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BOOTSTRAPPING THE CHANGE-POINT OF A HAZARD RATE

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Abstract. This paper concerns the asymptotic validity of the bootstrap method in a non-regular model. Specifically, it is shown that the parametric bootstrap of the change-point parameter in the change-point hazard rate model works.

Key words and phrases: Change-point, consistency of bootstrap method, parametric bootstrapping.

1. Introduction

There exists a large literature establishing the validity of the bootstrap method. In general, it works well when attention is paid to estimators whose limiting distribution is normal. This includes smooth functions of means (Hall (1988)) but also von Mises functionals and quantiles, etc. (Bickel and Freedman (1981)). However, there are many examples showing that bootstrap method is not always consistent (e.g. Athreya (1987), Hall *et al.* (1991)). Specifically, the method fails when bootstrapping the mean of a stable law, or the rank of some vector parameter for which ties occur. In such cases, the limiting distribution of the estimators is non normal. Thus, when such a situation arises one needs to be careful not to apply the bootstrap naively. In this paper, motivated by the estimation problem in the change-point hazard rate model (e.g. Miller (1960), Mathews and Farewell (1982), Nguyen *et al.* (1984), Matthews *et al.* (1985), Yao (1986, 1987), Pham and Nguyen (1990), Antoniadis and Grégroire (1991)), we will look at the validity of the bootstrap method in this special situation. The model is non-regular and yet a certain form of the maximum likelihood estimator exists and is strongly consistent (Pham and Nguyen (1990)). However, the limiting distribution of the change-point estimator is non normal and quite complicated. Thus, a bootstrap approach, if it works, would be useful to get an estimate of this distribution. Note that the same distribution arises also in a more general problem of estimating the location of a discontinuity in density (Chernoff and Rubin (1956)).

We will show that the parametric bootstrap of the change-point hazard rate model is indeed consistent. This seems to be due to the parametric nature of the problem, since the nonparametric version does not work. The proof for the consistency exploits this aspect, using an argument parallel to the one establishing the limiting distribution of the change-point estimator (Pham and Nguyen (1990)), bypassing the common method based on Mallows metric.

The model and the bootstrap method for obtaining the distribution of the estimator are described in Section 2. Section 3 presents the main results, the proofs of which are relegated to Section 4 to facilitate the reading.

2. The model and the bootstrap

In its simplest form, the change-point hazard rate model is described by the following parametric family of densities on the positive real line

$$f_\theta(t) = ae^{-a\tau}\mathbf{1}(0 \le t \le \tau) + be^{-a\tau - b(t-\tau)}\mathbf{1}(t > \tau)$$

where $\theta = (a, b, \tau) \in (0, \infty)^3$, and $\mathbf{1}(\cdot)$ denotes the set indicator function. A particular feature of this family is $f_\theta(t)$ has a discontinuity at the change-point parameter τ. Thus, this model does not fall into the standard setup in asymptotic maximum likelihood theory. However, a modified form of the maximum likelihood estimator has been successfully obtained and its asymptotic properties have been investigated (Yao (1986), Pham and Nguyen (1990)).

Let $X_1, \dots, X_n$ be random sample from f_{θ_0}, where $\theta_0 = (a_0, b_0, \tau_0)$ denotes the true value of the parameter. The log likelihood function of the model is

$$\sum_{i=1}^{n} \log f_\theta(X_i) = n \int_0^\infty \log f_\theta(t) dF_n(t)$$

where $F_n(\cdot)$ is the empirical cumulative distribution function (CDF) based on $X_1, \dots, X_n$. The above log likelihood can be maximized with respect to a and b for fixed τ, yielding the values

$$\begin{aligned} a_n(\tau) &= F_n(\tau) / \left[\int_0^\tau (t-\tau) dF_n(t) + \tau \right], \\ b_n(\tau) &= [1 - F_n(\tau)] / \left[\int_\tau^\infty (t-\tau) dF_n(t) \right]. \end{aligned} \tag{2.1}$$

Putting these values into the above log likelihood leads to the maximization (with respect to τ) of $L_n(\tau) = F_n(\tau) \log a_n(\tau) + [1 - F_n(\tau)] \log b_n(\tau)$. However, the above function is well defined only on $[\min_{i=1,\dots,n} X_i, \max_{i=1,\dots,n} X_i)$ and even by restricting to this interval it is unbounded (Nguyen *et al.* (1984)). This happens as τ approaches the upper bound of the interval. Note that the function is well-behaved in the neighbourhood of zero, with the convention that 0 log 0 = 0. Thus, a natural way to define the maximum likelihood estimator is to restrict the maximization of L_n to the interval $[0, X_{n-1,n}]$ where $X_{1,n}, \dots, X_{n,n}$ denotes the order statistics of the X_i's (see Yao (1986)). However, using other interval such as $[0, X_{n,n} - \epsilon]$, where ϵ is a fixed positive number, would do equally well (see Pham and Nguyen (1990)). Further, if one has reliable prior information on τ_0, one might want restrict the maximization of L_n to a fixed interval *known* to contain τ_0. Therefore, for reason of generality, we shall consider the maximization of L_n in some sub-interval $[T'_n, T''_n]$ of $[0, X_{n,n})$ where $T'_n = T'_n(X_1, \ldots, X_n)$ and $T''_n = T''_n(X_1, \ldots, X_n)$ are functions of the data. By definition, the maximum likelihood estimator $\hat{\tau}_n$ of τ, realizes the maximum of $L_n(\cdot)$ on $[T'_n, T''_n]$ and the one for θ is $\hat{\theta}_n = (a_n(\hat{\tau}_n), b_n(\hat{\tau}_n), \hat{\tau}_n)$. Note that there is a technical difficulty since the function L_n is not continuous at the data points and thus, may not admit a maximum on $[T'_n, T''_n]$. In this case, however, since this function is continuous in each open interval $(X_{i-1,n}, X_{i,n})$, its supremum equals its left or right limit at some data point, which is then taken as the $\hat{\tau}_n$. Under mild conditions on T'_n, T''_n, it has been shown in Pham and Nguyen (1990) that $\hat{\tau}_n$ is strongly consistent, which implies the strong consistency of $\hat{\theta}_n$ (an earlier weak consistency result has been obtained by Yao (1986)). Moreover, $n(\hat{\tau}_n - \tau_0)$ converges in distribution to a random variable R_I, where I is the index realizing the maximum of

$$S_i = i \log(a_0/b_0) + e^{-a_0\tau_0}(b_0 - a_0)R_i, \qquad -\infty < i < \infty,$$

and

$$R_i = \begin{cases} \displaystyle\sum_{j=i}^{0} (e^{a_0\tau_0}/a_0) Z_j, & \text{if } i \le 0, \\ \displaystyle\sum_{j=1}^{i} (e^{a_0\tau_0}/b_0) Z_j, & \text{if } i > 0, \end{cases}$$

Z_j, $-\infty < j < \infty$ being independent exponential variates with unit mean.

We will use the parametric bootstrap to approximate the distribution of $n(\hat{\tau}_n - \tau_0)$. Write $n(\hat{\tau}_n - \tau_0)$ in the form $U_n(X_1, \ldots, X_n, \tau_0)$ and let $X_1^*, \ldots, X_n^*$ be random variables which, conditionally on $X_1, \ldots, X_n$, are independently distributed according to $f_{\hat{\theta}_n}$. Then the bootstrap distribution for $n(\hat{\tau}_n - \tau_0)$ is simply the distribution of $U_n(X_1^*, \ldots, X_n^*, \hat{\tau}_n)$.

3. Consistency of the bootstrap

We show here the almost sure consistency of the bootstrap, i.e. for almost all sample sequences $X_1, X_2, \ldots$, the conditional distribution of $U_n(X_1^*, \ldots, X_n^*, \hat{\tau}_n)$ converges weakly to the distribution of R_I.

By reasoning conditionally on a realization of the sample $X_1, \ldots, X_n$, one is led to consider a sequence $\theta_n = (a_n, b_n, \tau_n) \in (0, \infty)^3$ converging to $\theta_0 = (a_0, b_0, \tau_0)$ and for each n, a random sample $X_1^*, \ldots, X_n^*$, of size n, from f_{θ_n}. Let $F_n^*(t)$ be the empirical CDF based on $X_1^*, \ldots, X_n^*$ and $L_n^*(\tau)$, $a_n^*(\tau)$, $b_n^*(\tau)$ be defined in the same way as $L_n(\tau)$, $a_n(\tau)$, $b_n(\tau)$, with $F_n(\tau)$ replaced by $F_n^*(\tau)$ and let $\hat{\tau}_n^*$ realize the maximum of L_n^* over $[T'_n(X_1^*, \ldots, X_n^*), T''_n(X_1^*, \ldots, X_n^*)]$. We will prove that the distribution of $n(\hat{\tau}_n^* - \tau_n) = U_n(X_1^*, \ldots, X_n^*, \tau_n)$, when the X_i^*'s are sampled from f_{θ_n}, converges weakly to that of R_I. This would imply that the conditional distribution of $U_n(X_1^*, \ldots, X_n^*, \hat{\tau}_n)$, given $X_1, \ldots, X_n$, when the X_i^*'s are sampled conditionally from $f_{\hat{\theta}_n}$, converges weakly almost surely to that of R_I.

We use a similar approach as in the proof for the limiting distribution of $n(\hat{\tau}_n - \tau_0)$ (Pham and Nguyen (1990)). We show that the result there remains valid for the random variable $n(\hat{\tau}_n^* - \tau_n)$ with f_{θ_0} replaced by f_{θ_n}. In the sequel, P_{θ_n} denotes the probability associated with f_{θ_n} and $X_{1,n}^*, \ldots, X_{n,n}^*$ denote the order statistics of the X_i^*'s. Our method consists of the following steps.

LEMMA 3.1. *As $n \to \infty$, $\|F_n^* - F_{\theta_0}\| \to 0$ and $\|H_n^* - H_{\theta_0}\| \to 0$ in P_{θ_0}-probability where $F_{\theta_0}(x) = \int_0^x f_{\theta_0}(t)dt$, $H_n^*(x) = \int_x^\infty [1-F_n^*(t)]dt$, $H_{\theta_0}(x) = \int_x^\infty [1-F_{\theta_0}(t)]dt$, and $\|\cdot\|$ denotes the sup norm.*

PROPOSITION 3.1. *Suppose that as $n \to \infty$, $P_{\theta_n}\{T_n'(X_1^*, \ldots, X_n^*) < \tau_0 < T_n''(X_1^*, \ldots, X_n^*)\} \to 1$ and $n^{-1}\log[X_{n,n}^* - T_n''(X_1^*, \ldots, X_n^*)] \to 0$ in P_{θ_n}-probability. Then $\hat{\tau}_n^* - \tau_n \to 0$ in P_{θ_n}-probability, as $n \to \infty$.*

Note. The second condition on T_n'' means that $1/[X_{n,n}^* - T_n''(X_1^*, \ldots, X_n^*)] = o(e^n)$ in P_{θ_n}-probability as $n \to \infty$, since $X_{n,n}^*/\log(n) \to 1/b_0$ in P_{θ_n}-probability (see the end of the proof of Proposition 3.1). This is a very mild condition and is satisfied in particular if $T_n'' = X_{n-1,n}^*$ (see the arguments in Pham and Nguyen (1990)).

LEMMA 3.2. *Let $a(\tau)$, $b(\tau)$ be defined as $a_n(\tau)$ and $b_n(\tau)$ with $F_n(x)$ replaced by $F_{\theta_0}(x) = \int_0^x f_{\theta_0}(t)dt$. Then $a(\tau) \to a_0$, $b(\tau) \to b_0$, as $\tau \to \tau_0$. Further*

$$
\begin{aligned}
&\text{(i)} \quad dL_n^*/d\tau = [b(\tau) - a(\tau)][1 - F_{\theta_0}(\tau)] + \epsilon_n(\tau), \qquad \tau \neq X_1^*, \ldots, X_n^*, \\
&\text{(ii)} \quad L_n^*(X_i^*+) - L_n^*(X_i^*-) = \{\log[a(X_i^*+)/b(X_i^*+)] + \epsilon_n(X_i^*)\}/n, \\
&\text{(iii)} \quad L_n^*(\tau + h) = L_n^*(\tau) + \int_\tau^{\tau+h} \{[b(t) - a(t)][1 - F_{\theta_0}(t)] + \epsilon_n(t)\}dt \\
&\qquad\qquad + \int_\tau^{\tau+h} \{\log[a(t)/b(t)] + \epsilon_n(t)\}dF_n^*(t)
\end{aligned}
$$

where $\epsilon_n(\tau)$ denotes a term tending to 0 in P_{θ_n}-probability as $n \to \infty$, uniformly in τ in any compact interval in $(0, \infty)$.

PROPOSITION 3.2. *$P_{\theta_n}\{n(\hat{\tau}_n^* - \tau_n) > c\}$ tends to 0, as $c \to \infty$, uniformly in n, for all n sufficiently large.*

LEMMA 3.3. *Denote by M^* the highest index $i \in \{1, \ldots, n\}$ such that $X_{i,n}^*$ is less than τ_n, then as $n \to \infty$, the pairs of random variables*

$$n(X_{M^*+i,n}^* - \tau_n), \quad n[L_n^*(X_{M^*+i,n}^*) - L_n^*(\tau_n)], \qquad i = 1 - k, \ldots, k,$$

for fixed k, converge jointly in law to the pairs R_i, S_i, $i = 1 - k, \ldots, k$.

LEMMA 3.4. *As $k \to \infty$, $\limsup_{n\to\infty} P_{\theta_n}\{\hat{\tau}_n^* \neq \hat{\tau}_n^*(k)\}$ converges to 0, where $\hat{\tau}_n^*(k)$ is the point $X_{M^*+i,n}^*$ realizing the maximum of $L_n^*(X_{M^*+i,n}^*+)$, $i = 1 -$*

$k,\ldots,k$. *The same result holds if* $L_n^*(\cdot+)$ *is replaced by* $L_n^*(\cdot-)$ *or* $\max\{L_n^*(\cdot+), L_n^*(\cdot-)\}$.

Details of proofs for the above results are presented in the next section.

The results of Lemma 3.1 show that $a_n^*(\cdot)$ and $b_n^*(\cdot)$ converge in P_{θ_n}-probability as $n \to \infty$ to $a(\cdot)$ and $b(\cdot)$, as defined in Lemma 3.2, uniformly in any compact interval in $(0, \infty)$, since by (2.1) and integration by parts, one also has

$$a_n^*(\tau) = F_n^*(\tau)/[H_n^*(0) - H_n^*(\tau)], \quad b_n^*(\tau) = [1 - F_n^*(\tau)]/H_n^*(\tau) \tag{3.1}$$

where H_n^* is as in Lemma 3.1. These convergence results and those of Lemma 3.1 play an important role in proving Proposition 3.1. They also help proving Lemma 3.2, part (iii) of which, together with the property of F_n^* is crucial for Proposition 3.2. This proposition together with Lemma 3.3 and (i) of Lemma 3.2 yield Lemma 3.4. The main result (consistency of the bootstrap) is a consequence of Lemmas 3.3 and 3.4, using an argument similar to the proof of Theorem 2 in Pham and Nguyen (1990). Explicitly, by Lemma 3.4, for any $\epsilon > 0$, there exists a positive integer K such that for all $k \geq K$, $P_{\theta_n}\{\hat{\tau}_n^* \neq \hat{\tau}_n^*(k)\} < \epsilon$ for all n sufficiently large. On the other hand, by Lemma 3.3, $n[\hat{\tau}_n^*(k) - \hat{\tau}_n]$ converges in distribution as $n \to \infty$ to $R_{I(k)}$, where $I(k)$ is the index realizing the maximum of S_i, $1 - k \leq i \leq k$. This means that for any real number t, $P_{\theta_n}\{n[\hat{\tau}_n^*(k) - \hat{\tau}_n] \leq t\} \to P\{R_{I(k)} \leq t\}$ as $n \to \infty$ (note that the CDF of $R_{I(k)}$ is continuous). Thus, $|P_{\theta_n}\{n(\hat{\tau}_n^* - \hat{\tau}_n) \leq t\} - P\{R_{I(k)} \leq t\}| < 2\epsilon$ for all n large enough. But $R_{I(k)} \to R_I$ almost surely as $k \to \infty$, implying that $|P\{R_{I(k)} \leq t\} - P\{R_I \leq t\}| < \epsilon$ for all k large enough. Hence, taking k as required, for any real number t, $|P_{\theta_n}\{n(\hat{\tau}_n^* - \hat{\tau}_n) \leq t\} - P\{R_I \leq t\}| < 3\epsilon$ for all n large enough, which is the desired result.

Note. If one uses the nonparametric bootstrap, then X_i^* would be one of the $X_1, \ldots, X_n$. Now, it can be shown that Lemma 3.4 is still valid, meaning that τ_n^* will be one of the $X_1, \ldots, X_n$ with probability tending to one as $n \to \infty$. Thus, the conditional distribution of $n(\tau_n^* - \tau_n)$, given $X_1, \ldots, X_n$, would have all its mass concentrated on the points $n(X_i - \tau_n)$ with probability tending to one as $n \to \infty$. Note that τ_n itself is also one of the points X_i with probability tending to one. But the points $n(X_{M+i,n} - \tau_0)$, M denoting the largest index m for which $X_{m,n} > \tau_0$, converge in distribution to the points of increase of an honogeneous Poisson process, as $n \to \infty$. Since these points are discrete, the conditional distribution of $n(\tau_n^* - \tau_n)$, given $X_1, \ldots, X_n$, has support converging to a *discrete random set*, and hence cannot converge in law to the distribution of R_I, which has support the whole real line. Thus, the nonparametric bootstrap is inconsistent.

4. Proofs of results

PROOF OF LEMMA 3.1. The proof for the pointwise convergence, in P_{θ_n}-probability as $n \to \infty$, of F_n^* to F_{θ_0} and of H_n^* to H_{θ_0} is standard, noting that $H_n^*(x) = \int_x^\infty (t - x) dF_n^*(t)$ by integration by parts. The convergence in the sup norm follows from the monotonicity of F_n^* and H_n^* and the convergence of $H_n^*(0)$

to $H_{\theta_0}(0)$ (one already has $F_n^*(0) = F_{\theta_0}(0) = 0 = H_n^*(\infty) = H_{\theta_0}(\infty)$ and $F_n^*(\infty) = F_{\theta_0}(\infty) = 1$). For completeness, we provide here a brief proof for this assertion concerning H_n^* (the one concerning F_n^* is similar). Let m be a large integer and define t_i by $H_{\theta_0}(t_i) = H_{\theta_0}(0) + [H_{\theta_0}(\infty) - H_{\theta_0}(0)]i/m$, $i = 0, \ldots, m$ (thus $t_0 = 0$, $t_m = \infty$). Then for $t \in [0, \infty)$, $t_{i-1} \le t < t_i$ for some $i \in \{1, \ldots, m\}$ and hence, H_n^* and H_{θ_0} being non increasing functions, $H_n^*(t) - H_{\theta_0}(t) \le H_n^*(t_{i-1}) - H_{\theta_0}(t_i)$ and $H_{\theta_0}(t) - H_n^*(t) \le H_{\theta_0}(t_{i-1}) - H_n^*(t_i)$. Thus

$$\|H_n^* - H_{\theta_0}\| \le \max_{i=1,\ldots,m} \max[H_n^*(t_{i-1}) - H_{\theta_0}(t_i), H_{\theta_0}(t_{i-1}) - H_n^*(t_i)].$$

But the above right-hand side converges in P_{θ_n}-probability as $n \to \infty$ (m fixed) to $\max_{i=1,\ldots,m}[H_{\theta_0}(t_{i-1}) - H_{\theta_0}(t_i)] = [H_{\theta_0}(0) - H_{\theta_0}(\infty)]/m$. Since m can be chosen arbitrarily, for any $\delta > 0$, $P_{\theta_n}(\|H_n^* - H_{\theta_0}\| < \delta) \to 0$ as $n \to \infty$, yielding the result. □

PROOF OF PROPOSITION 3.1. By Lemma 3.1, it is clear that $L_n^*(\tau)$ converges in P_{θ_n}-probability to

$$L(\tau) = F_{\theta_0}(\tau) \log a(\tau) + [1 - F_{\theta_0}(\tau)] \log b(\tau),$$

uniformly in any compact interval in $(0, \infty)$. Since L is uniquely maximized at τ_0 (see Lemma 3 in Pham and Nguyen (1990)), one may expect that the point $\hat{\tau}_n^*$ realizing the maximum of L_n^* over $[T_n'(X_1^*, \ldots, X_n^*), T_n''(X_1^*, \ldots, X_n^*)]$ converges to τ_0 in P_{θ_n}-probability. For this to happen, using the same arguments as in the proof of Lemma 1 in Pham and Nguyen (1990), one only needs the following further conditions, putting $T_n'^* = T_n'(X_1^*, \ldots, X_n^*)$, $T_n''^* = T_n''(X_1^*, \ldots, X_n^*)$,

(i) the convergence is uniform in the random interval $[T_n'^*, T_n''^*]$, in the sense that $\sup_{T_n' \le \tau \le T_n''} |L_n^*(\tau) - L(\tau)| \to 0$ in P_{θ_n}-probability,

(ii) $P_{\theta_n}\{T_n'^* < \tau_0 < T_n''^*\} \to 1$,

(iii) L is continuous and $L(\tau_0) > \max\{\limsup_{\tau\to\infty} L(\tau), \limsup_{\tau\to 0} L(\tau)\}$ (in addition to having τ_0 as the unique maximum).

Condition (ii) is part of the assumptions while (iii) is already proved in Lemma 1 in Pham and Nguyen (1990). Thus, one needs only to prove (i). Now, $L_n^*(\tau) = F_n^*(\tau) \log a_n^*(\tau) + [1 - F_n^*(\tau)] \log b_n^*(\tau)$ with $a_n^*(\tau)$ and $b_n^*(\tau)$ given by (2.2) and since F_n^* converges uniformly on $[0, \infty)$ to F_{θ_0} in P_{θ_n}-probability (Lemma 3.1), $F_n^* \log(F_n^*) + (1 - F_n^*) \log(1 - F_n^*)$ also converge uniformly to $F_{\theta_0} \log(F_{\theta_0}) + (1 - F_{\theta_0}) \log(1 - F_{\theta_0})$ (by convention $0 \log 0 = 0$). Thus, one needs only to prove the uniform convergence, on $[T_n'^*, T_n''^*]$ in P_{θ_n}-probability, of $F_n^* \log[H_n^*(0) - H_n^*]$ to $F_{\theta_0} \log[H_{\theta_0}(0) - H_{\theta_0}]$ and of $(1 - F_n^*) \log(H_n^*)$ to $(1 - F_{\theta_0}) \log(H_{\theta_0})$. However, the convergence is already uniform on any half interval $[\alpha, \infty)$ for the first random variable and on any half interval $(0, \beta]$ for the second. Since $F_{\theta_0}(\tau) \log[H_{\theta_0}(0) - H_{\theta_0}(\tau)]$ and $[1 - F_{\theta_0}(\tau)] \log[H_{\theta_0}(\tau)]$ converge to 0 as $\tau \to 0$ and $\tau \to \infty$, respectively, one needs only to show that for all $\delta > 0$,

$$\text{(i}'\text{)} \quad \limsup_{n\to\infty} P_{\theta_n} \left\{ \sup_{T_n'^* \le \tau \le t'} F_n^*(\tau) |\log[H_n^*(0) - H_n^*(\tau)]| > \delta \right\} \to 0 \quad \text{as} \quad t' \to 0,$$

$$\text{(i}''\text{)} \quad \limsup_{n\to\infty} P_{\theta_n}\left\{\sup_{t''\le\tau\le T_n''^*}[1-F_n^*(\tau)]|\log[H_n^*(\tau)]|>\delta\right\}\to 0 \qquad \text{as} \quad t''\to\infty.$$

To proceed further, we first prove the following results: for any $\epsilon > 0$, there exist positive numbers A, B such that

$$\sup_{\tau>0} F_n^*(\tau)/F_{\theta_n}(\tau) < A,$$
$$B < \inf_{0<\tau<X_{n,n}^*}[1-F_n^*(\tau)]/[1-F_{\theta_n}(\tau)] \le \sup_{\tau>0}[1-F_n^*(\tau)]/[1-F_{\theta_n}(\tau)] < A,$$

with P_{θ_n}-probability exceeding $1-\epsilon$, for all n. Indeed, observe that the random variables in the above inequalities have the same distribution when F_n^* is replaced by the empirical CDF of a sample $U_1,\dots,U_n$ from the uniform distribution over $[0,1]$, $F_{\theta_n}(\tau)$ is replaced by τ and $X_{n,n}^*$ by $\max(U_1,\dots,U_n)$. Since the order statistics $U_{1,n},\dots,U_{n,n}$ of the U_i's have the same distribution as $Z_1/(\sum_{k=1}^{n+1}Z_k),\dots,Z_n/(\sum_{k=1}^{n+1}Z_k)$ where $Z_1,\dots,Z_{n+1}$ are independent exponential variates with unit mean, the three considered random variables have the same distribution as A_n, A_n and B_n, respectively, where

$$A_n = \max_{i=1,\dots,n}\frac{i}{n}\frac{Z_1+\cdots+Z_{n+1}}{Z_1+\cdots+Z_i}, \qquad B_n = \min_{i=1,\dots,n}\frac{i}{n}\frac{Z_1+\cdots+Z_{n+1}}{Z_1+\cdots+Z_{i+1}}.$$

Now choose I, N high enough such that with probability exceeding $1-\epsilon/2$, $i/(\sum_{k=1}^{i}Z_k) < 2$, $(\sum_{k=1}^{n+1}Z_k)/n < 2$ for all $i > I$, $n > N$. Then choose $A > 4$ and high enough such that with probability exceeding $1-\epsilon/2$, $i/(\sum_{k=1}^{i}Z_i) < \sqrt{A}$, $(\sum_{k=1}^{i}Z_i)/n < \sqrt{A}$ for all $i = 1,\dots,I$, $n = 1,\dots,N$, one gets $A_n < A$ with probability exceeding $1-\epsilon$ for all n. By a similar argument, one can find $B > 0$ for which $B_n > B$ with probability exceeding $1-\epsilon$.

We now show (i$'$). From $[1-F_n^*(\tau)]\tau \le H_n^*(0)-H_n^*(\tau) \le \tau$, we get

$$F_n^*(\tau)|\log[H_n^*(0)-H_n^*(\tau)]| \le F_n^*(\tau)\{|\log\tau| + |\log[1-F_n^*(\tau)]|\}.$$

But with P_{θ_n}-probability exceeding $1-\epsilon$, $F_n^*(\tau) < A\,F_{\theta_n}(\tau)$, hence by choosing τ small enough, the above left-hand side can be bounded by an arbitrarily small number (with P_{θ_n}-probability exceeding $1-\epsilon$), yielding (i$'$).

We now show (i$''$). For any $\epsilon > 0$, let A, B be such that with P_{θ_n}-probability exceeding $1-\epsilon$, $[1-F_n^*(t)] < A[1-F_{\theta_n}(t)]$ for all $t \ge 0$ and $[1-F_n^*(t)] \ge B[1-F_{\theta_n}(t)]$ for all $t \in [0, X_{n,n}^*)$. Then

$$B\int_\tau^{X_{n,n}^*}[1-F_{\theta_n}(t)]dt \le H_n^*(\tau) \le A\int_\tau^\infty[1-F_{\theta_n}(t)]dt.$$

Using the fact that $1-F_{\theta_n}(t) = \exp(-tb_n)$ for $t \ge \tau_n$, the above inequality yields

$$B[F_{\theta_n}(X_{n,n}^*)-F_{\theta_n}(\tau)]/b_n \le H_n^*(\tau) \le A[1-F_{\theta_n}(\tau)]/b_n \le A/b_n, \qquad \text{for} \quad \tau \ge \tau_n.$$

Since $1 - F_n^*(X_{n-1,n}^*) = 1/n$, we also have $n[1 - F_{\theta_n}(X_{n-1,n}^*)] \le 1/B$ with P_{θ_n}-probability exceeding $1-\epsilon$, for all n. Let $c(n)$ be defined by $1-F_{\theta_n}[c(n)] = 2/(nB)$, then for $\tau \le c(n)$, $[1 - F_{\theta_n}(X_{n,n}^*)] \le [1 - F_{\theta_n}(\tau)]/2$, hence with P_{θ_n}-probability exceeding $1 - 2\epsilon$:

$$B[1 - F_{\theta_n}(\tau)]/(2b_n) \le H_n^*(\tau) \le A/b_n, \qquad \text{for all} \quad \tau \in [\tau_n, c(n)].$$

Again, with P_{θ_n}-probability exceeding $1 - \epsilon$, $1 - F_n^*(\tau) \le A[1 - F_{\theta_n}(\tau)]$. Thus, taking $t'' > \tau_n$ and large enough, for all $\tau \in [t'', c(n)]$, $[1 - F_n^*(\tau)]|\log H_n^*(\tau)|$ is bounded by an arbitrarily small number with P_{θ_n}-probability exceeding $1-3\epsilon$, for all n.

On the other hand, for $\tau \le T_n''^* < X_{n,n}^*$, $1/n \le 1 - F_n^*(\tau) \le 1$ yielding $(X_{n,n}^* - T_n''^*)/n \le (X_{n,n}^* - \tau)/n \le H_n^*(\tau) \le X_{n,n}^* - \tau \le X_{n,n}^*$ and hence if moreover $\tau \ge c(n)$,

$$\begin{aligned}&[1 - F_n^*(\tau)]|\log[H_n^*(\tau)]| \\ &\quad \le (1 - F_n^*[c(n)]) \max[|\log(X_{n,n}^* - T_n''^*)| + \log n, |\log X_{n,n}^*|].\end{aligned}$$

But $1 - F_n^*[c(n)] \le A\{1 - F_{\theta_n}[c(n)]\} = 2A/(nB)$ with P_{θ_n}-probability exceeding $1 - \epsilon$. Also, for $X_{n,n}^* \ge \tau_n$, $1 - F_{\theta_n}(X_{n,n}^*) = \exp(-X_{n,n}^* b_n)$ and hence $1/(nA) \le \exp(-X_{n,n}^* b_n) \le 1 - F_{\theta_n}(X_{n-1,n}^*) \le 1/(nB)$, with P_{θ_n}-probability exceeding $1 - \epsilon$, for all n, yielding that $X_{n-1,n}^*/(\log n) \to 1/b_0$ in P_{θ_n}-probability. Thus, by the assumption of Proposition 3.1 and the fact that $n^{-1}\log n \to 0$, $[1 - F_n^*(\tau)]|\log H_n^*(\tau)|$ can be bounded, for all $\tau \in [c(n), T_n''^*]$, by an arbitrarily small number with P_{θ_n}-probability exceeding $1 - \epsilon$, for all n large enough. This completes the proof of (i$''$) and hence of Proposition 3.1. □

PROOF OF LEMMA 3.2. The convergence, as $\tau \to \tau_0$, of $a(\tau)$ and $b(\tau)$ to a_0 and b_0 is clear from their explicit expression, as obtained in Lemma 2 of Pham and Nguyen (1990). Now, direct computation shows that $dL_n^*/d\tau = [1-F_n^*(\tau)][b_n^*(\tau) - a_n^*(\tau)]$ for $\tau \ne X_1^*, \dots, X_n^*$. Then, from Lemma 3.1 and (3.1), F_n^*, $a_n^*(\cdot)$ and $b_n^*(\cdot)$ converges to F_{θ_0}, $a(\cdot)$ and $b(\cdot)$ in P_{θ_n}-probability, uniformly on any compact interval of $(0, \infty)$, yielding (i). Also, since $F_n^*(X_i^*+) = F_n^*(X_i^*-) + 1/n$,

$$\begin{aligned}&F_n^*(X_i^*+)\log a_n^*(X_i^*+) - F_n^*(X_i^*-)\log a_n^*(X_i^*-) \\ &\quad = \frac{1}{n}\log a_n^*(X_i^*+) + F_n^*(X_i^*-)\log\frac{a_n^*(X_i^*+)}{a_n^*(X_i^*-)} \\ &\quad = \frac{1}{n}\log a_n^*(X_i^*+) + F_n^*(X_i^*-)\log[1 + n^{-1}F_n^*(X_i^*-)^{-1}].\end{aligned}$$

Similarly,

$$\begin{aligned}&[1 - F_n^*(X_i^*+)]\log b_n^*(X_i^*+) - [1 - F_n^*(X_i^*-)]\log b_n^*(X_i^*-) \\ &\quad = -\frac{1}{n}\log b_n^*(X_i^*+) + [1 - F_n^*(X_i^*-)]\log\{1 - n^{-1}[1 - F_n^*(X_i^*-)]^{-1}\}.\end{aligned}$$

It follows that

$$\begin{aligned} L_n^*(X_i^*+) - L_n^*(X_i^*-) = {} & \log[a_n^*(X_i^*+)/b_n^*(X_i^*+)]/n \\ & + F_n^*(X_i^*-)\log[1+n^{-1}F_n^*(X_i^*-)^{-1}] \\ & + [1-F_n^*(X_i^*-)]\log\{1-n^{-1}[1-F_n^*(X_i^*-)]^{-1}\} \\ \le {} & \log[a_n^*(X_i^*+)/b_n^*(X_i^*+)]/n. \end{aligned}$$

By a similar computation, $L_n^*(X_i^*-) - L_n^*(X_i^*+) \le -\log[a_n^*(X_i^*-)/b_n^*(X_i^*-)]/n$. This yields (ii). The result (iii) then follows from integration, taking into account of (i) and (ii). □

PROOF OF PROPOSITION 3.2. Put $D_n^*(h) = F_n^*(\tau_n + h) - F_n^*(\tau_n)$. By (iii) of Lemma 3.2 and the convergence, as $\tau \to \tau_0$, of $a(\tau)$, $b(\tau)$ and $1 - F_{\theta_0}(\tau)$ to a_0, b_0 and $\exp(-a_0\tau_0)$, for any $\epsilon > 0$, $\rho > 0$, one has with P_{θ_n}-probability exceeding $1-\epsilon$,

$$\begin{aligned} & |L_n^*(\tau_n+h) - L_n^*(\tau_n) - h[(b_0-a_0)\exp(-a_0\tau_0) - \log(a_0/b_0)D_n^*(h)| \\ & \quad \le \rho[|h| + |D_n^*(h)|] \end{aligned}$$

for all h sufficiently small and all n large enough. On the other hand, by the same argument as in the proof of Lemma A2 in Pham and Nguyen (1990), noting that $E[D_n^*(h)] = F_{\theta_n}(\tau_n+h) - F_{\theta_n}(\tau_n) = D_n(h)$ and $\mathrm{var}\{D_n^*(h)\} = |D_n(h)|(1 - |D_n(h)|)/n$, for all $\eta > 0$,

$$P_{\theta_n}\left\{\sup_{h:c/n\le|h|\le 1/c} |D_n^*(h)/h - d_n(h)| > \eta\right\} \to 0, \qquad \text{as} \qquad c \to \infty,$$

uniformly in n, where $d_n(h) = f_{\theta_n}(\tau_n+)$ or $-f_{\theta_n}(\tau_n-)$ according to h positive or not. Clearly $d_n(h) \to d(h)$ defined in the same way as $d_n(h)$ with θ_n replaced by θ_0. Thus, one may choose $c = c(\epsilon,\rho)$ large enough such that for all $h \in [-1/c, -c/n] \cup [c/n, 1/c]$,

$$|L_n^*(\tau_n+h) - L_n^*(\tau_n) - h[(b_0-a_0)\exp(-a_0\tau_0) + \log(a_0/b_0)d(h)]| \le 2\rho|h|,$$

with F_{θ_n}-probability greater than $1-\epsilon$, for all n sufficiently large. The last term in the above left-hand side is negative since $1 - b_0/a_0 < \log(a_0/b_0) < a_0/b_0 - 1$ ($a_0 \ne b_0$). Thus, taking ρ small enough, with P_{θ_n}-probability exceeding $1-\epsilon$, $L_n^*(\tau_n+h) < L_n^*(\tau_n)$ for all $h \in [-1/c, -c/n] \cup [c/n, 1/c]$, all n large enough, implying $P_{\theta_n}\{c/n < |\tau_n^* - \tau_n| < 1/c\} < \epsilon$ for all n sufficiently large. But by Proposition 3.1, $P_{\theta_n}\{|\tau_n^* - \tau_n| > 1/c\} < \epsilon$ for all n large enough, giving the result. □

PROOF OF LEMMA 3.3. As in the proof of Lemma A3 in Pham and Nguyen (1990), using the fact that $F_{\theta_n}(\tau_n) \to F_{\theta_0}(\tau_0)$,

$$\begin{aligned} & n[F_{\theta_n}(X_{M^*+i+1,n}^*) - F_{\theta_n}(X_{M^*+i,n}^*)], \qquad i = 1-k, \ldots, -1, \\ & n[F_{\theta_n}(\tau_n) - F_{\theta_n}(X_{M^*,n}^*)], \qquad n[F_{\theta_n}(X_{M^*+1,n}^*) - F_{\theta_n}(\tau_n)], \\ & n[F_{\theta_n}(X_{M^*+i,n}^*) - F_{\theta_n}(X_{M^*+i-1,n}^*)], \qquad i = 2, \ldots, k, \end{aligned}$$

converge jointly in distribution to $Z_{1-k}, \ldots, Z_k$. On the other hand, for $i \leq -1$,

$$n[F_{\theta_n}(X^*_{M^*+i+1,n}) - F_{\theta_n}(X^*_{M^*+i,n})] - n[X^*_{M^*+i+1,n} - X^*_{M^*+i,n}]f_{\theta_0}(\tau_0-) \to 0$$

in P_{θ_n}-probability (mean value theorem). Similarly for other variables. The result follows. □

Proof of Lemma 3.4. By the same argument as in the first part of the proof of Lemma A4 in Pham and Nguyen (1990), using the results of Proposition 3.2 and Lemma 3.3,

$$\limsup_{n\to\infty} P_{\theta_n}(\hat{\tau}^*_n \notin [X^*_{M^*+1-k,n}, X^*_{M^*+k,n}]) \to 0, \qquad \text{as} \quad k \to \infty.$$

On the other hand, by (i) of Lemma 3.2, there exists a constant $\gamma > 0$ such that $|dL^*_n/d\tau| \geq \gamma$ for all τ in $[X^*_{M^*+1-k,n}, X^*_{M^*+k,n}]$ and distinct from $X^*_{M^*+i,n}$, $i = 1-k, \ldots, k$, with P_{θ_n}-probability tending to one as $n \to \infty$ (k fixed). Thus, for any fixed k, the probability P_{θ_n} that $\hat{\tau}^*_n$ is in $[X^*_{M^*+1-k,n}, X^*_{M^*+k,n}]$ and differs from $\hat{\tau}^*_n(k)$ can be made arbitrarily small for n large enough. The result follows. □

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