it is easier to implement.

5. Conclusions and future aspects

Our numerical examples in the preceding section lead to the following tentative conclusions:

(i) The global order of numerical scheme appears well in the deterministic part of error.

(ii) We emphasize that approximation for sample path of solution $X(t)$ is completely determined by sequences of pseudo-random numbers. Therefore, how to generate pseudo-random numbers is the most crucial problem in the simulation of SDEs. Namely, the problem reduces how well the Wiener process $W(t)$ and the random process $\int_0^t e^s dW(s)$ can be realized by using pseudo-random numbers in digital computer. Henceforce, the program for simulation of SDE will have to require both numerical scheme and specific pseudo-random number generator.

(iii) Case 3 of Examples 4 and 5 show that Runge-Kutta scheme proposed by Rümelin (1982) may be used if the random process $\int_0^t e^s dW(s)$ is realized by only one pseudo-random number at each step.

We investigated only the deterministic part of global error in this paper. However, so far it seems very difficult to study the stochastic part. In general, it requires a finer analysis through good pseudo-random numbers.

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