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A GENERALIZATION OF THE RESULTS OF PILLAI

YASUHIRO FUJITA

Department of Mathematics, Toyama University, Toyama 930, Japan

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Abstract. In a recent article Pillai (1990, *Ann. Inst. Statist. Math.*, **42**, 157–161) showed that the distribution $1-E_\alpha(-x^\alpha)$, $0<\alpha\le 1$; $0\le x$, where $E_\alpha(x)$ is the Mittag-Leffler function, is infinitely divisible and geometrically infinitely divisible. He also clarified the relation between this distribution and a stable distribution. In the present paper, we generalize his results by using Bernstein functions. In statistics, this generalization is important, because it gives a new characterization of geometrically infinitely divisible distributions with support in $[0,\infty)$.

Key words and phrases: Bernstein function, Laplace-Stieltjes transform, infinite divisibility, geometric infinite divisibility, Lévy process.

1. Introduction and results

Pillai (1990) showed that the distribution $1 - E_\alpha(-x^\alpha)$, $0 < \alpha \le 1$; $0 \le x$, where $E_\alpha(x) = \sum_{n=0}^{\infty} x^n/\Gamma(1+n\alpha)$ is the Mittag-Leffler function, is infinitely divisible and geometrically infinitely divisible (for the definition of geometric infinite divisibility, see below). He also showed that this distribution is equal to the distribution of $Z_\alpha(S(1))$, where $Z_\alpha(t)$ is the stable process with $\boldsymbol{E}\exp\{-uZ_\alpha(t)\} = \exp\{-tu^\alpha\}$, $u \ge 0$, and $S(t)$ is the gamma process with the density $x^{t-1}e^{-x}dx/\Gamma(t)$, $x > 0$.

The aim of the present paper is to generalize his results by using Bernstein functions. In statistics, this generalization is important, because it gives a new characterization of geometrically infinitely divisible distributions with support in $[0,\infty)$.

A C^∞-function f from $(0,\infty)$ to $\boldsymbol{R}$ is said to be a *Bernstein function*, if $f(x) \ge 0$, $x > 0$, and $(-1)^p d^p f/dx^p \le 0$, $x > 0$, for all integers $p \ge 1$ (cf. Def. 9.1 of Berg and Forst (1975)). Thus df/dx becomes a completely monotone function. Such a function f is characterized by

$$f(x) = a + bx + \int_0^\infty (1 - e^{-sx})\mu(ds), \qquad x > 0,$$

where a, b are non-negative constants and $\mu(ds)$ is a positive measure on $(0,\infty)$

such that

$$\int_0^\infty \frac{s}{1+s}\mu(ds) < \infty$$

(see Theorem 9.8 of Berg and Forst (1975)). In the present paper we assume that

$$\lim_{x\downarrow 0} f(x) = 0, \qquad \lim_{x\to\infty} f(x) = \infty. \tag{1.1}$$

It is easy to see that $\lim_{x\downarrow 0} f(x) = 0$ if and only if $a = 0$, and that $\lim_{x\to\infty} f(x) = \infty$ if and only if $b > 0$ or $\mu((0,\infty)) = \infty$. Then, since f is a non-zero Bernstein function, $f(x) > 0$, $x > 0$, and $1/f$ is completely monotone (cf. Exercise 9.9 of Berg and Forst (1975)). Thus, there exists a unique positive measure $W(dx)$ on $[0,\infty)$ such that

$$\frac{1}{f(x)} = \int_0^\infty e^{-sx}W(ds), \qquad x > 0. \tag{1.2}$$

We denote by $W^{n*}(dx)$ the n-times convolution measure of $W(dx)$. For $\lambda > 0$ define the function $U_\lambda(x)$ on $\boldsymbol{R}$ by

$$U_\lambda(x) = \begin{cases} -\displaystyle\sum_{n=1}^{\infty}(-\lambda)^n W^{n*}([0,x]), & x \geq 0, \\ 0, & x < 0. \end{cases} \tag{1.3}$$

We shall show that $U_\lambda(x)$ is a distribution function and it is a generalization of the distribution function $1 - E_\alpha(-x^\alpha)$, $0 < \alpha \leq 1$; $0 \leq x$. Remark that the function $f(x) = x^\alpha$ is a Bernstein function. In this case $W^{n*}([0,x]) = \{1/\Gamma(1+n\alpha)\}x^{n\alpha}$ and $U_1(x) = 1 - E_\alpha(-x^\alpha)$.

Now, we state the main results of the present paper. All theorems of this section are proved in Section 2. The first theorem shows that U_λ is infinitely divisible.

THEOREM 1.1. *Let f be a Bernstein function with* (1.1). *Then, for every $\lambda > 0$, U_λ is an infinitely divisible distribution with the Laplace-Stieltjes transform $\lambda\,(\lambda + f(\cdot))^{-1}$.*

In Theorem 1.2 below, we construct the Lévy process which has the distribution U_λ for $t = 1$. This theorem corresponds to Theorem 4.3 of Pillai (1990), which clarifies the relation between the distribution $1 - E_\alpha(-x^\alpha)$ and a stable process.

THEOREM 1.2. *Let f be a Bernstein function with* (1.1). *Then, the Lévy process with the distribution function U_λ for $t = 1$ is $Z(S_\lambda(\cdot))$. Here $Z(t)$ is the non-negative and non-decreasing Lévy process such that $\boldsymbol{E}\exp\{-uZ(t)\} = \exp\{-tf(u)\}$ for $u \geq 0$; $S_\lambda(t)$ is the gamma process with the probability density $\lambda^t s^{t-1}e^{-\lambda s}ds/\Gamma(t)$, $s > 0$; Z and S_λ are independent.*

Next, we show that U_λ, $\lambda > 0$, is geometrically infinitely divisible. A distribution function G with $G(0) = 0$ is said to be geometrically infinitely divisible if for every p, $0 < p < 1$, there exists a distribution function H_p with $H_p(0) = 0$ such that

$$G(x) = \sum_{j=1}^{\infty} p(1-p)^{j-1} H_p^{j*}(x), \qquad x > 0,$$

where H_p^{j*} is the j-times convolution of H_p. For more details, see Klebanov *et al.* (1984).

THEOREM 1.3. *Let f be a Bernstein function with* (1.1). *Then, for every $\lambda > 0$ and $0 < p < 1$,*

$$U_\lambda(x) = \sum_{j=1}^{\infty} p(1-p)^{j-1} U_{\lambda/p}^{j*}(x), \qquad x > 0. \tag{1.4}$$

Thus U_λ is geometrically infinitely divisible.

Theorems 1.1, 1.2 and 1.3 are a generalization of the corresponding results of Pillai (1990). As shown in Theorem 1.4 below, this generalization is important in statistics, because it gives a new characterization of geometrically infinitely divisible distributions with support in $[0, \infty)$.

THEOREM 1.4. *A distribution function G with $G(0) = 0$ is geometrically infinitely divisible, if and only if G is expressed by*

$$G(x) = -\sum_{n=1}^{\infty} (-1)^n W^{n*}([0, x]), \qquad x > 0, \tag{1.5}$$

where $W(dx)$ is the positive measure satisfying (1.2) *for some Bernstein function f with property* (1.1).

2. Proofs

PROOF OF THEOREM 1.1. Choose $z_0 > 0$ so that $f(z) > \lambda$ for $z > z_0$. This is possible, since f is non-decreasing and $f(z) \to \infty$ as $z \to \infty$. For $z > z_0$, we have

$$0 \leq \sum_{n=1}^{\infty} \lambda^n \int_0^{\infty} e^{-zx} W^{n*}(dx) = \sum_{n=1}^{\infty} (\lambda/f(z))^n = \frac{\lambda}{f(z) - \lambda} < \infty.$$

Thus the dominated convergence theorem yields

$$\begin{aligned}
\int_0^{\infty} e^{-zx} dU_\lambda(x) &= -\sum_{n=1}^{\infty} (-\lambda)^n \int_0^{\infty} e^{-zx} W^{n*}(dx) \\
&= -\sum_{n=1}^{\infty} (-\lambda/f(z))^n \\
&= \frac{\lambda}{\lambda + f(z)} = \exp\left\{-\log\left(1 + \frac{f(z)}{\lambda}\right)\right\}.
\end{aligned} \tag{2.1}$$

Let $g(z) = \log(1 + f(z)/\lambda)$. In order to complete the proof, it is sufficient to show that $g'(z) = f'(z)/[\lambda + f(z)]$ is completely monotone (see Theorem 1 of XIII 7 of Feller (1971)). Since f is a Bernstein function, Criteria 1 and 2 of XIII 4 of Feller (1971) shows that both f' and $1/[\lambda + f(z)]$ are completely monotone so g' is also completely monotone. This completes the proof. □

PROOF OF THEOREM 1.2. The existence of $Z(t)$ follows from Theorem 9.18 of Berg and Forst (1975). Then the theorem is clear, since

$$\begin{aligned} \boldsymbol{E}\exp\{-uZ(S_\lambda(1))\} &= \lambda \int_0^\infty \boldsymbol{E}\exp\{-uZ(s)\}e^{-\lambda s}ds \\ &= \frac{\lambda}{\lambda + f(u)} = \int_0^\infty e^{-ux}dU_\lambda(x), \qquad u \geq 0. \end{aligned}$$

In the last equality we used Theorem 1.1. This completes the proof. □

PROOF OF THEOREM 1.3. An easy calculation and Theorem 1.1 show that

$$\begin{aligned} \int_0^\infty e^{-zx}d_x\left[\sum_{j=1}^\infty p(1-p)^{j-1}U_{\lambda/p}^{j*}(x)\right] &= \sum_{j=1}^\infty p(1-p)^{j-1}\left[\int_0^\infty e^{-zx}dU_{\lambda/p}(x)\right]^j \\ &= \sum_{j=1}^\infty p(1-p)^{j-1}\left[\frac{\lambda}{\lambda + pf(z)}\right]^j \\ &= \frac{\lambda}{\lambda + f(z)}, \qquad z \geq 0. \end{aligned}$$

Thus (1.4) follows from the uniqueness of the Laplace-Stieltjes transforms. This completes the proof. □

PROOF OF THEOREM 1.4. First, we show "if part". Suppose that G is expressed by (1.5). Here $W(dx)$ is the positive measure satisfying (1.2) for some Bernstein function f with property (1.1). Letting x tend to ∞ in (1.2), we have by (1.1) $W(\{0\}) = 0$. Thus $G(0) = 0$. Then "if part" follows from Theorem 1.3. Second we show "only if part". Suppose that G is geometrically infinitely divisible distribution with $G(0) = 0$. By Theorem 2 of Klebanov *et al.* (1984), there exists a Lévy process $X(t)$ such that

$$\boldsymbol{E}e^{-zX(t)} = \exp\left\{-t\left(\left[\int_0^\infty e^{-zx}dG(x)\right]^{-1} - 1\right)\right\}, \qquad \mathrm{Re}z = 0. \tag{2.2}$$

For $\mathrm{Re}z = 0$, let

$$f(z) = \left[\int_0^\infty e^{-zx}dG(x)\right]^{-1} - 1. \tag{2.3}$$

We shall show that f is extended to the domain $\mathrm{Re} z \geq 0$ and it is a Bernstein function on $(0, \infty)$. We have by (2.2) and (2.3)

$$\begin{aligned}(2.4)\qquad \int_0^\infty e^{-zx} dG(x) &= \frac{1}{1+f(z)} = \exp\{-\log(1+f(z))\} \\ &= \exp\left\{\int_0^\infty (e^{-sf(z)}-1)s^{-1}e^{-s}ds\right\} \\ &= \exp\left\{\int_{\boldsymbol{R}} (e^{-zx}-1)\rho(dx)\right\}, \qquad \mathrm{Re} z = 0,\end{aligned}$$

where $\rho(dx) = \int_0^\infty \boldsymbol{P}(X(s) \in dx)s^{-1}e^{-s}ds$. Thus G is infinitely divisible. Since the support of G is in $[0, \infty)$, ρ must satisfy $\rho((-\infty, 0)) = 0$. Then the support of $X(t)$, $t \geq 0$, is also in $[0, \infty)$, and (2.2) is extended to the domain $\mathrm{Re} z \geq 0$. This means that the function $e^{-tf(z)} = \boldsymbol{E}e^{-zX(t)}$, $z > 0$, is completely monotone for each $t > 0$. By Proposition 9.2 of Berg and Forst (1975), f is a Bernstein function. By (2.3) and $G(0) = 0$, f satisfies (1.1). Then we find a positive measure $W(dx)$ on $[0, \infty)$ satisfying (1.2). By Theorem 1.1, $H(x) = -\sum_{n=1}^\infty (-1)^n W^{n*}([0, x])$ satisfies

$$\int_0^\infty e^{-zx} dH(x) = \frac{1}{1+f(z)}, \qquad z > 0.$$

By (2.4), the uniqueness of the Laplace-Stieltjes transforms shows that $G = H$. This completes the proof. □

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