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A NOTE ON ASYMPTOTIC EXPANSIONS FOR SUMS OVER A WEAKLY DEPENDENT RANDOM FIELD WITH APPLICATION TO THE POISSON AND STRAUSS PROCESSES

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Abstract. Previous results on Edgeworth expansions for sums over a random field are extended to the case where the strong mixing coefficient depends not only on the distance between two sets of random variables, but also on the size of the two sets. The results are applied to the Poisson and the Strauss point processes, giving rise also to local limit results.

Key words and phrases: Conditional Cramér condition, local limit theorem, Poisson process, Strauss process.

1. Introduction

Asymptotic expansions for the distribution of a sum of dependent random variables is a fairly recent subject, especially so for sums over a random field. If the random variables X_i are indexed by $i \in \boldsymbol{Z}^\nu$ we measure the dependency by

$$\alpha(n,k,l) = \sup |P(A \cap B) - P(A)P(B)|, \tag{1.1}$$

where $A \in \sigma(X_i, i \in I_1)$, $B \in \sigma(X_i, i \in I_2)$, $I_1, I_2 \subset \boldsymbol{Z}^\nu$ with $|I_1| = k$ and $|I_2| = l$, and the Euclidean distance $d_E(I_1, I_2)$ is greater than n. In the linear case $\nu = 1$ Götze and Hipp (1983) obtained general results for the case $\alpha(n,k,l) = c\exp(-dn)$. For Markov chains it is possible to obtain expansions also in cases with $\alpha(n,k,l) = cn^{-d}$. Actually, the value of d determines how many terms can be included in the expansion, see Jensen (1989) and the references given there. When $\nu > 1$ the results are more sparse. Jensen (1986) generalized the results in Götze and Hipp (1983) in a direct way for the case $\alpha(n,k,l) = ce^{-dn}$. An outline of the generalization in Jensen (1986) is given in Jensen (1988). When dealing with the characteristic function for large values of its argument the point of view in Jensen (1986), as well as here, is to make necessary conditions of practical importance. In the m-dependent case, i.e. $\alpha(n,k,l) = 0$ for $n > m$, a considerable amount of work has been done by Heinrich (1987, 1990) and Götze and Hipp (1989) with a view towards necessary and sufficient conditions.

In this note we point out an easy generalization of the results in Jensen (1986, 1988) to the case

$$\alpha(n,k,l) = ck^{\delta_1} l^{\delta_2} e^{-dn} \tag{1.2}$$

for some constants δ_1, δ_2 and $d > 0$. The generalization to (1.2) is of relevance for the study of Gibbs processes, see Section 4 below. There will be no proofs for the main results here, because the results follow fairly easily by going through the detailed calculations in Jensen (1986, 1988). It seems natural to enquire into the possibility to relax (1.2) to

$$\alpha(n,k,l) = ck^{\delta_1} l^{\delta_2} n^{-\theta} \tag{1.3}$$

for a sufficiently high value of θ. A careful analysis of the proof in Jensen (1986, 1988) shows, that it is possible from (1.3) to get the necessary bounds on the cumulants and the remainder term of an expansion of the characteristic function for the standardized sum (2.2) below in the region $\|t\| < c_1 (\log n)^{1/2}$, where t is the argument of the characteristic function. However, the region $\|t\| < c_1(\log n)^{1/2}$ is not sufficiently large, that the method of estimation for large values of the argument t will produce a sufficiently small bound. Using instead (1.2) the estimation in the inner region can be used for $\|t\| < c_1 n^{\epsilon}$ for a suitable small $\epsilon > 0$, and then the estimation for large values, i.e. $\|t\| > c_1 n^{\epsilon}$, also works.

As a particular example we apply the results to the Poisson process. If S is the number of pairs of points with a distance less than some fixed number r, Heinrich (1986) proved asymptotic normality of S and raised the problem of the possibility of obtaining local limit results and asymptotic expansion. Such results are established in Section 3. Finally, in Section 4 we study the possibility of using the general results for the Strauss process.

2. Results

Let $\mathcal{D}_j$ for $j \in \boldsymbol{Z}^\nu$ be σ-fields with mixing coefficients given by (1.1), where now $A \in \sigma(\mathcal{D}_i, i \in I_1)$ and $B \in \sigma(\mathcal{D}_i, i \in I_2)$. The mixing coefficients will throughout be assumed to satisfy (1.2). Let Λ be a finite set of indices which we think of as a neighbourhood of $0 \in \boldsymbol{Z}^\nu$, and which is symmetric in the sense that $j \in \Lambda$ implies that $-j \in \Lambda$. For each $j \in \boldsymbol{Z}^\nu$ we have a random vector $X_j \in \boldsymbol{R}^\mu$ measurable with respect to $\sigma(\mathcal{D}_i : i \in j + \Lambda)$. In particular if the σ-fields $\mathcal{D}_j$, $j \in \boldsymbol{Z}^\nu$, are independent the X_j's are "m-dependent". We assume that for some $s \geq 2$

$$EX_i = 0 \quad \text{and} \quad E\|X_i\|^{s+1} \leq \beta_{s+1}. \tag{2.1}$$

We let $V_n \subseteq \boldsymbol{Z}^\nu$ be an increasing sequence of indices with the number of elements being equal to n. Our interest will be with the sum

$$S_n = \frac{1}{\sqrt{n}} \sum_{i \in V_n} X_i. \tag{2.2}$$

For this sum we denote the cumulants by

$$\kappa_{j_1\cdots j_r} = \frac{1}{i^r}\frac{\partial^r}{\partial t_{j_1}\cdots\partial t_{j_r}} \ln E\exp(it\cdot S_n) \mid_{t=0}.$$

Then the formal Edgeworth expansion (Bhattacharya and Rao (1976)) for the distribution of S_n has a density of the form

$$E_{s-2}(x;\{\kappa\}) = \phi_{\Sigma_n}(x)\left\{1+\sum_{r=1}^{s-2} n^{-r/2}P_r(x;\{\tilde{\kappa}\})\right\}, \tag{2.3}$$

where P_r is a polynomial in x and the sequence of standardized cumulants $\tilde{\kappa}$, $\tilde{\kappa}_{j_1\cdots j_r} = n^{(r-2)/2}\kappa_{j_1\cdots j_r}$, $\Sigma_n = \{\kappa_{ij}\}$ is the variance, and $\phi_\Sigma(x)$ is the normal density with mean zero and variance Σ. We note here, that the terms in the sum (2.3) have decreasing orders of magnitude, since we have from Jensen (1986) that

$$|\kappa_{j_1\cdots j_r}| \le c_1 n^{-(r-2)/2}\beta_{s+1}^{1+r/(s+1)} \tag{2.4}$$

for $2 \le r \le s$, where c_1 is a constant.

The assumptions made above are sufficient for establishing (2.4), and also to get a suitable bound on the derivatives of the remainder term in the expansion of the characteristic function in the region $\|t\| < c_2 n^{\epsilon_1}$ for some constants c_2 and ϵ_1. To handle the characteristic function for large values of the argument we need some further assumptions. The first assumption says that we almost have a Markov property: for any k we have

$$E|E(Y \mid \mathcal{D}_j : j \ne k) - E(Y \mid \mathcal{D}_j : 0 < |j-k| \le m)| \le ce^{-dm} \tag{2.5}$$

for any $\sigma(\mathcal{D}_j : j \in k+\Lambda+\Lambda)$ measurable random variable Y with $|Y| \le 1$. The second condition is a conditional Cramér condition. There exist $b > 0$, $c > 0$ and $\rho < 1$ such that for at least cn values of k

$$E\left|E\left(\exp\left(it\cdot\sum_{j\in k+\Lambda} X_j\right) \mid \mathcal{D}_j : j \ne k\right)\right| \le \rho \quad \text{for} \quad \|t\| > b. \tag{2.6}$$

The condition (2.6) is suitable when the random variables X_j have a continuous component. In the lattice case, say X_j is concentrated on the minimal lattice $\mu_j + \boldsymbol{Z}^\mu$, we need instead

(2.7) (2.6) holds for $\|t\| > b$ and $|t_i| \le \pi$, $i = 1, \ldots, \mu$, for some $b < \pi/2$.

In the theorem below Φ_Σ in the normal distribution with mean zero and variance Σ, and $(\partial A)^\delta$ denotes the δ-boundary of a set A.

THEOREM 2.1. (Continuous case) *Assume* (1.2), (2.1), (2.5), (2.6) *and that the eigenvalues of* Σ_n *are bounded away from zero. Then for any* $\epsilon > 0$ *and* $c > 0$ *we have*

$$\left| P(S_n \in A) - \int_A E_{s-2}(x; \{\kappa\}) dx \right| = O(n^{-(s-1)/2+\epsilon})$$

uniformly over Borel sets A *satisfying* $\Phi_{\Sigma_n}((\partial A)^\delta) \leq c\delta$.

THEOREM 2.2. (Lattice case) *Let* $X_j = \tilde{X}_j - \mu_j$, *where* $\mu_j = E\tilde{X}_j$ *and* $\tilde{X}_j$ *is a lattice variable with values in* $\boldsymbol{Z}^\mu$, *and assume* (1.2), (2.1), (2.5), (2.7) *and that the eigenvalues of* Σ_n *are bounded away from zero. For* $z \in \boldsymbol{Z}^\mu$ *let* $y_{z,n} = n^{-1/2}(z - \sum_{V_n} \mu_j)$. *Then for any* $\epsilon > 0$ *we have*

$$\sup_{z \in \boldsymbol{Z}^\mu} (1 + \|y_{z,n}\|^s) \left| P\left(\sum_{V_n} \tilde{X}_j = z \right) - n^{-\mu/2} E_{s-2}(y_{z,n}; \{\kappa\}) \right|$$
$$= O(n^{-\mu/2-(s-1)/2+\epsilon}).$$

Remark on proof. The only change from Jensen (1986, 1988) is the relaxation of the mixing condition to the one given in (1.2). Theorem 1.1 is obtained therefore by simply going through the calculations in Jensen (1986, 1988). For the lattice case the proof is also as in Jensen (1986, 1988), except that the truncation function used there has to be redefined slightly, such that the truncated variable is again a lattice variable. The restriction on b in (2.7) appears in order to be able to use the proof of Theorem 1.1 in Petrov (1975). The estimate (2.4) on the cumulants can be established under the mixing assumption (1.3) with $\theta > (s-1)(s+1)\nu$.

Remark on non-zero variance. The condition that the eigenvalues of Σ_n are bounded away from zero can in some cases be proved by using the formula $V(X) = V(E(X \mid Y)) + E(V(X \mid Y))$ for random variables X and Y. In some Gibbs models, by using a coding set for Y, the random variable X becomes conditionally a sum of independent terms. A lower bound for the variance can then be established.

3. Application to the Poisson process

Let $Z = (Z_1, \ldots, Z_N)$ be the points of a Poisson process with intensity γ in the bounded region $W_n \subseteq \boldsymbol{R}^\nu$, and define the two statistics

$$N = \text{number of points}, \qquad S = \sum_{i \neq j} 1(\|Z_i - Z_j\| < r), \tag{3.1}$$

where r is a fixed number. When conditioning on $N = m$ the points $Z_1, \ldots, Z_m$ will be uniformly distributed on W_n, and S can be used as a test statistic for this uniformity assumption. This is the point of view in Heinrich (1986), where the asymptotic normality of S is established. Here we establish a local limit theorem for S under the uniformity assumption by using conditioning in the Poisson process.

Let $\kappa > r$ be a fixed number and define for each $j = (j_1, \ldots, j_\nu) \in \boldsymbol{Z}^\nu$ the region $U_j \subset \boldsymbol{R}^\nu$ as $U_j = \{x \in \boldsymbol{R}^\nu \mid \kappa j_i \le x_i \le \kappa(j_i + 1)\}$. We let $\mathcal{D}_j$ be the σ-field generated by the Poisson process in U_j, and define

$$N_j = \text{number of points in } W_n \cap U_j,$$

$$S_j = \sum_{\substack{Z_{i_1}, Z_{i_2} \in W_n \cap U_j \\ Z_{i_1} \ne Z_{i_2}}} 1(\|Z_{i_1} - Z_{i_2}\| < r) + \sum_{\substack{Z_{i_1} \in W_n \cap U_j \\ Z_{i_2} \in W_n \cap U_j^c}} 1(\|Z_{i_1} - Z_{i_2}\| < r). \tag{3.2}$$

Let V_n be the set of indices j with $W_n \cap U_j \ne \emptyset$ and let $|V_n| = n$. We will assume that the regions W_n are regular in the sense that $n/\text{volume}(W_n) \to \lambda$ with $0 < \lambda < \infty$ and $W_n \cap U_j = U_j$ for at least cn values of j for some constant $c > 0$. It is clear that

$$N = \sum_{j \in V_n} N_j \quad \text{and} \quad S = \sum_{j \in V_n} S_j$$

so that we may try to apply Theorem 2.2 with $X_j = (N_j - EN_j, S_j - ES_j)$.

Since the σ-fields $\mathcal{D}_j$ are independent we of course have (1.2). The set Λ is $\{j \in \boldsymbol{Z}^\nu \mid |j_i| \le 1\}$ and (2.5) is also trivially satisfied. The main problem is therefore to check (2.7). Define

$$S_k^0 = \sum_{\substack{Z_{i_1}, Z_{i_2} \in W_n \cap U_k \\ Z_{i_1} \ne Z_{i_2}}} 1(\|Z_{i_1} - Z_{i_2}\| < r) \quad \text{and} \quad S_k^1 = \sum_{j \in k + \Lambda} S_j.$$

Then

$$\begin{aligned} &E|E(\exp(it_1 N_k + it_2 S_k^1) \mid \mathcal{D}_j : j \ne k)| \\ &\quad = E|E(\exp(it_1 N_k + it_2 S_k^1) \mid \mathcal{D}_j : j \ne k, |j_i - k_i| \le 2)| \\ &\quad \le |E \exp(it_1 N_k + it_2 S_k^0)| P(N_j = 0, j \ne k, |j_i - k_i| \le 2) \\ &\qquad + 1 - P(N_j = 0, j \ne k, |j_i - k_i| \le 2) \le \rho \end{aligned} \tag{3.3}$$

for some $\rho < 1$ and for those k with $W_n \cap U_k = U_k$. Thus (2.7) is established.

Since all the moments of (N_j, S_j) exist, we have from Theorem 2.2 an Edgeworth expansion to any order s for (N, S). Now choose the Poisson intensity γ such that $E_\gamma N = n$, and divide the Edgeworth expansion for $P(N = n, S = k)$ with the Edgeworth expansion for $P(N = n)$. We then have an expansion for $P(S = k \mid N = n)$ or, equivalently, an expansion for $P(S = k)$ when the points are uniformly distributed in W_n. This solves a problem raised in Heinrich (1986). The first two terms of the expansion are

$$P(S = k) = n^{-1/2} \left\{ \phi(y_n) + \frac{\gamma_n}{6\sigma_n^3}(y_n^3 - 3y_n)\phi(y_n) + O(n^{-1+\epsilon}) \right\}$$

where $y_n = (k - \mu_n)/\sigma_n$ with

$$\mu_n = n\left\{\frac{n-1}{A}K_{12}\right\},$$

$$\sigma_n^2 = n\left\{2\frac{n-1}{A}K_{12} + 4\frac{(n-1)(n-2)}{A^2}K_{13} - 4\frac{n(n-1)}{A^2}K_{12}^2\right\},$$

$$\begin{aligned}\gamma_n = n\Bigg\{&4\frac{n-1}{A}K_{12} + 24\frac{(n-1)(n-2)}{A^2}K_{13} + 8\frac{(n-1)(n-2)(n-3)}{A^3}K_{14}\\ &+ 24\frac{(n-1)(n-2)(n-3)}{A^3}K_{24} + 8\frac{(n-1)(n-2)}{A^2}K_{23}\\ &+ \frac{(n-1)(n-2)(-72n+144)}{A^3}K_{12}K_{13} + \frac{(n-1)(-24n+36)}{A^2}K_{12}^2\\ &+\frac{23n^3 - 11n^2 + 24n}{A^3}K_{12}^3\Bigg\}.\end{aligned}$$

Here $A = \text{volume}(W_n)$ and

$$K_{1m} = \int_{W_n}\left\{\int_{W_n^{m-1}}\prod_{j=2}^{m} 1(\|x_1 - x_j\| < r)dx_j\right\}\frac{dx_1}{A},$$

$$\begin{aligned}K_{23} = \int_{W_n}\Bigg\{\int_{W_n^2} &1(\|x_1 - x_2\| < r)\\ &\cdot 1(\|x_1 - x_3\| < r)1(\|x_2 - x_3\| < r)dx_2dx_3\Bigg\}\frac{dx_1}{A},\end{aligned}$$

$$\begin{aligned}K_{24} = \int_{W_n}\Bigg\{\int_{W_n^3} &1(\|x_1 - x_2\| < r)\\ &\cdot 1(\|x_1 - x_3\| < r)1(\|x_2 - x_4\| < r)dx_2dx_3dx_4\Bigg\}\frac{dx_1}{A}.\end{aligned}$$

4. Application to the Strauss process

The Strauss point process $P_{\zeta,\beta}$ (Strauss (1975)) in the region W_n has the density

$$\frac{dP_{\zeta,\beta}}{d\nu_{W_n}}(z) = \frac{1}{Z_n(\zeta,\beta)}\zeta^N e^{-\beta(1/2)S}, \tag{4.1}$$

where N is the number of points in the configuration z, $z = \{z_1, \ldots, z_N\}$, $S = \sum_{i\neq j} 1(\|z_i - z_j\| < r)$ as in (3.1), and the density is w.r.t. the Poisson measure ν_{W_n} with unit intensity.

For convenience in the calculations below let us take $n = (2n_1 + 1)^\nu$ and $W_n = [-n_1\kappa, n_1\kappa]^\nu$. We define U_j as in Section 3 and similarly $\mathcal{D}_j$ is the σ-field generated by the process in U_j. Also define N_j and S_j as in (3.2).

In Jensen (1993) the mixing coefficients $\alpha(r,k,l)$ for the σ-fields $\{\mathcal{D}_j : j \in \boldsymbol{Z}^\nu\}$ generated by (4.1), and more generally generated by a translation invariant Gibbs measure, are estimated by

$$\alpha(r,k,l) \le ck \sum_{d_E(0,k)\ge r} \tilde{D}_k. \tag{4.2}$$

Here

$$\tilde{D}_k = \sum_{n=0}^{\infty} \tilde{C}_k^n, \qquad \tilde{C}_k^n = \sum_{k_1,\dots,k_{n-1}\in \boldsymbol{Z}^\nu} \tilde{C}_{k_1}\cdots \tilde{C}_{k_{n-1}} \tilde{C}_{k-k_1-\dots-k_{n-1}},$$

and $\tilde{C}_k$ is a certain number measuring how much the conditional distribution in U_0 given $\{\mathcal{D}_j : j \neq 0\}$ depends on the point configuration in U_k, $k \in \boldsymbol{Z}^\nu$. The details of this setup is given in Jensen (1993). What we need here is the following lemma.

LEMMA 4.1. *Assume that* $\rho = \sum_k \tilde{C}_k < 1$ *and that* $\sum_k e^{\alpha\|k\|}\tilde{C}_k < \infty$ *for some* $\alpha > 0$. *Then there exists* c_1, $d > 0$ *such that*

$$\sum_{d_E(0,k)\ge r} \tilde{D}_k \le c_1 e^{-dr}. \tag{4.3}$$

PROOF. Let $p_k = \tilde{C}_k/\rho$ and let $Y_1, Y_2, \dots$ be an i.i.d. sequence with $P(Y_1 = k) = p_k$. Then $\tilde{C}_k^n = \rho^n P(Y_1 + \dots + Y_n = k)$. Using $P(|Y_1 + \dots + Y_n| \ge y) \le P(|Y_1| + \dots + |Y_n| \ge y)$ and the estimate $P(|Y_1| + \dots + |Y_n| \ge y) \le \phi(s)^n \exp(-sy)$, where $\phi(s) = \sum \exp(s\|k\|)p_k$, we get

$$\begin{aligned}\sum_{\|k\|\ge r} \tilde{D}_k &= \sum_{n=0}^{\infty} \rho^n P(|Y_1 + \dots + Y_n| \ge r) \le \sum_{n=0}^{\infty} \rho^n \phi(s)^n e^{-sr} \\ &= \frac{1}{1-\rho\phi(s)} e^{-sr},\end{aligned}$$

when s is taken so small that $\rho\phi(s) < 1$. □

It now follows from (4.2), (4.3) and the estimates in Jensen (1993) for $\tilde{C}_k$ that the strong mixing condition (1.2) holds for a set of parameter values on the form $\{(\zeta,\beta) \mid \beta > 0, \zeta < \zeta_0(\beta)\}$, for some function $\zeta_0(\beta)$ tending to infinity for $\beta \to 0$. Also (2.5) holds because of the Markov property. Again then we want to establish (2.7). We can proceed as in (3.3), but now the probability

$$P_{\zeta,\beta}(N_j = 0, j \neq k, |j_i - k_i| \le 2) \tag{4.4}$$

depends on k. However, because of the factor $e^{-\beta S} \le 1$ in (4.1) the probability (4.4) will be greater than the corresponding probability in the Poisson process with rate ζ. Thus, we have a bound as in (3.3) and an Edgeworth expansion for

(N, S) or $S \mid N = n$ can be established. This is, however, a theoretical result since the coefficients in the expansion cannot be explicitly given.

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