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INTERPRETATION AND MANIPULATION OF EDGEWORTH EXPANSIONS

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Abstract. With a given Edgeworth expansion sequences of i.i.d. r.v.'s are associated such that the Edgeworth expansion for the standardized sum of these r.v.'s agrees with the given Edgeworth expansion. This facilitates interpretation and manipulation of Edgeworth expansions. The theory is applied to the power of linear rank statistics and to the combination of such statistics based on subsamples. Complicated expressions for the power become more transparent. As a consequence of the sum-structure it is seen why splitting the sample causes no loss of first order efficiency and only a small loss of second order efficiency.

Key words and phrases: Asymptotic expansions, combination of test statistics, one-sample problem, two-sample problem, power, contiguous alternatives, second order efficiency, second order standardization.

1. Introduction

For many statistics it is impossible to present the exact distribution in a tractable form. Therefore much effort has been made to derive limit theorems in order to approximate the exact distribution by an easily computable asymptotic distribution. In particular the normal distribution is often used for this kind of approximations, due to the fact that many statistics are at first order asymptotically equivalent to a sum of i.i.d. r.v.'s.

Subsequently, second order limit theorems are derived to improve the approximations. Indeed, using Edgeworth expansions far more accurate approximations may be obtained, as has been showed by many authors. The validity of formal Edgeworth expansions has been proved under suitable assumptions by Bhattacharya and Ghosh (1978). As a disadvantage, however, sometimes the approximations become more difficult to interpret. For instance, it is hard to see in the Edgeworth expansion for the power of the two-sample linear rank test (cf. Bickel and van Zwet (1978)) what the effect is of (small) changes in the alternative.

Also manipulation of the more complicated expressions is harder. Sometimes we want to take a convolution of a finite number of statistics, where for each statistic an Edgeworth expansion of its distribution function (d.f.) is available.

Such a situation occurs for example in testing theory when one wants to combine several test statistics (cf. Albers (1992)) or in reliability where one has a superposition of several component streams (cf. Kroese and Kallenberg (1989)). If we restrict attention to first order results such manipulations are easy, since sums of independent normal variables are again normal. However, for second order results one needs evaluation of convolutions of Edgeworth expansions, which is in general more difficult.

Working out the convolution for a finite number of subsamples, frequently it turns out that the resulting Edgeworth expansion has a similar form as the Edgeworth expansion based on the whole sample. At first sight it seems a happy coincidence that a lot of terms have exactly the right form to cooperate in a nice way. Conditions for such a right form are given in Albers (1992). These conditions look rather peculiar, but are motivated by considering the example of sums of i.i.d. r.v.'s. Here we go a step further. *With each given Edgeworth expansion of an individual statistic based on a subsample we associate sequences of i.i.d. r.v.'s in such a way that the Edgeworth expansion for the standardized sum of these r.v.'s agrees (up to the given order) with the given Edgeworth expansion.* So for each subsample we have a sum of r.v.'s. Adding these sums we end up again with a sum and hence the required expansion for the convolution is immediately obtained, using the well-known structure of Edgeworth expansions for sums of r.v.'s. In this way sums of independent r.v.'s are not only motivating particular examples, but typically generate the general case.

From this interpretation it becomes clear why the resulting Edgeworth expansion has a similar form as the Edgeworth expansion of the undivided statistic based on the whole sample.

Moreover, the Edgeworth expansion of the undivided statistic itself is more easily understood, since *the situation is the same as if our test statistic is a sum of i.i.d. r.v.'s*. The latter description is much more familiar. This enables us for example to judge the effect on the power of small changes in the alternative.

At the same time a *second order standardization* is introduced. The usual standardization of a statistic ensures that its mean and variance are 0 and 1, respectively, under the null hypothesis. The second order standardization makes also the coefficient of skewness and the kurtosis equal to zero under the null hypothesis.

Particular attention is paid to the papers by Albers *et al.* (1976) and Bickel and van Zwet (1978). These papers, in the sequel denoted by ABZ and BZ, respectively, present asymptotic expansions for the power of distribution free tests in the one- and the two-sample case. In Section 2 the one-sample case is treated, in Section 3 a general approach is presented, which is applied to the more complicated two-sample case.

2. One-sample linear rank statistics

In this section we consider the power of one-sample linear rank statistics to illustrate the ideas in a relative simple but non-trivial situation.

Let $X_1, \dots, X_n$ be i.i.d. r.v.'s with continuous symmetric d.f. $F(x-\theta)$. The one-sample linear rank statistic for testing the null hypothesis $\theta = 0$ is denoted by T. (When there is no confusion we suppress the index n.) Denote by Φ the standard normal d.f. and by ϕ its density. Write H_k for the Hermite polynomial of degree k ($k = 0, 1, \ldots$). Under suitable regularity conditions there exist for each $c > 0$ positive numbers $\delta_1, \delta_2, \ldots$ such that $\lim_{n\to\infty} \delta_n = 0$ and for every n and $0 \le \theta \le cn^{-1/2}$

$$|\pi(\theta) - \pi_1(\theta)| \le \delta_n n^{-1}, \tag{2.1}$$

where $\pi(\theta)$ denotes the power of the one-sided level-α test based on T at the alternative θ and $\pi_1(\theta)$ is given in (4.11) of ABZ. For convenience we repeat here the formulae (4.7), (4.8), (4.10) and (4.11) of ABZ in a condensed form.

$$\tilde{K}_\theta(x) = \Phi(x) + \phi(x) \sum_{k=0}^{3} n^{(1-k)/2} \theta^{3-k} b_k H_k(x), \tag{4.7 ABZ}$$

$$K_{\theta,1}(x) = \tilde{K}_\theta(x) + \phi(x) \frac{1}{2} n^{-1} \tilde{\eta} \tilde{b}_0(n), \tag{4.8 ABZ}$$

$$\tilde{\eta} = n^{1/2}\theta h, \tag{4.10 ABZ}$$

$$\pi_1(\theta) = 1 - K_{\theta,1}(u_\alpha - \tilde{\eta}) + \phi(u_\alpha - \tilde{\eta}) n^{-1} b_3 H_3(u_\alpha). \tag{4.11 ABZ}$$

Explicit expressions for the constants $b_0, \ldots, b_3$, h and $\tilde{b}_0(n)$ are given in ABZ. We only need the expression for b_3. Writing J for the score function of T it is given by

$$b_3 = \int_0^1 J^4(t)dt \left\{ 12 \int_0^1 J^2(t)dt \right\}^{-2}. \tag{2.2}$$

Inspecting (4.11), (4.8) and (4.7) of ABZ and using

$$(u_\alpha - \tilde{\eta})^3 - 3(u_\alpha - \tilde{\eta}) - u_\alpha^3 + 3u_\alpha = -3\tilde{\eta} H_2(u_\alpha - \tilde{\eta}) - 3\tilde{\eta}^2 H_1(u_\alpha - \tilde{\eta}) - \tilde{\eta}^3,$$

$\pi_1(\theta)$ may be written in the form

$$\pi_1(\theta) = 1 - G(u_\alpha - \tilde{\eta}), \tag{2.3}$$

where

$$G(x) = \Phi(x) + \phi(x) \sum_{k=0}^{2} a_k(n) H_k(x), \tag{2.4}$$

$$a_0(n) = n^{1/2}\theta^3 b_0 + \frac{1}{2} n^{-1} \tilde{\eta} \tilde{b}_0(n) - \tilde{\eta}^3 n^{-1} b_3,$$
$$a_1(n) = \theta^2 b_1 - 3\tilde{\eta}^2 n^{-1} b_3,$$
$$a_2(n) = n^{-1/2}\theta b_2 - 3\tilde{\eta} n^{-1} b_3, \qquad u_\alpha = \Phi^{-1}(1-\alpha), \qquad \tilde{\eta} = n^{1/2}\theta h.$$

The regularity conditions and the proof of (2.1) can be found on p. 126, 127 of ABZ. Here we take the result for granted, give a new interpretation of it and show

how we can use this new interpretation. We therefore associate with G a sequence of independent r.v.'s.

Let $Y_{1n}, \ldots, Y_{nn}$ be i.i.d. r.v.'s with

$$EY_{in} = n^{-1/2}\{\tilde{\eta} - a_0(n)\}, \qquad \operatorname{var} Y_{in} = (1 + \theta^2 b_1 - 3\tilde{\eta}^2 n^{-1} b_3)^{-2},$$
$$\kappa_3 = \kappa_3(Y_{in}) = -6\theta b_2 + 18\tilde{\eta} n^{-1/2} b_3, \qquad \kappa_4 = \kappa_4(Y_{in}) = 0.$$

Further we choose Y_{in} such that $E|Y_{in}|^5$ is uniformly bounded and the densities of Y_{in} are functions of uniformly bounded variation. It is easily seen that such Y_{in} exist. Note that here and in the following, the distribution of Y_{in} depends on θ, and hence also the expectations and variances.

By straightforward calculations (cf. Taniguchi (1986), pp. 5–6, Petrov (1975), Theorem 7 on p. 175) we get $P(n^{-1/2} \sum_{i=1}^n Y_{in} \le u_\alpha) = G(u_\alpha - \tilde{\eta}) + o(n^{-1})$ as $n \to \infty$. Hence, in view of (2.1) and (2.3),

$$\pi(\theta) = P\left\{n^{-1/2} \sum_{i=1}^{n} Y_{in} > u_\alpha\right\} + o(n^{-1}) \quad \text{as} \quad n \to \infty. \tag{2.5}$$

So the power of the one-sided level-α test based on the linear rank statistic T is up to $o(n^{-1})$ asymptotically equivalent to the power of the one-sided level-α test based on the sample mean of the Y's, where the Y's are given above. Changing from null hypothesis to contiguous alternatives corresponds to a shift, mainly given by $\tilde{\eta} n^{-1/2}$ (as suggested in (2.1) and (2.3)), but slightly different from it. Further there is a change of order θ^2 in the variance and of order θ in the skewness, while the kurtosis remains 0. Interpreting the behaviour of the one-sample linear rank statistic as if we were dealing with the sample mean of the Y's facilitates the interpretation of Theorem 4.1 in ABZ.

Next consider on the one hand r independent one-sample linear rank statistics $T_1, \ldots, T_r$ based on r subsamples with sample sizes $n_1, \ldots, n_r$, where $n_j n^{-1} \ge \epsilon$ for some $\epsilon > 0$ and all $j = 1, \ldots, r$, and on the other hand consider the unsplitted situation. In order to investigate the penalty to be paid for dividing the total sample into subgroups we want to compare the performance of a combined statistic with the statistic, which would have been used when the total sample was available. Let $S_1, \ldots, S_r$ be the standardized version of $T_1, \ldots, T_r$ as given in Theorem 4.1 of ABZ. Since S_j is asymptotically normal with mean $\tilde{\eta}_j = n_j^{1/2}\theta h$ and variance 1, a first order optimal combination of the S_j is given by $S^* = n^{-1/2} \sum_{j=1}^r n_j^{1/2} S_j$. It is immediately seen that at first order S^* and S are equivalent. To study the second order behaviour, the idea is that S_j is asymptotically equivalent up to order $o(n^{-1})$ to $n_j^{-1/2} \sum_{i=1}^{n_j} Y_i^{(j)}$, where $Y_i^{(j)} = Y_{in_j}$. Hence S^* is asymptotically equivalent up to order $o(n^{-1})$ to $n^{-1/2} \sum_j \sum_i Y_i^{(j)}$, which as a sum of independent r.v.'s is easily comparable to S. However, this is not quite true due to the fact that the power *behaviour* of the S's is equivalent to the power *behaviour* of the sample means of the Y's, but the statistics themselves are *not* equivalent. This is clearly seen from (4.7) in ABZ. The coefficient b_3 of H_3 (cf. (2.2)) is not 0 (even under the null hypothesis) as it should be if the S's correspond to the sample means of the Y's.

We present two solutions to solve this problem. The first one is to modify S a little bit to make it still closer to normality. The transformation which makes disappear the difference in the fourth cumulant is given by

$$\tilde{S} = S + b_3(S^3 - 3S)n^{-1}.$$

This transformation may be called a *second order standardization*: not only the first two moments, but also the third and fourth moment are standardized. Now the d.f. of $\tilde{S}_j = S_j + b_3(S_j^3 - 3S_j)n^{-1}$ is both under the null hypothesis and contiguous alternatives asymptotically equivalent up to order $o(n^{-1})$ to the d.f. of $n_j^{-1/2}\sum_{i=1}^{n_j} Z_i^{(j)}$, where $Z_i^{(j)} = Z_{in_j}$ and Z_{in} is defined similarly as Y_{in}, but with b_3 replaced by 0. For example by (2.9) in Albers (1992), it follows that the d.f. of $\tilde{S}^* = n^{-1/2}\sum_{j=1}^{r} n_j^{1/2}\tilde{S}_j$ is asymptotically equivalent up to order $o(n^{-1})$ to the d.f. of $n^{-1/2}\sum_j\sum_i Z_i^{(j)}$. To compare $\tilde{S}^*$ and $\tilde{S}$ we note that the only difference between $Z_i^{(j)}$ and Z_{in} is in its first moment under alternatives. (Note that $n_j^{-1/2}\tilde{\eta}_j = n^{-1/2}\tilde{\eta}$.) Using $n^{-1}\tilde{\eta}\tilde{b}_0(n) = o(n^{-1/2})$ (cf. p. 155 in ABZ) the only change we have to make when replacing $n^{-1/2}\sum_i Z_i$ by $n^{-1/2}\sum_j\sum_i Z_i^{(j)}$ is therefore to replace $a_0(n)$ by $n^{-1/2}\sum_j n_j^{1/2}a_0(n_j) = a_0(n) + (1/2)n^{-1}\tilde{\eta}\{\sum_j \tilde{b}_0(n_j) - b_0(n)\}$, resulting in

$$(2.6)\quad P(\tilde{S}^* \le x) = P(\tilde{S} \le x) + \phi(x - \tilde{\eta})\frac{1}{2}n^{-1}\tilde{\eta}\left\{\sum_j \tilde{b}_0(n_j) - b_0(n)\right\} + o(n^{-1})$$

uniformly in x as $n \to \infty$, both under the null hypothesis and contiguous alternatives. This shows that there is only a very small difference between $\tilde{S}^*$ and $\tilde{S}$ and hence not much need to reconstruct T, even if the original observations are available, and not merely the T_j, $j = 1, \ldots, r$, cf. also Remark 2.2.

Note that for sufficiently large n the test based on $\tilde{S}$ is exactly the same as the test based on S. (Therefore there is no problem in describing the *power behaviour* of S in terms of sample means.)

The second solution to the problem is based on the fact that if U_j has a d.f. of the form

$$G_j(x) = \Phi(x) - \phi(x)\left\{\frac{\kappa_3}{6\sqrt{n_j}}H_2(x) + \frac{\kappa_4}{24n_j}H_3(x) + \frac{\kappa_3^2}{72n_j}H_5(x)\right\} + o(n^{-1})$$

then $n^{-1/2}\sum_j n_j^{1/2}U_j$ has the same d.f. with n_j replaced by n. For $\kappa_4 > -2$ this follows by relating G_j with the d.f. of standardized sample means. But if it holds for any $\kappa_4 > -2$, then it holds for any κ_4. Using this argument it is now easily seen that as in (2.6) we only have to bother about the coefficient of H_0, when comparing the d.f. of S^* and S, leading to

$$\begin{aligned}(2.7)\quad P(S^* \le x) &= P(S \le x) + \frac{1}{2}n^{-1}\tilde{\eta}d_n\phi(\tilde{x}) + o(n^{-1})\\ &= \Phi(\tilde{x}) + \phi(\tilde{x})\sum_{k=0}^{3} b_k(n)H_k(\tilde{x}) + \frac{1}{2}n^{-1}\tilde{\eta}d_n\phi(\tilde{x}) + o(n^{-1}),\end{aligned}$$

uniformly in x as $n \to \infty$, where

$$b_0(n) = n^{1/2}\theta^3 b_0 + \frac{1}{2}n^{-1}\tilde{\eta}\tilde{b}_0(n), \qquad b_1(n) = \theta^2 b_1, \qquad b_2(n) = \theta n^{-1/2} b_2,$$

$$b_3(n) = n^{-1}b_3, \qquad d_n = \sum_{j=1}^{r} \tilde{b}_0(n_j) - \tilde{b}_0(n), \qquad \tilde{x} = x - \tilde{\eta}.$$

This provides a more direct proof of Theorem 3.2 in Albers (1992).

Remark 2.1. If we replace $n^{-1/2}n_j^{1/2}$ in S^* by γ_j with $\sum_{j=1}^{r}\gamma_j^2 = 1$ and $\gamma_j = n^{-1/2}n_j^{1/2} + o(n^{-1/2})$ as $n \to \infty$, then by the same argument as above (2.7) continues to hold.

Remark 2.2. As is seen in the preceding approach the one-sample linear rank statistics behave like sums of independent r.v.'s with only a small difference in expectation and no difference in the other relevant moments for the several subsamples. Therefore the d.f. of the combined statistic is almost the same as the d.f. of the statistic based on the whole sample. A translation of this result into terms of powers and deficiencies can be found in Albers (1992).

By the interpretation of Edgeworth expansions the tedious direct computation of convolutions of these expansions is avoided. Moreover, it is seen that it is not a happy coincidence that a lot of terms have exactly the right form to cooperate in a nice way, but that the sum-structure of the Edgeworth expansion makes it very easy to write down and evaluate convolutions.

3. General case

Let, uniformly in x as $n \to \infty$, G satisfy the following Edgeworth expansion

$$G(x) = \Phi(x) + \phi(x)\sum_{k=0}^{5} a_k(n)H_k(x) + o(n^{-1}), \tag{3.1}$$

where

$$\begin{aligned} &a_k(n) = O(n^{-1/2}), \qquad k = 0,1,2, \qquad a_k(n) = O(n^{-1}), \qquad k = 3,4,5, \\ &a_4(n) = -a_1(n)a_2(n) + o(n^{-1}), \qquad a_5(n) = -\frac{1}{2}a_2^2(n) + o(n^{-1}), \end{aligned} \tag{3.2}$$

$$-24n\left\{a_3(n) + \frac{1}{2}a_1^2(n) + a_0(n)a_2(n)\right\} > -2. \tag{3.3}$$

Referring to Taniguchi ((1986), pp. 5–6), Petrov ((1975), Theorem 7 on p. 175) we may associate with G i.i.d. r.v.'s $Y_{1n}, \ldots, Y_{nn}$ such that

$$\sup_x \left| G(x) - P\left(n^{-1/2}\sum_{i=1}^{n} Y_{in} \le x\right)\right| = o(n^{-1}) \tag{3.4}$$

as $n \to \infty$, where $EY_{in} = \mu_n$, $\operatorname{var} Y_{in} = \sigma_n^2$, $\kappa_3(Y_{in}) = \kappa_{3n}$ and $\kappa_4(Y_{in}) = \kappa_{4n}$ should satisfy

$$
\begin{aligned}
a_0(n) &= -\sigma_n^{-1} n^{1/2}\mu_n + \sigma_n^{-1} n^{1/2}\mu_n(\sigma_n^{-1} - 1) + o(n^{-1}),\\
a_1(n) &= (\sigma_n^{-1} - 1) - \frac{1}{2} n\sigma_n^{-2}\mu_n^2 - \frac{3}{2}(\sigma_n^{-1} - 1)^2 + o(n^{-1}),\\
a_2(n) &= -\frac{1}{6} n^{-1/2}\kappa_{3n} + \frac{1}{2} n^{-1/2}\kappa_{3n}(\sigma_n^{-1} - 1)\\
&\quad + n^{1/2}\mu_n\sigma_n^{-1}(\sigma_n^{-1} - 1) + o(n^{-1}),\\
a_3(n) &= -\frac{1}{2}(\sigma_n^{-1} - 1)^2 - \frac{1}{24} n^{-1}\kappa_{4n} - \frac{1}{6}\sigma_n^{-1}\kappa_{3n}\sigma_n^{-1}\mu_n + o(n^{-1}),\\
a_4(n) &= \frac{1}{6} n^{-1/2}\kappa_{3n}(\sigma_n^{-1} - 1) + o(n^{-1}), \qquad a_5(n) = -\frac{1}{72} n^{-1}\kappa_{3n}^2 + o(n^{-1})
\end{aligned}
\tag{3.5}
$$

as $n \to \infty$ and can be chosen as

$$
\begin{aligned}
\mu_n &= -n^{-1/2} a_0(n), \qquad \sigma_n = 1 - a_1(n) - \frac{1}{2} a_0^2(n) - \frac{1}{2} a_1^2(n),\\
\kappa_{3n} &= 6n^{1/2}\{-a_2(n) + 3a_4(n) - a_0(n)a_1(n)\},\\
\kappa_{4n} &= -24n\left\{a_3(n) + \frac{1}{2} a_1^2(n) + a_0(n)a_2(n)\right\},
\end{aligned}
\tag{3.6}
$$

and where $E|Y_{in}|^5$ is uniformly bounded and the densities of Y_{in} are functions of uniformly bounded variation. Similarly as in Section 2 the power of linear rank tests in the two-sample problem can be discussed.

Remark 3.1. We do not assume that $a_0(n) = a_1(n) = 0$, corresponding to $\mu_n = 0$, $\sigma_n = 1$, since in practice the given Edgeworth expansions do not fulfill this condition, due to the fact that usually only a first order standardization has been carried out.

Here we treat the combination problem in the general case. Let S be the statistic in the unsplitted situation. Assume that

$$
\sup_x |P(S \le x) - G(x - \eta)| = o(n^{-1})
\tag{3.7}
$$

as $n \to \infty$, where G is of the form (3.1) and $\eta = \eta(n)$ for some function η. Suppose that the coefficients $a_k(n)$ in (3.1) satisfy (3.2). Let $S_1, \ldots, S_r$ be r statistics based on r subsamples with sample sizes $n_1, \ldots, n_r$ satisfying $\sum_{j=1}^r n_j = n$ and $n_j n^{-1} \ge \epsilon$ for some $\epsilon > 0$ and all $j = 1, \ldots, r$. Assume that for each $j = 1, \ldots, r$

$$
\sup_x |P(S_j \le x) - G_j(x - \eta_j)| = o(n^{-1})
\tag{3.8}
$$

as $n \to \infty$, where G_j equals G with n replaced by n_j and $\eta_j = \eta(n_j)$. Note that the functions a_k are supposed to be the same for all $j = 1, \ldots, r$.

It follows by the same argument as in the so called "second solution" in Section 2 that

$$\sup_x \left| P\left(n^{-1/2}\sum_{j=1}^{r} n_j^{1/2} S_j \le x\right) - G^*(x-\eta^*)\right| = o(n^{-1})$$

as $n \to \infty$, where $\eta^* = \sum_{j=1}^{r} n_j^{1/2} n^{-1/2}\eta_j$ and G^* is of the form (3.1) with coefficients $a_k^*(n)$, say, given by (3.5) if we replace μ_n by $\sum_{j=1}^{r} n_j n^{-1}\mu_{n_j}$, σ_n by $(\sum_{j=1}^{r} n_j n^{-1}\sigma_{n_j}^2)^{1/2}$, κ_{3n} by

$$\frac{\sum_{j=1}^{r} n_j n^{-1}\kappa_{3n_j}\sigma_{n_j}^3}{(\sum_{j=1}^{r} n_j n^{-1}\sigma_{n_j}^2)^{3/2}},$$

κ_{4n} by

$$\frac{\sum_{j=1}^{r} n_j n^{-1}\kappa_{4n_j}\sigma_{n_j}^4}{(\sum_{j=1}^{r} n_j n^{-1}\sigma_{n_j}^2)^{2}} = \sum_{j=1}^{r} n_j n^{-1}\kappa_{4n_j} + o(1).$$

Here μ_{n_j}, σ_{n_j}, κ_{3n_j} and κ_{4n_j} are given by (3.6) with $a_k(n_j)$ coming from G_j. (Note that we can do the calculations as if G_j and G correspond to Edgeworth expansions of sums of i.i.d. r.v.'s, although such r.v.'s do not necessarily exist, since we do not require (3.3) or, equivalently, $\kappa_{4n} > -2$.)

In order to compare the combined statistic $n^{-1/2}\sum_{j=1}^{r} n_j^{1/2} S_j$ with the unsplitted statistic S, we compare a_k^*, η^* with a_k, η.

Now we put the following conditions on the coefficients a_k (here and in the sequel κ_{3n}, κ_{3n_j}, μ_n etc. are defined by (3.6))

$$\begin{aligned} &a_0(n_j) = n_j^{1/2} n^{-1/2} a_0(n) + o(n^{-1/2}), \qquad a_1(n_j) = a_1(n) + o(n^{-1/2}),\\ &\kappa_{3n_j} = \kappa_{3n} + o(n^{-1/2}), \qquad \kappa_{4n_j} = \kappa_{4n} + o(1) \end{aligned} \tag{3.9}$$

as $n \to \infty$. Under these conditons we have

$$\sum_{j=1}^{r} n_j n^{-1}\mu_{n_j} = \mu_n + o(n^{-1}), \qquad \sigma_{n_j} = \sigma_n + o(n^{-1/2}),$$

$$\left(\sum_{j=1}^{r} n_j n^{-1}\sigma_{n_j}^2\right)^{1/2} = \sigma_n + o(n^{-1/2}),$$

$$\frac{\sum_{j=1}^{r} n_j n^{-1}\kappa_{3n_j}\sigma_{n_j}^3}{(\sum_{j=1}^{r} n_j n^{-1}\sigma_{n_j}^2)^{3/2}} = \kappa_{3n} + o(n^{-1/2}), \qquad \sum_{j=1}^{r} n_j n^{-1}\kappa_{4n_j} = \kappa_{4n} + o(1),$$

implying (cf. (3.5))

$$\begin{aligned} &a_0^*(n) = a_0(n) - \sigma_n^{-1} n^{1/2}\left\{\sum_{j=1}^{r} n_j n^{-1}\mu_{n_j} - \mu_n\right\} + o(n^{-1}),\\ &a_1^*(n) = a_1(n) + \left(\sum_{j=1}^{r} n_j n^{-1}\sigma_{n_j}^2\right)^{-1/2} - \sigma_n^{-1} + o(n^{-1}),\\ &a_k^*(n) = a_k(n) + o(n^{-1}) \qquad k = 2, \dots, 5. \end{aligned} \tag{3.10}$$

If moreover

$$\eta(n_j) = n_j^{1/2} n^{-1/2} \eta(n) + o(n^{-1}), \tag{3.11}$$

then

$$\eta^* = \eta(n) + o(n^{-1}). \tag{3.12}$$

In view of (3.10) and (3.12) we have proved the following theorem.

THEOREM 3.1. *Suppose that* (3.7) *and* (3.8) *hold, where the coefficients* $a_k(n)$ *satisfy* (3.2) *and* (3.9) *and* $\eta(n)$ *satisfies* (3.11). *Then*

$$\begin{aligned}\sup_x \Bigg| P\left(n^{-1/2}\sum_{j=1}^{r} n_j^{1/2} S_j \leq x\right) \\ - P(S \leq x) - \phi(x-\eta)\{v_0(n) - v_1(n)(x-\eta)\}\Bigg| = o(n^{-1})\end{aligned} \tag{3.13}$$

iff

$$\begin{aligned} v_0(n) &= \sigma_n^{-1} n^{1/2}\left\{\mu_n - \sum_{j=1}^{r} n_j n^{-1}\mu_{n_j}\right\} + o(n^{-1}), \\ v_1(n) &= \sigma_n^{-1} - \left(\sum_{j=1}^{r} n_j n^{-1}\sigma_{n_j}^2\right)^{-1/2} + o(n^{-1}) \end{aligned} \tag{3.14}$$

as $n \to \infty$.

As an application of this theorem we consider (5.14) of BZ, where an Edgeworth expansion is presented for the d.f. of the linear rank statistic in the two-sample problem. The structure of the coefficients $a_k(n)$ is as follows (cf. (5.8) and (5.9) in BZ)

$$\begin{aligned} a_0(n) &= n^{1/2}\theta^2 a_0^{(1)} + n^{-1/2}\theta a_0^{(2)} + n^{1/2}\theta^3 a_0^{(3)} + \frac{1}{2} n^{-1}\eta\tilde{b}_0(n), \\ a_1(n) &= \theta a_1^{(1)} + \theta^2 a_1^{(2)} + n\theta^4 a_1^{(3)} + n^{-1} a_1^{(4)}, \\ a_2(n) &= n^{-1/2} a_2^{(1)} + n^{-1/2}\theta a_2^{(2)} + n^{1/2}\theta^3 a_2^{(3)}, \\ a_3(n) &= n^{-1} a_3^{(1)} + \theta^2 a_3^{(2)}, \quad a_4(n) = n^{-1/2}\theta a_4^{(1)}, \quad a_5(n) = n^{-1} a_5^{(1)} \end{aligned} \tag{3.15}$$

and

$$\eta(n) = n^{1/2}\theta h. \tag{3.16}$$

Since $\theta = O(n^{-1/2})$ and $n^{-1}\eta\tilde{b}_0(n) = o(n^{-1/2})$ it is easily seen that $a_k(n) = O(n^{-1/2})$ $(k = 0, 1, 2)$ and $a_k(n) = O(n^{-1})$ $(k = 3, 4, 5)$. Inspection of (5.8) in BZ yields

$$a_4^{(1)} = -a_1^{(1)}a_2^{(2)} \quad \text{and} \quad a_5^{(1)} = -\frac{1}{2}(a_2^{(1)})^2 \tag{3.17}$$

and therefore (3.2) is satisfied.

The first two conditions of (3.9) are immediately seen from (3.15), since only the first order terms $n^{1/2}\theta^2 a_0^{(1)}$ and $\theta a_1^{(1)}$ are involved. Inspection of (5.8) in BZ yields

$$a_2^{(3)} = -a_0^{(1)}a_1^{(1)} \tag{3.18}$$

and hence up to order $o(n^{-1/2})$ κ_{3n} is a function of θ alone, and not of n, implying $\kappa_{3n_j} = \kappa_{3n} + o(n^{-1/2})$. Finally it follows from (5.8) of BZ that

$$a_3^{(2)} = -\frac{1}{2}(a_1^{(1)})^2 - a_0^{(1)}a_2^{(1)} \tag{3.19}$$

and hence $\kappa_{4n} = -24a_3^{(1)} + o(1)$ and therefore $\kappa_{4n_j} = \kappa_{4n} + o(1)$. Condition (3.11) follows from (3.16) and thus Theorem 3.1 may be applied. Using (cf. (5.8) in BZ)

$$n^{-1/2}\theta a_0^{(2)} = -n^{-1}\eta, \qquad a_1^{(3)} = -\frac{1}{2}(a_0^{(1)})^2, \qquad a_1^{(4)} = -\frac{1}{2} \tag{3.20}$$

straightforward calculation yields

$$\begin{aligned} v_0(n) &= \frac{\eta}{2n}\left[\left\{\sum_{j=1}^{r}\tilde{b}_0(n_j) - b_0(n)\right\} + 2(r-1)\right], \\ v_1(n) &= \frac{1}{2}(r-1)n^{-1}. \end{aligned} \tag{3.21}$$

Theorem 4.2 of Albers (1992) is now established by application of Theorem 3.1.

Remark 3.2. There are several relations between the constants $a_i^{(j)}$ appearing in (3.15). These relations are given in (3.17), (3.18), (3.19) and (3.20). Note that the relations are of a different nature. In a sense (3.17) is very basic. If this relation should not hold, the sum-structure was not present. The other relations clean up the answer. In view of (3.18) and (3.19) we only have to take into account the coefficients of H_0 and H_1, or, equivalently, the expectation μ and variance σ^2. Relation (3.20) ensures that v_0 is a multiple of η and that v_1 does not depend on θ.

The different roles of the relations (3.17)–(3.20) with their different interpretations are not very clear in the approach of Albers (1992).

Remark 3.3. The method presented here is a constructive method. As soon as the mild condition (3.2) is fulfilled, we get an explicit answer. This enables us to consider also cases where the combined statistic is further away from the unsplitted statistic.

Remark 3.4. If we replace $n^{-1/2}n_j^{1/2}$ in (3.13) by γ_j with $\sum_{j=1}^{r}\gamma_j^2 = 1$ and $\gamma_j = n^{-1/2}n_j^{1/2} + o(n^{-1/2})$ as $n \to \infty$, then Theorem 3.1 continues to hold.

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