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THE LEVEL PROBABILITIES FOR A SIMPLE LOOP ORDERING*

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Abstract. Bartholomew's statistics for testing homogeneity of normal means with ordered alternatives have null distributions which are mixtures of chi-squared or beta distributions depending on whether the variances are known or not. The mixing coefficients depend on the sample sizes and the order restriction. If a researcher knows which mean is smallest and which is largest, but does not know how the other means are ordered, then a loop ordering is appropriate. Exact expressions for the mixing coefficients for a loop ordering and arbitrary sample sizes are given for five or fewer populations and approximations are developed for more than five populations. Also, the mixing coefficients for a loop ordering with equal sample sizes are computed. These mixing coefficients also arise in testing the ordering as the null hypothesis, in testing order restrictions in exponential families and in testing order restrictions nonparametrically.

Key words and phrases: Level probabilities, likelihood ratio tests, order restricted inference, simple loop ordering.

1. Introduction

We consider an experimental situation in which one wishes to compare several treatments with a control when it is believed a priori that all of the treatments are as effective as the control and that a particular one of the treatments is as effective as the others. For instance, suppose that one wished to study the effects of diet, a drug and exercise on patients suffering from a heart condition, but were not able to consider all combinations of the three treatments. One could let treatment 1 be a control, treatment 2 consist of diet alone, treatment 3 the drug alone, treatment 4 exercise alone and treatment 5 consists of all three. If μ_i is the mean response for treatment i, $1 \leq i \leq 5$; if larger means are desirable; and if it is believed that diet, the drug and exercise have a positive effect, then one could test the statistical

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significance of these effects by testing $\mu_1 = \mu_2 = \cdots = \mu_5$ with the alternative constrained by $\mu_1 \leq \mu_j \leq \mu_5$ for $j = 2, 3, 4$. Of course, if the effects of the diet, drug and exercise are additive, that is the interaction terms are all zero, then the testing situation is simpler. We consider the case in which interactions may be present and assume nothing is known about the signs of these interaction terms.

In general, with k treatments, we consider tests of $H_0 : \mu_1 = \mu_2 = \cdots = \mu_k$ versus $H_1 - H_0$, i.e. H_1 holds but H_0 does not, with $H_1 : \mu_1 \leq \mu_j \leq \mu_k$ for $j = 2, 3, \ldots, k-1$. The order restriction given by H_1 is called a (simple) loop, cf. Robertson *et al.* ((1988), p. 84). Bartholomew (1959, 1961) developed the likelihood ratio tests (LRTs) of H_0 versus $H_1 - H_0$ for normal observations with common variance, σ^2. Robertson and Wegman (1978) developed the LRTs of H_1 versus $H_2 :\sim H_1$. In fact, the LRTs for both testing situations were developed for arbitrary partial order restrictions. The null distributions of the LR statistics are mixtures of chi-squared or beta distributions depending on whether σ^2 is known or not. In this paper, we study the mixing coefficients, which are also called level probabilities, for the loop ordering.

In Section 2, the level probabilities are computed for the case of a balanced design, i.e. for equal sample sizes. In Section 3, we use the approach in Chase (1974) to obtain an approximation to the level probabilities for the case in which the control has a larger number of observations, but the other treatments have (nearly) equal sample sizes. Section 3 also contains an approximation for arbitrary weights which is based on the pattern of small and large weights. Analogous approximations have been developed for the simple order ($\mu_1 \leq \mu_2 \leq \cdots \leq \mu_k$) by Robertson and Wright (1983), the simple tree order ($\mu_1 \leq \mu_j$ for $j = 2, 3, \ldots, k$) by Wright and Tran (1985) and for the unimodal order ($\mu_1 \leq \mu_2 \leq \cdots \leq \mu_h \geq \mu_{h+1} \geq \cdots \geq \mu_k$ with $1 < h < k$) by Lucas *et al.* (1989).

Barlow *et al.* (1972) and Robertson *et al.* (1988) demonstrate that these order restricted tests can be substantially more powerful than their omnibus counterparts. They also argue that unless additional information concerning the spacings among the means is available, the LRTs are preferred over contrast tests. Singh and Schell (1990) study the power functions of these LRTs for the loop ordering.

2. The level probabilities

Let $\bar{X} = (\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_k)$ be the vector of sample means of independent random samples from k normal populations with a common variance σ^2. Let $\bar{\mu} = (\bar{\mu}_1, \bar{\mu}_2, \ldots, \bar{\mu}_k)$ be the maximum likelihood estimate of $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ subject to the restriction imposed by H_1. $\bar{\mu}$ can be computed by relabeling so that $\bar{X}_2 \leq \bar{X}_3 \leq \cdots \leq \bar{X}_{k-1}$ and then applying the pool-adjacent-violators algorithm (PAVA) to all k sample means, cf. Robertson *et al.* ((1988), p. 8) or by the algorithm given in Singh and Schell (1992), which is derived from the minimum lower set algorithm, cf. Robertson *et al.* ((1988), p. 24). The first algorithm is called the modified PAVA.

If σ^2 is known, then the LRT of H_0 versus $H_1 - H_0$ (H_1 versus H_2) rejects H_0 (H_1) for large values of $\bar{\chi}^2_{01}$ ($\bar{\chi}^2_{12}$) and for σ^2 unknown, the test statistics are denoted by $\bar{E}^2_{01}$ and $\bar{E}^2_{12}$. These statistics are defined in Robertson *et al.* ((1988), pp. 61–64). Robertson and Wegman (1978) show that for $\bar{\chi}^2_{12}$ and $\bar{E}^2_{12}$, H_0 is least favorable within H_1 and that, with χ^2_j denoting a chi-square variable with j degrees of freedom ($\chi^2_0 \equiv 0$), under H_0

$$(2.1) \qquad P[\bar{\chi}^2_{01} \geq s, \bar{\chi}^2_{12} \geq t] = \sum_{l=1}^{k} P_L(l,k;n) P[\chi^2_{l-1} \geq s] P[\chi^2_{k-l} \geq t],$$

where $n = (n_1, n_2, \ldots, n_k)$ and $P_L(l,k;n)$ is the probability, under H_0, that $\bar{\mu}$ contains exactly l distinct values. Similarly, with $B_{a,b}$ denoting a beta variable ($B_{0,b} \equiv 0$),

$$(2.2) \qquad P[\bar{E}^2_{01} \geq s, \bar{E}^2_{12} \geq t] = \sum_{l=1}^{k} P_L(l,k;n) P[B_{(l-1)/2,(\nu+k-l)/2} \geq s] P[B_{(k-l)/2,\nu/2} \geq t]$$

when H_0 is true. It should be noted that p-values for $\bar{\chi}^2_{01}$ and $\bar{E}^2_{01}$ ($\bar{\chi}^2_{12}$ and $\bar{E}^2_{12}$) could be obtained from (2.1) and (2.2) by taking $t = 0$ ($s = 0$) if the $P_L(l,k;n)$ were known.

The $P_L(l,k;n)$ also arise when testing hypotheses involving a loop ordering in exponential families and when using the analogue of Chacko's nonparametric test for a loop ordering, see Robertson *et al.* ((1988), p. 163 and p. 204).

The estimate $\bar{\mu}$ is the vector in $I_L(k) = \{x \in R^k : x_1 \leq x_j \leq x_k \text{ for } j = 2, 3, \ldots, k-1\}$, the simple loop cone, which is closest to $\bar{X}$ in the sense that it minimizes $\sum n_i(\bar{X}_i - \mu_i)^2$ among all $\mu \in I_L(k)$. For w a vector of positive weights, let $P_w(x \mid I_L(k))$ denote the projection of x into $I_L(k)$ with distance $d_w(x,y) = \sqrt{\sum w_i(x_i - y_i)^2}$. For $Y = (Y_1, Y_2, \ldots, Y_k)$ a vector of independent random variables with $Y_i \sim \mathcal{N}(0, 1/w_i)$ and $w_i > 0$ for $i = 1, 2, \ldots, k$, let

$$(2.3) \qquad P_L(l,k;w) = P[P_w(Y \mid I_L(k)) \text{ has } l \text{ distinct values}].$$

Of course, the $P_L(l,k;w)$ are not changed if the $Y_i \sim \mathcal{N}(a, 1/w_i)$ with a any real number or if all of the weights w_i are multiplied by the same positive constant.

2.1 *Expressions for the level probabilities: arbitrary weights*

Expressions for the level probabilities, $P_L(l,k;w)$, with $k = 3$ and 4 are given in Robertson *et al.* ((1988), p. 78 and p. 84).

The expressions given below for $k = 5$ are tedious, but are easily programmed. The level probabilities, $P_L(l,k;w)$, for the loop ordering depend on those for the simple order and simple tree ordering via equation (2.4.4) in Robertson *et al.* (1988). Because expressions have not been determined for the simple order with arbitrary weights and $k \geq 6$, we only consider $k = 5$.

Let $I_S(k) = \{x \in R^k : x_1 \leq x_2 \leq \cdots \leq x_k\}$; $I_T(k) = \{x \in R^k : x_1 \leq x_j \text{ for } j = 2, 3, \ldots, k\}$; $P_w(x \mid I_S(k))$ and $P_w(x \mid I_T(k))$ denote the projections of x, with distance $d_w(\cdot,\cdot)$ onto $I_S(k)$ and $I_T(k)$, respectively; and $P_S(l,k;w) = P[P_w(Y \mid I_S(k)) \text{ has } l \text{ distinct values}]$ and $P_T(l,k;w) = P[P_w(Y \mid I_T(k)) \text{ has } l \text{ distinct values}]$.

If $B_1, B_2, \ldots, B_l$ partition $\Gamma = \{1, 2, \ldots, k\}$ and the projection is constant on each B_j and increases with j over the B_j, then the B_j are the ordered level sets for the projection. Now consider the case of five populations with arbitrary weights.

There are six subcases in which five ordered level sets can occur; these are as follows: (a) $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}$; (b) $\{1\}, \{2\}, \{4\}, \{3\}, \{5\}$; (c) $\{1\}, \{3\}, \{2\}, \{4\}, \{5\}$; (d) $\{1\}, \{3\}, \{4\}, \{2\}, \{5\}$; (e) $\{1\}, \{4\}, \{2\}, \{3\}, \{5\}$; and (f) $\{1\}, \{4\}, \{3\}, \{2\}$ $\{5\}$. In subcase (a), the contribution to the level probability, $P_L(5, 5; w)$, is given by

$$\begin{aligned}(2.4)\qquad & \frac{1}{16} + \frac{1}{8\pi}(\sin^{-1}\rho_{12} + \sin^{-1}\rho_{23} + \sin^{-1}\rho_{34}) \\ & + \frac{1}{4\pi^2}\int_0^{-\rho_{34}} (1-x^2)^{-1/2} \sin^{-1}\left[-\frac{(1-x^2)^{1/2}\rho_{12}}{(1-x^2-\rho_{23}^2)^{1/2}}\right] dx\end{aligned}$$

where

$$(2.5)\qquad \rho_{i,i+1} = \rho_{i+1,i} = -\left[\frac{w_i w_{i+2}}{(w_i + w_{i+1})(w_{i+1} + w_{i+2})}\right]^{1/2}$$

for $i = 1, 2, 3$, cf. Robertson *et al.* ((1988), p. 75). The contribution to the level probability for subcase (b) can be obtained from the above subcase by interchanging w_3 and w_4; and the contributions of the remaining four subcases can be obtained analogously. The level probability, $P_L(5, 5; w)$ is the sum of the six probabilities mentioned above.

There are twelve subcases in which four ordered level sets can occur, and these are divided into six pairs of cases which are convenient to manipulate mathematically to obtain $P_L(4, 5; w)$: (a) (i) $\{1, 2\}, \{3\}, \{4\}, \{5\}$ and (ii) $\{1, 2\}, \{4\}, \{3\}, \{5\}$; (b) (i) $\{1, 3\}, \{2\}, \{4\}, \{5\}$ and (ii) $\{1, 3\}, \{4\}, \{2\}, \{5\}$; (c) (i) $\{1, 4\}, \{2\}, \{3\}, \{5\}$ and (ii) $\{1, 4\}, \{3\}, \{2\}, \{5\}$; (d) (i) $\{1\}, \{2\}, \{3\}, \{4, 5\}$ and (ii) $\{1\}, \{3\}, \{2\}$, $\{4, 5\}$; (e) (i) $\{1\}, \{2\}, \{4\}, \{3, 5\}$ and (ii) $\{1\}, \{4\}, \{2\}, \{3, 5\}$; and (f) (i) $\{1\}, \{3\}$ $\{4\}, \{2, 5\}$ and (ii) $\{1\}, \{4\}, \{3\}, \{2, 5\}$. Consider the pair of cases in subcase (a). The required probability in this case is given by

$$\begin{aligned}& P_S(1, 2; w_1, w_2)[P_S(4, 4; w_1 + w_2, w_3, w_4, w_5) + P_S(4, 4; w_1 + w_2, w_4, w_3, w_5)] \\ & \quad = \frac{1}{8\pi}\left[\left(\frac{\pi}{2} - \theta_1 - \theta_3\right) + \left(\frac{\pi}{2} - \theta_2 - \theta_4\right)\right]\end{aligned}$$

where

$$\theta_1 = \sin^{-1}\left[\frac{(w_1 + w_2)w_4}{(w_1 + w_2 + w_3)(w_3 + w_4)}\right]^{1/2},$$

$$\theta_2 = \sin^{-1}\left[\frac{w_3 w_5}{(w_3 + w_4)(w_4 + w_5)}\right]^{1/2}$$

and $\theta_3 = \theta_1$ and $\theta_4 = \theta_2$ with w_3 and w_4 interchanged. Using $\sin(\pi/2 - \theta_1 - \theta_3) = \cos\theta_1 \cos\theta_3 - \sin\theta_1 \sin\theta_3$, the above probability can be written as

$$\frac{1}{8\pi}(\sin^{-1}\tau_{3(1,2),(1,2)4} + \sin^{-1}\tau_{35,54})$$

where

$$\tau_{r(u,v),(u,v)s} = \left[\frac{w_r w_s}{(w_u + w_v + w_r)(w_u + w_v + w_s)}\right]^{1/2} \tag{2.6}$$

and

$$\tau_{ru,us} = \left[\frac{w_r w_s}{(w_u + w_r)(w_u + w_s)}\right]^{1/2}. \tag{2.7}$$

The probabilities in the remaining five case pairs can be obtained analogously. Hence,

$$\begin{aligned}(2.8)\ P_L(4,5;w) = \frac{1}{8\pi}[&\sin^{-1}\tau_{3(1,2),(1,2)4} + \sin^{-1}\tau_{35,54} + \sin^{-1}\tau_{2(1,3),(1,3)4} \\ &+ \sin^{-1}\tau_{25,54} + \sin^{-1}\tau_{2(1,4),(1,4)3} + \sin^{-1}\tau_{25,53} \\ &+ \sin^{-1}\tau_{3(5,4),(5,4)2} + \sin^{-1}\tau_{31,12} + \sin^{-1}\tau_{4(5,3),(5,3)2} \\ &+ \sin^{-1}\tau_{41,12} + \sin^{-1}\tau_{4(5,2),(5,2)3} + \sin^{-1}\tau_{41,13}].\end{aligned}$$

There are also twelve subcases in which three ordered level sets can occur, and these are as follows: (a) $\{1,2,3\},\{4\},\{5\}$; (b) $\{1,2,4\},\{3\},\{5\}$; (c) $\{1,3,4\},\{2\},\{5\}$; (d) $\{1\},\{2\},\{3,4,5\}$; (e) $\{1\},\{3\},\{2,4,5\}$; (f) $\{1\},\{4\},\{2,3,5\}$; (g) $\{1,3\},\{2\}$, $\{4,5\}$; (h) $\{1,4\},\{2\},\{3,5\}$; (i) $\{1,2\},\{3\},\{4,5\}$; (j) $\{1,4\},\{3\},\{2,5\}$; (k) $\{1,2\}$, $\{4\},\{3,5\}$; and (l) $\{1,3\},\{4\},\{2,5\}$. In subcase (a) applying the results in Robertson *et al.* ((1988), p. 75 and p. 83), the contribution to the level probability, $P_L(3,5;w)$, is given by

$$\begin{aligned}&P_T(1,3;w_1,w_2,w_3)P_S(3,3;w_1+w_2+w_3,w_4,w_5) \\ &\quad= \left(\frac{1}{2} - P_T(3,3;w_1,w_2,w_3)\right)\left(\frac{1}{4} + \frac{1}{2\pi}\sin^{-1}\rho_{12}^S\right) \\ &\quad= \left(\frac{1}{4} - \frac{1}{2\pi}\sin^{-1}\rho_{12}^T\right)\left(\frac{1}{4} + \frac{1}{2\pi}\sin^{-1}\rho_{12}^S\right)\end{aligned}$$

where

$$\rho_{12}^T = \left[\frac{w_2 w_3}{(w_1 + w_2)(w_1 + w_3)}\right]^{1/2}, \qquad \rho_{12}^S = -\left[\frac{w_1^* w_3^*}{(w_1^* + w_2^*)(w_2^* + w_3^*)}\right]^{1/2} \tag{2.9}$$

and $w_1^* = (w_1 + w_1 + w_3)$, $w_2^* = w_4$, $w_3^* = w_5$.

The contributions to the level probability from subcases (b) and (c) are obtained from that of subcase (a) by interchanging w_3 and w_4, w_2 and w_4, respectively. Likewise, the contributions from subcases (d), (e) and (f) are obtained from (a), (b) and (c) with the following substitutions: $w_1 = w_5$, $w_2 = w_4$, $w_3 = w_3$, $w_4 = w_2$ and $w_5 = w_1$. The contribution to the probability $P_L(3,5;w)$ from subcase (g) is given by

$$\begin{aligned}&P_S(1,2;w_1,w_3)P(1,2;w_4,w_5)P_S(3,3;w_1+w_3,w_2,w_4+w_5) \\ &\quad= \frac{1}{4}\left(\frac{1}{4} + \frac{1}{2\pi}\sin^{-1}\rho_{12}^S\right)\end{aligned}$$

where ρ_{12}^S is given by (2.9) with the substitutions $w_1^* = (w_1 + w_3)$, $w_2^* = w_2$ and $w_3^* = (w_4 + w_5)$. The contributions to the probability from subcases (h) to (l) are obtained in analogous manner. Thus, the level probability $P_L(3, 5; w)$ is equal to the sum of the twelve aforesaid probabilities. Because $1/2 = P_L(1, 5; w) + P_L(3, 5; w) + P_L(5, 5; w) = P_L(2, 5; w) + P(4, 5; w)$ (see Robertson *et al.* ((1988), p. 115)).

$$P_L(2, 5; w) = \frac{1}{2} - P_L(4, 5; w) \quad \text{and}$$
$$P_L(1, 5; w) = \frac{1}{2} - P_L(3, 5; w) - P_L(5, 5; w).$$

2.2 *The level probabilities: equal weights*

If the sample sizes are equal, i.e. $n_1 = n_2 = \cdots = n_k$, then the weights are equal and the $P_L(l, k; w)$ can be computed rather easily. We will use the notation $P_L(l, k)$, $P_S(l, k)$ and $P_T(l, k)$ when the weights are all equal. Robertson and Wright (1983) for the simply ordered case, Wright and Tran (1985) for simple tree orderings, and Lucas *et al.* (1989) for unimodal orderings found that using the equal weights level probabilities in (2.1) or (2.2) provides a reasonable approximation to the true value of tail probabilities (2.1) or (2.2) provided the weights are not too different, the reader should consult those papers for details. Because similar results are expected for the loop ordering, one also could use the values of $P_L(l, k)$ obtained in this subsection for weights that are nearly equal.

Using the approach in Robertson *et al.* ((1988), p. 76),

$$P_L(k, k) = (k - 2)!/k! = [k(k - 1)]^{-1}.$$

Next we compute $P_L(l, k)$ for $2 \le l \le k - 1$. If $B_1, B_2, \ldots, B_l$ denote the ordered level sets for a loop ordering, then card. $(B_j) = 1$ for $1 < j < l$. Let $i = \text{card.}(B_1 - \{1\})$ and $j = \text{card.}(B_l - \{k\})$. Clearly, $i + j = k - l$ or $j = k - l - i$. If $w_1' = i + 1$, $w_l' = j+1$, $w_\alpha' = 1$ for $2 \le \alpha \le l-1$ and ϕ and Φ denote the density and distribution function for the standard normal distribution, then $P_L(l, l; w') = P[Y_1 \le Y_\alpha \le Y_l$ for $\alpha = 2, 3, \ldots, l - 1]$ with $Y_1, Y_2, \ldots, Y_l$ independent normal random variables with zero means, $\text{Var}(Y_1) = 1/(i + 1)$, $\text{Var}(Y_\alpha) = 1$ for $\alpha = 2, 3, \ldots, l - 1$ and $\text{Var}(Y_l) = 1/(j + 1)$. Conditioning on Y_1 and Y_l with $Y_1 < Y_l$, this can be written as

$$\text{(2.10)} \quad \int_{-\infty}^{\infty} \phi(x) \int_{\sqrt{(j+1)/(i+1)}x}^{\infty} \phi(y) \left[\Phi\left(\frac{y}{\sqrt{j+1}}\right) - \Phi\left(\frac{x}{\sqrt{i+1}}\right)\right]^{l-2} dydx,$$

which we denote by $I(i + 1, j + 1)$. The values of $I(i + 1, j + 1)$ were obtained by numerical integration. Applying Theorem 2.4.1 of Robertson *et al.* (1988), with $j = k - l - i$,

$$\text{(2.11)} \quad P_L(l, k) = \sum_{i=0}^{k-l} \binom{k-2}{i} \binom{k-2-i}{j} P_T(1, i + 1) P_T(1, j + 1) I(i + 1, j + 1)$$

Table 1. Equal-weights level probabilities for a loop ordering, $P_L(l,k)$, and the associated cumulants, δ_1 and δ_2.

l	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$	$k=11$	$k=12$	$k=13$	$k=14$	$k=15$
1	0.1041	0.0508	0.0230	0.0097	0.0039	0.0015	0.0005	0.0002	0.0001	0.0001	0.0000
2	0.2939	0.1809	0.0995	0.0498	0.0230	0.0099	0.0040	0.0015	0.0006	0.0002	0.0001
3	0.3459	0.2995	0.2144	0.1329	0.0733	0.0368	0.0170	0.0073	0.0030	0.0011	0.0004
4	0.2061	0.2858	0.2864	0.2300	0.1567	0.0937	0.0504	0.0248	0.0112	0.0048	0.0019
5	0.0500	0.1497	0.2388	0.2673	0.2353	0.1733	0.1109	0.0633	0.0328	0.0156	0.0069
6		0.0333	0.1141	0.2023	0.2472	0.2345	0.1844	0.1251	0.0752	0.0409	0.0203
7			0.0238	0.0901	0.1736	0.2279	0.2303	0.1915	0.1366	0.0860	0.0487
8				0.0179	0.0731	0.1507	0.2101	0.2240	0.1956	0.1459	0.0957
9					0.0139	0.0606	0.1322	0.1940	0.2167	0.1975	0.1533
10						0.0111	0.0511	0.1170	0.1795	0.2089	0.1978
11							0.0091	0.0438	0.1044	0.1665	0.2010
12								0.0076	0.0379	0.0938	0.1549
13									0.0064	0.0332	0.0848
14										0.0055	0.0293
15											0.0048
δ_1	1.804	2.403	3.056	3.752	4.481	5.238	6.016	6.812	7.624	8.448	9.284
δ_2	4.686	6.212	7.838	9.531	11.272	13.047	14.848	16.671	18.510	20.364	22.229

and $P_L(1,k) = 1 - \sum_{l=2}^{k} P_L(l,k)$. Table 1 contains the values of $P_L(l,k)$ for $1 \le l \le k$ and $5 \le k \le 15$.

Because (2.1) is tedious to compute even for moderate k, Bartholomew (1961) proposed a two moment gamma approximation. With $\bar{G}_b$, the survival function (i.e. one minus the distribution function) of the gamma distribution with parameters b and 1,

$$P[\bar{\chi}_{01}^2 \ge s] \doteq p\bar{G}_b(s/\rho) \quad \text{and} \quad P[\bar{\chi}_{12}^2 \ge s] \doteq p\bar{G}_b(s/\rho) \tag{2.12}$$

where $p = 1 - P_L(1,k)$ for $\bar{\chi}_{01}^2$ or $p = 1 - P_L(k,k)$ for $\bar{\chi}_{12}^2$ and b and ρ are given in terms of the first two cumulants, under H_0, of $\bar{\chi}_{01}^2$ or $\bar{\chi}_{12}^2$, cf. (3.2.3) of Robertson *et al.* (1988). Table 1 contains the values of δ_1 and δ_2, the first two cumulants of $\bar{\chi}_{01}^2$ for a simple loop ordering under H_0 and the corresponding cumulants for $\bar{\chi}_{12}^2$ are $\delta_1^* = k - 1 - \delta_1$ and $\delta_2^* = 2(k-1) - 4\delta_1 + \delta_2$.

Two-moment beta approximations are given for the distributions of $\bar{E}_{01}^2$ and $\bar{E}_{12}^2$ under H_0. Let $\bar{H}_{c,d}$ be the survival function of a beta distribution with parameters c and d. The approximation for $\bar{E}_{12}^2$ involves b and ρ given above, see (3.2.7) of Robertson *et al.* (1988). With p as in (2.2) and ν the degrees of freedom on the estimator of the common σ^2,

$$P[\bar{E}_{01}^2 \ge s] \doteq p\bar{H}_{cd}(s) \tag{2.13}$$

where $a = \delta_1/[(\nu + k - 1)p]$, $b = (\delta_2 + \delta_1^2)/[(\nu + k + 1)p]$, $c = a(a-b)/(b-a^2)$ and $d = (1-a)(a-b)/(b-a^2)$.

3. Approximating the level probabilities

Following Chase (1974) we first consider the situation in which the sample size for the control population, n_1, is larger than the other sample sizes which are equal, or at least nearly equal. Then an approximation is derived for arbitrary weights for those situations in which the equal-weights approximation might not seem adequate.

3.1 *Large sample size on the control*

Suppose that $w_1 = W$ and $w_2 = w_3 = \cdots = w_k$ and define

$$Q_L(l,k) = \lim_{W\to\infty} P_L(l,k;w) \quad \text{for} \quad 1 \le l \le k. \tag{3.1}$$

As in the last section, $P_L(k,k;w) = (k-2)!P_S(k,k:w)$, and applying (3.3.4) of Robertson *et al.* (1988), $Q_L(k,k) = [2^{k-1}(k-1)]^{-1}$. Furthermore, from Theorem 3.3.3 of Robertson *et al.* (1988), we see that $\lim_{W\to\infty} P_T(1,k;w) = 2^{-(k-1)}$. Repeating the proof that led to (2.11) with i, j and $I(a,b)$ defined as there, $j = k-l-i$, and

$$\lim_{W\to\infty} I(W+i, j+1) = \int_0^\infty \phi(y)\left[\Phi\left(\frac{y}{\sqrt{j+1}}\right) - \frac{1}{2}\right]^{l-2} dy,$$

which we denote by $I(\infty, j+1)$, for $2 \le l \le k-1$,

$$Q_L(l,k) = \sum_{i=0}^{k-l} \binom{k-2}{i}\binom{k-2-i}{j} 2^{-i} P_T(1,j+1) I(\infty, j+1). \tag{3.2}$$

and $Q_L(1,k) = 1 - \sum_{l=2}^{k} Q_L(l,k)$. The values of $I(\infty, j+1)$ were obtained by numerical integration. Table 2 gives the values of $Q_L(l,k)$ for $1 \le l \le k$ and $5 \le k \le 15$.

Chase (1974) for the simple order, and Robertson and Wright (1985) for the simple tree ordering, found that interpolating on $1/\sqrt{W}$ between $W = 1$ and $W = \infty$ provided a reasonable approximation to the tail probabilities of these test statistics with weight vector, w, of the form considered in this subsection. In particular, let

$$\bar{\chi}^2_{01}(s;w) = \sum_{l=1}^{k} P_L(l,k;w) P[\chi^2_{l-1} \ge s], \tag{3.3}$$

$\bar{\chi}^2_{01}(s)$ be as in (3.3) with $P_L(l,k;w)$ replaced by $P_L(l,k)$ and $\bar{\chi}^2_{01}(s;\infty)$ be as in (3.3) with $P_L(l,k;w)$ replaced by $Q_L(l,k)$. Because results like those obtained for the simple order and the simple tree ordering are expected, we recommend the approximation

$$\bar{\chi}^2_{01}(s;w) = (1 - 1/\sqrt{W})\bar{\chi}^2_{01}(s;\infty) + 1/\sqrt{W}\bar{\chi}^2_{01}(s). \tag{3.4}$$

The two-moment gamma approximation can be used to approximate $\bar{\chi}^2_{01}(s)$. If one replaces δ_1 and δ_2 by θ_1 and θ_2, the first two cumulants of the distribution

Table 2. Limiting level probabilities for a loop ordering with first weight approaching ∞, $Q_L(l,k)$ and cumulants, Θ_1 and Θ_2.

l	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$	$k=11$	$k=12$	$k=13$	$k=14$	$k=15$
1	0.2020	0.1456	0.1039	0.0735	0.0516	0.0359	0.0248	0.0171	0.0117	0.0079	0.0054
2	0.3969	0.3300	0.2659	0.2090	0.1610	0.1219	0.0909	0.0669	0.0487	0.0351	0.0250
3	0.2824	0.3057	0.3013	0.2785	0.2456	0.2087	0.1722	0.1387	0.1094	0.0848	0.0647
4	0.1031	0.1637	0.2108	0.2389	0.2483	0.2420	0.2246	0.2004	0.1732	0.1456	0.1196
5	0.0156	0.0486	0.0921	0.1367	0.1746	0.2010	0.2143	0.2154	0.2065	0.1904	0.1699
6		0.0063	0.0233	0.0509	0.0853	0.1209	0.1524	0.1763	0.1906	0.1953	0.1913
7			0.0026	0.0112	0.0278	0.0518	0.0805	0.1102	0.1373	0.1588	0.1731
8				0.0011	0.0054	0.0150	0.0308	0.0520	0.0768	0.1025	0.1262
9					0.0005	0.0027	0.0081	0.0180	0.0328	0.0520	0.0739
10						0.0005	0.0013	0.0043	0.0104	0.0203	0.0343
11							0.0001	0.0006	0.0023	0.0059	0.0124
12								0.0000	0.0003	0.0012	0.0033
13									0.0000	0.0002	0.0006
14										0.0000	0.0001
15											0.0000
Θ_1	1.333	1.658	2.001	2.360	2.731	3.113	3.504	3.904	4.310	4.723	5.142
Θ_2	3.593	4.527	5.515	6.544	7.607	8.697	9.807	10.934	12.075	13.227	14.388

determined by $\bar{\chi}^2_{01}(s;\infty)$, which are given in Table 2, and replaces $P_L(1,k)$ by $Q_L(1,k)$, then a two-moment gamma approximation is obtained for $\bar{\chi}^2_{01}(s;\infty)$ also. In the same manner, approximations to $\bar{\chi}^2_{12}(t;w)$, $\bar{E}^2_{01}(s;w)$ and $\bar{E}^2_{12}(t;w)$ are obtained.

3.2 *A pattern approximation*

In this subsection, an approximation to the $P_L(l,k;w)$ for arbitrary w is derived. Each weight is classified as small or large; the limit of the level probabilities with the small weights fixed and equal to unity, and the large weights equal and growing without bound is obtained; and the approximate level probabilities are obtained by interpolating between the equal-weights level probabilities and the limiting level probabilities. The details are given at the end of this subsection.

For the pattern approximation, we need to determine the limit as $W \to \infty$ of $P_L(l,k;w)$ for all possible patterns of 1's and W's in the weight. Let w denote a vector of weights which are all 1's or W's. Let A be the number of large weights. Of course, we may assume $1 \le A \le k-1$, for if not the weights are equal. We consider three cases. While the limiting level probabilities depend on w, this will not be made explicit in the notation.

Case I. $(w_1 = w_k = W)$

THEOREM 3.1. *Let* $w_1 = w_k = W$, $w_i = 1$ *or* W *for* $i = 2,3,\ldots,k-1$, *and* $2 \le A \le k-1$. *If* $P_L^{(1)}(l,k) = \lim_{W\to\infty} P_L(l,k;w)$, *then*

$$P_L^{(1)}(l,k) = P_L(l,A), \tag{3.5}$$

where $P_L(l,k)$ *denotes the equal-weights level probabilities for a simple loop ordering with* k *populations and it is understood that* $P_L(l,A)=0$ *for* $l>A$.

PROOF. Dividing all the weights by W, the argument given in Robertson *et al.* (1988) for the Lemma on p. 149 shows that $\lim_{W\to\infty} P_L(l,k;w)=P_L(l,A)$. □

Case II. ($w_1=W$ and $w_k=1$; and by symmetry the case $w_1=1$ and $w_k=W$)

THEOREM 3.2. *Let* $w_1=W$, $w_k=1$, $w_i=1$ *or* W *for* $i=2,3,\ldots,k-1$, *and* $1\le A\le k-1$. *If* $P_L^{(2)}(l,k)=\lim_{W\to\infty}P_L(l,k;w)$ *then* $P_L^{(2)}(l,k)$ *is the* $(l+1)$*-st term of the convolution* $\{P_T(j,A)\}*\{Q_L(j,k-A+1)\}$.

PROOF. In Subsection 3.1, the case $A=1$ was proved. Next, we consider $A>1$. By relabeling, we may assume that $w_1=w_2=\cdots=w_A=W$ and $w_{A+1}=w_{A+2}=\cdots=w_k=1$. Let $Z_0,Z_1,\ldots,Z_k$ be i.i.d. standard normal random variables defined on some probability space, and set $Y_i=Z_i/\sqrt{w_i}$ for $i=0,1,\ldots,k$ with $w_0=W$. Set $V=(Y_1,Y_2,\ldots,Y_A)$, $V'=(Y_0,Y_{A+1},\ldots,Y_k)$, $v=(w_1,w_2,\ldots,w_A)$ and $v'=(w_0,w_{A+1},\ldots,w_k)$. We will argue that for each ω in a set with probability one, there exists a $W_0(\omega)$ such that for all $W\ge W_0(\omega)$,

$$M[P_w(Y\mid I_L(k))]=M[P_v(V\mid I_T(A))]+M[P_{v'}(V'\mid I_L(k-A+1))]-1$$

where $M[x]$ denotes the number of distinct elements in the vector x. Because V and V' are independent this will prove the theorem.

Let $\Gamma_0=\{0,1,\ldots,A\}$ and for $\phi\ne C\subset\{0,1,\ldots,k\}$ let $Av(C)=\sum_{i\in C}w_iY_i/\sum_{i\in C}w_i$. The event $\Omega_0=\{\omega: Av(C)\ne 0$ for all $\phi\ne C\subset\Gamma-\Gamma_0$ and $Av(C_1)\ne Av(C_2)$ for every C_1 and C_2 different, nonempty subsets of $\Gamma_0\}$ has probability one. Thus, we suppose $\omega\in\Omega_0$. As $W\to\infty$, $Y_j(\omega)\to 0$ for $j=0,1,\ldots,A$, and so there exist a $W_1(\omega)$ such that for all $W\ge W_1(\omega)$ and each $\phi\ne C\subset\Gamma-\Gamma_0$, $Av(C)<\min\{Y_i(\omega): i\in\Gamma_0\}$ or $Av(C)>\max\{Y_i(\omega): i\in\Gamma_0\}$. Also, choose $W_2(\omega)$ such that for $W\ge W_2(\omega)$, C_1 and C_2 nonempty subsets of Γ_0 and $\phi\ne D\subset\Gamma-\Gamma_0$, $Av(C_1\cup D)<Av(C_2\cup D)$ if $Av(C_1)<Av(C_2)$. We will choose $W_0(\omega)\ge\max(W_1(\omega),W_2(\omega))$.

For an arbitrary quasi-order, we can use either the minimum lower sets algorithm or the maximum upper sets algorithm to compute a projection onto $I_{\ll}(k)=\{x\in R^k: x_i\le x_j$ if $i\ll j\}$. These algorithms are discussed in Robertson *et al.* ((1988), pp. 24–26). It is clear that at each step one is finding a level set in the projection for the restricted problem in which one only considers the elements of Γ for which the projection has not been determined with $\ll$ restricted to this set. Thus, at each stage one may select either the largest lower set, L, in the restricted problem, which minimizes $Av(L)$ or the largest upper set, U, in the restricted problem, which maximizes $Av(U)$. In projecting onto $I_L(k)$, we will first find an upper set U_1 and if $U_1\ne\Gamma$, then we will find a lower set L_1.

Let $I_H(k)=\{x\in R^k: x_j\le x_k$ for $j=1,2,\ldots,k-1\}$. In computing the projection of $(Y_{A+1},Y_{A+2},\ldots,Y_k)$ onto $I_H(k-A)$ with weights $(w_{A+1},w_{A+2},\ldots,w_k)$, let U_1 be the largest upper set in $\{A+1,A+2,\ldots,k\}$ which maximizes $Av(U)$

among such upper sets. We consider the case $Av(U_1) > \max\{Y_i(\omega) : i \in \Gamma_0\}$ first. In this case, U_1 is also the first upper set in determining the projections $P_{v'}(V' \mid I_L(k-A+1))$ and $P_w(Y \mid I_L(k))$. With $C^- = \{i : A+1 \le i \le k, Y_i(\omega) < 0\}$ and $C^+ = \{i : A+1 \le i \le k; Y_i(\omega) > 0\}$, U_1 contains k but no element in $C^- - \{k\}$. Clearly $\{1\} \notin U_1$. Thus in computing $P_w(Y \mid I_L(k))$, let L_1 be the lower set determined after U_1. L_1 includes all of $C^- - \{k\}$. In fact, if L_1^* is the first lower set found in computing $P_v(V \mid I_T(A))$, then $L_1 = L_1^* \cup (C^- - \{k\})$. Furthermore, the lower set determined after U_1, when computing $P_{v'}(V' \mid I_L(k-A+1))$ is $(C^- - \{k\}) \cup \{0\}$. Thus, card.$(C^+ - U_1) = M[P_{v'}(V' \mid I_L(k-A+1))] - 2$. Also, $M[P_w(Y \mid I_L(k))] = M[P_v(V \mid I_T(A))] +$ card.$(C^+ - U_1) + 1$. The result is proved in this case.

Next, we consider the case $Av(U_1) < \min\{Y_i(\omega) : i \in \Gamma_0\}$. Clearly $k \in C^-$ and $U_1 \supset C^+$. Let $B_1, B_2, \ldots, B_l$ be the ordered level sets in the projection $P_v(V \mid I_T(A))$, and recall that $Av(B_1) < Av(B_2) < \cdots < Av(B_l)$ and if $l > 1$ then $B_2, \ldots, B_l$ are singletons. In determining the first upper set for $P_w(Y \mid I_L(k))$, one considers Γ and $U \cup C$ with U an upper set in $\{A+1, A+2, \ldots, k\}$ and $C \subset \{2, 3, \ldots, A\}$. Therefore, the first upper set in $P_w(Y \mid I_L(k))$ is $\{k\} \cup B_l \cup C^+$ and the lower set determined next in computing $P_w(Y \mid I_L(k))$ is $B_1 \cup (C^- - \{k\})$. Thus, $M[P_w(Y \mid I_L(k))] = M[P_v(V \mid I_T(A))]$ and the proof is completed by showing that $M[P_{v'}(V' \mid I_L(k-A+1))] = 1$. The upper sets in $\Gamma' = \{0, A+1, A+2, \ldots, k\}$ are Γ' and the upper sets in $\{A+1, A+2, \ldots, k\}$. Because $Av(\Gamma') \to 0$, we choose $W_0(\omega) \ge \max(W_1(\omega), W_2(\omega))$ so that $Av(\Gamma') > Av(U_1)$ for all $W \ge W_0(\omega)$. Hence, the first upper set obtained in determining the projection $P_{v'}(V' \mid I_L(k-A+1))$ is Γ'. $\square$

Case III. $(w_1 = w_k = 1)$ The situation with only one large weight which is on an interior element is basic to Case III. Of course, it does not matter which interior weight is large. Let $w_2 = W$ and $w_i = 1$ for $i \in \Gamma - \{2\}$ and set

$$Q'_L(l, k) = \lim_{W \to \infty} P_L(l, k; w). \tag{3.6}$$

Using the techniques employed in the last section,

$$Q'_L(k, k) = \int_{-\infty}^{0} \phi(x) \int_{0}^{\infty} \phi(y)[\Phi(y) - \Phi(x)]^{k-3} dy dx = \frac{1 - 1/2^{k-2}}{(k-1)(k-2)}. \tag{3.7}$$

For $k \ge 2$ and w defined as above ($w_1 = w_3 = \cdots = w_k = 1$ and $w_2 = W$), let $Q_T(l, k) = \lim_{W \to \infty} P_T(l, k; w)$ with $P_T(l, k; w)$ denoting the level probabilities for a simple tree ordering. The $Q_T(l, k)$ are given in Table A.13 of Robertson *et al.* (1988). To determine $Q'_L(l, k)$ with $2 \le l \le k-1$, we consider separately the cases with 2 pooled with 1, 2 pooled with k (which has the same probability as the past case), and 2 is a level set by itself. Using the techniques of Subsection 3.1, this yields

$$Q'_L(l, k) = 2 \sum_{i=0}^{k-l-1} \binom{k-3}{i} \binom{k-3-i}{k-l-i-1} Q_T(1, i+2) P_T(1, k-l-i) \tag{3.8}$$

$$\times \int_0^\infty \phi(y) \left[\Phi\left(\frac{y}{\sqrt{k-l-i}}\right) - \frac{1}{2}\right]^{l-2} dy$$
$$+ \sum_{i=0}^{k-l} \binom{k-3}{i} \binom{k-3-i}{k-l-i} P_T(1, i+1) P_T(1, k-l-i+1)$$
$$\times \int_{-\infty}^0 \phi(x) \int_0^\infty \phi(y)$$
$$\times \left[\Phi\left(\frac{y}{\sqrt{k-l-i+1}}\right) - \Phi\left(\frac{x}{\sqrt{i+1}}\right)\right]^{l-3} dydx.$$

Of course $Q'_L(1,k) = 1 - \sum_{l=2}^k Q'_L(l,k)$. For $5 \le k \le 15$, the values of $Q'_L(l,k)$ are given in Table 3. Case III with $A > 1$ is treated in the next result.

Table 3. Limiting level probabilities for a loop ordering with second weight approaching ∞, $Q'_L(l,k)$ and cumulants, α_1 and α_2.

l	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$	$k=11$	$k=12$	$k=13$	$k=14$	$k=15$
1	0.0583	0.0255	0.0105	0.0041	0.0015	0.0005	0.0002	0.0001	0.0000	0.0000	0.0000
2	0.2354	0.1302	0.0652	0.0300	0.0129	0.0052	0.0020	0.0007	0.0002	0.0001	0.0000
3	0.3688	0.2824	0.1820	0.1031	0.0526	0.0246	0.0107	0.0044	0.0017	0.0006	0.0002
4	0.2646	0.3229	0.2904	0.2126	0.1337	0.0745	0.0376	0.0175	0.0076	0.0031	0.0012
5	0.0729	0.1922	0.2752	0.2808	0.2281	0.1566	0.0942	0.0508	0.0250	0.0114	0.0048
6		0.0469	0.1444	0.2339	0.2642	0.2339	0.1729	0.1110	0.0635	0.0330	0.0157
7			0.0323	0.1120	0.2000	0.2455	0.2336	0.1841	0.1251	0.0753	0.0409
8				0.0234	0.0892	0.1725	0.2270	0.2297	0.1912	0.1366	0.0860
9					0.0177	0.0727	0.1502	0.2096	0.2237	0.1954	0.1459
10						0.0138	0.0604	0.1320	0.1937	0.2166	0.1974
11							0.0111	0.0511	0.1169	0.1794	0.2088
12								0.0091	0.0437	0.1044	0.1664
13									0.0076	0.0379	0.0938
14										0.0064	0.0332
15											0.0055
α_1	2.058	2.667	3.317	4.003	4.719	5.461	6.225	7.008	7.807	8.621	9.447
α_2	5.138	6.634	8.214	9.860	11.557	13.294	15.065	16.861	18.680	20.515	22.363

THEOREM 3.3. *Let* $w_1 = w_k = 1$, $w_i = 1$ *or* W *for* $i = 2, 3, \ldots, k-1$ *and* $1 \le A \le k-2$. *If* $P_L^{(3)}(l,k) = \lim_{W\to\infty} P_L(l,k;w)$, *then* $P_L^{(3)}(l,k) = 0$ *for* $l < A$ *and*

$$P_L^{(3)}(l,k) = Q'_L(l-A+1, k-A+1) \quad \text{for} \quad A \le l \le K. \tag{3.9}$$

We give a hueristic justification for Theorem 3.3. A formal proof like the one given for Theorem 3.2 would provide no new insights. Let $Z_1, Z_2, \ldots, Z_k$ be independent standard normal random variables and set $Y_i = Z_i/\sqrt{w_i}$ for $i \in \Gamma$. Set $G = \{2, 3, \ldots, A+1\}$ and let $C^- = \{i : Z_i < 0 \text{ with } i \in \Gamma - (G \cup \{1, k\})\}$ and $C^+ = \{i : Z_i > 0 \text{ with } i \in \Gamma - (G \cup \{1, k\})\}$. Consider ω in the underlying probability space with $Av(C) \neq 0$ for each $\phi \neq C \subset \Gamma - G$ and $Z_i \neq Z_j$ for $1 \leq I \neq j \leq k$. Let π be a permutation of the integers $2, 3, \ldots, A+1$ for which $Z_{\pi(2)} < Z_{\pi(3)} < \cdots < Z_{\pi(A+1)}$. Suppose that W is large enough so that for $\phi \neq C \subset \Gamma - G$, $Av(C) < Z_{\pi(2)}/\sqrt{W}$ or $Av(C) > Z_{\pi(A+1)}/\sqrt{W}$.

Let $B_1, B_2, \ldots, B_l$ be the ordered level sets of the projection of $(Y_1, Y_2, Y_{A+2}, Y_{A+3}, \ldots, Y_k)$ onto $I_L(k - A + 1)$ with weight vector $(w_1, w_2, w_{A+2}, w_{A+3}, \ldots, w_k)$. In the three cases below, we argue that the number of level sets in the projection of Y onto $I_L(k)$ with weight vector w is $l + A - 1$ for sufficiently large W.

Case 1. ($B_j = \{2\}$ for some $1 < j < k$) For $1 \leq i \leq j-1$, $Av(B_i) < Z_{\pi(2)}/\sqrt{W}$ and for $j < i \leq k$, $Av(B_i) > Z_{\pi(A+1)}/\sqrt{W}$. Applying the modified PAVA, one obtains the ordered level sets $B_1, B_2, \ldots, B_{j-1}$, $\{\pi(2)\}, \{\pi(3)\}, \ldots, \{\pi(A+1)\}$, $B_{j+1}, \ldots, B_l$.

Case 2. ($l = 1$) Consider how the single level set B_1 was obtained via the modified PAVA. Ordering $Y_2, Y_{A+2}, Y_{A+3}, \ldots, Y_{k-1}$ in increasing order, one obtains the Y_i with $i \in C^-$ in increasing order, Y_2, and then the Y_i with $i \in C^+$ in increasing order. We first claim that $Av(\{1\} \cup C^-) > Z_{\pi(A+1)}/\sqrt{W}$ for if not $Av(\{1\} \cup C^-) < Z_{\pi(2)}/\sqrt{W}$. We suppose the latter and seek a contradiction. Starting the pooling process at the left, i.e. with Y_1, and only considering the indices 1, C^-, 2, this process will stop before coming to Y_2. For sufficiently large W, $Av(\{k, 2\} \cup C^+)$ will be essentially zero. Thus for sufficiently large W, if one starts pooling at the right, i.e. with Y_k, and considers the indices 1, C^-, 2, C^+, k, the process will stop as or before Y_2 is reached. This contradicts the fact that $l = 1$. Hence $Av(\{1\} \cup C^-) > Z_{\pi(A+1)}/\sqrt{W}$ and it can be shown similarly that $Av(\{k\} \cup C^+) < Z_{\pi(2)}/\sqrt{W}$. If one computes the projection of Y onto $I_L(k)$ with weight vector w, then for sufficiently large W, the level sets are $\{1, \pi(2)\} \cup C^-$, $\{\pi(3)\}, \ldots, \{\pi(A)\}$, $C^+ \cup \{\pi(A+1), k\}$.

Case 3. ($l > 1$ and $2 \in B_1$; $2 \in B_l$ is similar) Order $Y_2, Y_{A+2}, \ldots, Y_{k-1}$ as in Case 2. Start pooling from the right, i.e. with Y_k, to determine B_l. The pooling process must stop before reaching Y_2 because $2 \in B_1$, $l > 1$ and once two elements are pooled they remain together throughout the process. Thus $Av(B_l) > 0$. Next, pool from the left to obtain B_1. The pooling process must include Y_2 because $2 \in B_1$ and it must stop with Y_2, because for W sufficiently large $Av(B_1)$ is essentially zero. Therefore, the level sets in the projection of Y onto $I_L(k)$ with weight vector w, for sufficiently large W, is $(B_1 - \{2\}) \cup \{\pi(2)\}, \{\pi(3)\}, \ldots, \{\pi(A+1)\}$, $B_2, \ldots, B_l$. The argument is complete.

Following Robertson and Wright (1983), we show how one may approximate tail probabilities of $\bar{\chi}^2_{01}$ under H_0, i.e.

$$\bar{\chi}^2_{01}(s; w) = \sum_{l=1}^{k} P_L(l, k; w) P[\chi^2_{l-1} \geq s]. \tag{3.10}$$

First, classify each weight, w_i, as large if

$$w_i > \bar{w} = 0.65 \min(w_1, w_2, \ldots, w_k) + 0.35 \max(w_1, w_2, \ldots, w_k)$$

(otherwise it is small) and define λ to be the 1/3 power of the ratio of the average of the small weights to the average of the large weights. Next determine to which of the three cases above the weight vector belongs. Compute $\bar{\chi}^2_{01}(s)$ as in (3.11) with $P_L(l, k; w)$ replaced by $P_L^{(1)}(l, k)$ in Case I, by $P_L^{(2)}(l, k)$ in Case II or by $P_L^{(3)}(l, k)$ in Case III. Then, $\bar{\chi}^2_{01}(s; w)$ is approximated by

$$\lambda \bar{\chi}^2_{01}(s) + (1 - \lambda) \chi^2_{01}(s; \infty). \tag{3.11}$$

Of course, $\bar{\chi}^2_{01}(s)$ and $\bar{\chi}^2_{01}(s; \infty)$ could be approximated by two moment approximation, (2.12), with the cumulants given in Tables 1–3.

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