Ann. Inst. Statist. Math.
Vol. 45, No. 2, 233–242 (1993)

THE ASYMPTOTIC NORMALITY OF AN INTERMEDIATE ORDER STATISTIC OF THE RANGES OF SUB-SAMPLES*

J. W. H. SWANEPOEL

Department of Statistics and O.R., Potchefstroom University for CHE, Potchefstroom 2520, South Africa

(Received March 28, 1991; revised April 13, 1992)

Abstract. An intermediate order statistic based on the ranges of sub-samples is introduced and its limiting distribution derived under certain regularity conditions.

Key words and phrases: Intermediate order statistic, sample range, weak convergence.

1. Introduction and statement of results

Let $X_1, X_2, \dots, X_n$ $(n \geq 3)$ be a sequence of independent random variables with common distribution function F, which is assumed to be absolutely continuous (w.r.t. Lebesgue measure) with probability density function f. For a given integer m $(2 \leq m \leq n-1)$ denote by

$$W_{(1)} \leq W_{(2)} \leq \cdots \leq W_{(N)},$$

the order statistics of the sample ranges

$$r(X_{i(1)}, \dots, X_{i(m)}) = \max(X_{i(1)}, \dots, X_{i(m)}) - \min(X_{i(1)}, \dots, X_{i(m)})$$

taken over the $N = \binom{n}{m}$ combinations $1 \leq i(1) < \cdots < i(m) \leq n$. Consider the statistic given by

$$T_n = a(n)\{b(n)W_{(c(n))} - (\gamma + d(n))\} \tag{1.1}$$

for sequences $\{a(n)\}$, $\{b(n)\}$, $\{c(n)\}$ and $\{d(n)\}$ of real numbers, all depending on m, which will be specified below. $W_{(c(n))}$ is an intermediate order statistic (see, e.g. Reiss (1989), p. 12) since we will be dealing only with sequences $\{c(n)\}$ satisfying $c(n) \to \infty$ and $c(n)/N \to 0$ as $n \to \infty$. Throughout the discussion

* This research was supported by grants from the FRD of South Africa.

γ is the parameter of interest. If $m = 2$, γ plays an important role in nonparametric inference. It appears in the Pitman asymptotic efficacy of nonparametric tests based on Wilcoxon scores. These include the Wilcoxon signed rank, Mann-Whitney-Wilcoxon rank sum, Kruskal-Wallis and Friedman tests in addition to asymptotically nonparametric tests in the linear model. It also appears in the asymptotic variance of R-estimators derived from these test statistics and in the standardizing constant for the test statistics in the linear model. The purpose of this paper is to derive the limiting distribution of T_n under certain regularity conditions imposed on F and the above sequences of constants.

Throughout the discussion below the density f will be assumed to satisfy the following conditions

(1.2) f has support (a, b) for some $-\infty \leq a < b \leq \infty$,

(1.3) f is absolutely continuous w.r.t. Lebesgue measure,

(1.4) $f(a+) := \lim_{s \downarrow a} f(s)$ exists and is finite,

$f(b-) := \lim_{s \uparrow b} f(s)$ exists and is finite,

(1.5) the first two derivatives of f exist and are bounded on (a, b), almost everywhere,

(1.6) $f \in L^{2m-1}(a, b)$, where $L^p(a, b)$ denotes the space of all Lebesgue p-integrable functions.

To state our main results precisely, we first introduce some further notation. Let $\{\beta_n\}$ be a sequence of real numbers such that $0 < \beta_n \to \infty$ as $n \to \infty$, and define

$$\gamma := 1 \Big/ \left(m! \int_a^b (f(x))^m dx \right), \tag{1.7}$$

$$b(n) := \beta_n^{1/(m-1)} \gamma^{(m-2)/(m-1)}, \tag{1.8}$$

$$c(n) := \max\{n : n \text{ integer}, n \leq N/((m-1)!\beta_n)\}, \tag{1.9}$$

$$d(n) := (m-1)!\gamma^{(2m-1)/(m-1)}\{(f(a+))^m + (f(b-))^m\}/2\beta_n^{1/(m-1)}. \tag{1.10}$$

We write "$\to_d$" to denote convergence in distribution.

Our main results regarding the asymptotic distribution of T_n defined by (1.1) are stated in the following theorem and the necessary proofs are postponed until Section 2.

THEOREM 1.1. *Suppose $n^{m-1}/\beta_n^4 \to 0$ and $n^{m-1}/\beta_n \to \infty$ as $n \to \infty$. If f is a non-uniform density satisfying* (1.2)–(1.6), *then*

$$T_n \to_d N(0, 1) \tag{1.11}$$

as $n \to \infty$, where

$$a(n) := (n^{1/2}/m^2(m-2)!\gamma^2) \cdot \left\{ \int_a^b (f(x))^{2m-1} dx - \left(\int_a^b (f(x))^m dx \right)^2 \right\}^{-1/2}. \tag{1.12}$$

If $n^{m-1}/\beta_n^3 \to 0$ *and* $n^{m-1}/\beta_n^2 \to \infty$ *as* $n \to \infty$ *and* f *is the uniform density on* (a, b), *then* (1.11) *continues to hold with*

$$a(n) := \{3n\beta_n^{1/(m-1)}/2\gamma^{(4m-3)/(m-1)}\}^{1/2}(b-a)^{m-1/2}/m!. \tag{1.13}$$

2. Proof of theorem

In order to prove Theorem 1.1 we first have to prove some lemmas. Define, for each $m = 2, 3, \ldots$, a function $r(x_1, \ldots, x_m)$ by

$$r(x_1, \ldots, x_m) := \max(x_1, \ldots, x_m) - \min(x_1, \ldots, x_m). \tag{2.1}$$

Suppose $\{\epsilon_n\}$ is a sequence of real numbers such that $0 < \epsilon_n \to 0$ as $n \to \infty$, and define

$$f_n(x_1, \ldots, x_m) := I(r(x_1, \ldots, x_m) \le \epsilon_n) \tag{2.2}$$

where $I(A) = 1$ or 0 according as the event A holds or not. For each $x \in (a, b)$, let

$$l_n(x) := Ef_n(x, X_2, \ldots, X_m), \tag{2.3}$$

$$g_n(x) := l_n(x) - \mu_n, \tag{2.4}$$

$$\mu_n := Ef_n(X_1, \ldots, X_m). \tag{2.5}$$

We now derive the following two lemmas

LEMMA 2.1. *Under* (1.2)–(1.6), *we have as* $n \to \infty$

$$Eg_n^2(X_1) = m^2\epsilon_n^{2m-2}\left\{\int_a^b (f(x))^{2m-1}dx - \left(\int_a^b (f(x))^m dx\right)^2\right\} + O(\epsilon_n^{2m-1}), \tag{2.6}$$

$$Eg_n^4(X_1) = O(\epsilon_n^{4m-4}), \tag{2.7}$$

and if f *is the uniform density on* (a, b) *then*

$$Eg_n^2(X_1) = \{2(m-1)^2\epsilon_n^{2m-1}/3(b-a)^{2m-1}\} - (m-1)^2\epsilon_n^{2m}/(b-a)^{2m}, \tag{2.8}$$

$$Eg_n^4(X_1) = O(\epsilon_n^{4m-3}). \tag{2.9}$$

PROOF. Application of result (2.2.1) of David (1981) together with some elementary probability calculations yield

$$l_n(x) = (m-1)\int_a^x [F(y+\epsilon_n) - F(y)]^{m-2}f(y)dy + [F(x+\epsilon_n) - F(x)]^{m-1}, \quad a \le x \le a + \epsilon_n, \tag{2.10}$$

$$l_n(x) = (m-1)\int_{x-\epsilon_n}^x [F(y+\epsilon_n) - F(y)]^{m-2}f(y)dy + [F(x+\epsilon_n) - F(x)]^{m-1}, \quad a + \epsilon_n \le x \le b - \epsilon_n, \tag{2.11}$$

$$l_n(x) = (m-1)\int_{x-\epsilon_n}^{b-\epsilon_n} [F(y+\epsilon_n) - F(y)]^{m-2}f(y)dy + [1 - F(b-\epsilon_n)]^{m-1}, \quad b - \epsilon_n \le x \le b. \tag{2.12}$$

In the proofs of (2.13), (2.15) and (2.23) below, $0 < \theta_1 < 1$ and $0 < \theta_2 < 1$ will denote two generic constants. Then (2.10) can be rewritten as follows by applying Taylor series expansions together with (1.2)–(1.6),

$$\begin{aligned}
l_n(x) &= (m-1)\epsilon_n^{m-2}\int_a^x [f(y)+f'(y)\epsilon_n/2+f''(y+\theta_1\epsilon_n)\epsilon_n^2/6]^{m-2}f(y)dy \\
&\quad + [f(x)\epsilon_n + f'(x+\theta_2\epsilon_n)\epsilon_n^2/2]^{m-1} \\
&= (m-1)\epsilon_n^{m-2}\int_a^x [(f(y))^{m-2}+(m-2)(f(y))^{m-3}f'(y)\epsilon_n/2]f(y)dy \\
&\quad + \epsilon_n^{m-1}(f(x))^{m-1} + O(\epsilon_n^m) \\
&= (m-1)\epsilon_n^{m-2}\int_a^x (f(y))^{m-1}dy + (m-2)(f(x))^{m-1}\epsilon_n^{m-1}/2 \\
&\quad - (m-2)(f(a+))^{m-1}\epsilon_n^{m-1}/2 + \epsilon_n^{m-1}(f(x))^{m-1} + O(\epsilon_n^m) \\
&= (m-1)\epsilon_n^{m-2}(f(a+))^{m-1}(x-a) + (m-2)(f(x))^{m-1}\epsilon_n^{m-1}/2 \\
&\quad - (m-2)(f(a+))^{m-1}\epsilon_n^{m-1}/2 + \epsilon_n^{m-1}(f(x))^{m-1} + O(\epsilon_n^m).
\end{aligned}$$

Note that the $O(\epsilon_n^m)$ terms do not depend on x, since f, f' and f'' are assumed to be bounded. Hence, we have that

$$\begin{aligned}
(2.13)\quad l_n(x) &= (m-1)\epsilon_n^{m-2}(f(a+))^{m-1}(x-a) + m\epsilon_n^{m-1}(f(x))^{m-1}/2 \\
&\quad - (m-2)\epsilon_n^{m-1}(f(a+))^{m-1}/2 + O(\epsilon_n^m), \qquad a \le x \le a+\epsilon_n.
\end{aligned}$$

Now, consider the case where $a+\epsilon_n \le x \le b-\epsilon_n$. As in the proof of (2.13) above, (2.11) can be rewritten as follows

$$\begin{aligned}
(2.14)\quad l_n(x) &= (m-1)\epsilon_n^{m-2}\int_{x-\epsilon_n}^x (f(y))^{m-1}dy \\
&\quad + (m-2)\epsilon_n^{m-1}\{(f(x))^{m-1} - (f(x-\epsilon_n))^{m-1}\}/2 \\
&\quad + \epsilon_n^{m-1}(f(x))^{m-1} + O(\epsilon_n^m) \\
&= (m-1)\epsilon_n^{m-2}\int_{x-\epsilon_n}^x (f(y))^{m-1}dy + \epsilon_n^{m-1}(f(x))^{m-1} + O(\epsilon_n^m).
\end{aligned}$$

Applying Taylor's theorem once again, we also find that

$$\begin{aligned}
&\int_{x-\epsilon_n}^x (f(y))^{m-1}dy \\
&\quad = (f(x))^{m-1}\epsilon_n - (m-1)\epsilon_n^2(f(x-\theta_1\epsilon_n))^{m-2}f'(x-\theta_1\epsilon_n)/2 \\
&\quad = (f(x))^{m-1}\epsilon_n + O(\epsilon_n^2),
\end{aligned}$$

and substituting this into (2.14), yields

$$(2.15)\qquad l_n(x) = m\epsilon_n^{m-1}(f(x))^{m-1} + O(\epsilon_n^m), \qquad a+\epsilon_n \le x \le b-\epsilon_n.$$

It is clear from the calculations above that the $O(\epsilon_n^m)$ term in (2.15) does not depend on x, since f, f' and f'' are bounded almost everywhere. In exactly the same way as (2.13), one can also rewrite (2.12) as follows

$$\begin{aligned}(2.16)\quad l_n(x) &= (m-1)\epsilon_n^{m-2}(f(b-))^{m-1}(b-x) + m\epsilon_n^{m-1}(f(x))^{m-1}/2 \\ &\quad - (m-2)\epsilon_n^{m-1}(f(b-))^{m-1}/2 + O(\epsilon_n^m), \qquad b-\epsilon_n \le x \le b,\end{aligned}$$

where, once again, $O(\epsilon_n^m)$ does not depend on x. Also, note from the proofs above that, if f is the uniform density on (a,b), (2.13), (2.15) and (2.16) hold with $O(\epsilon_n^m) = 0$.

We now claim that if f is not the uniform density, then as $n \to \infty$

$$\begin{aligned}(2.17)\quad &\int_a^{a+\epsilon_n} (l_n(x))^2 f(x)dx \\ &\qquad = (m^2+m+1)\epsilon_n^{2m-1}(f(a+))^{2m-1}/3 + O(\epsilon_n^{2m}),\end{aligned}$$

$$(2.18)\quad \int_{a+\epsilon_n}^{b-\epsilon_n} (l_n(x))^2 f(x)dx = m^2\epsilon_n^{2m-2}\int_a^b (f(x))^{2m-1}dx + O(\epsilon_n^{2m-1}),$$

$$\begin{aligned}(2.19)\quad &\int_{b-\epsilon_n}^{b} (l_n(x))^2 f(x)dx \\ &\qquad = (m^2+m+1)\epsilon_n^{2m-1}(f(b-))^{2m-1}/3 + O(\epsilon_n^{2m}),\end{aligned}$$

and if f is the uniform density on (a,b), we have that

$$(2.20)\quad \int_a^{a+\epsilon_n} (l_n(x))^2 f(x)dx = (m^2+m+1)\epsilon_n^{2m-1}(f(a+))^{2m-1}/3,$$

$$\begin{aligned}(2.21)\quad &\int_{a+\epsilon_n}^{b-\epsilon_n} (l_n(x))^2 f(x)dx \\ &\qquad = m^2\epsilon_n^{2m-2}\int_a^b (f(x))^{2m-1}dx \\ &\qquad\quad - m^2\epsilon_n^{2m-1}\{(f(a+))^{2m-1} + (f(b-))^{2m-1}\},\end{aligned}$$

$$(2.22)\quad \int_{b-\epsilon_n}^{b} (l_n(x))^2 f(x)dx = (m^2+m+1)\epsilon_n^{2m-1}(f(b-))^{2m-1}/3.$$

We will prove only (2.17), since the proofs of (2.19), (2.20) and (2.22) are similar, while (2.18) and (2.21) follow directly from (2.15).

Since f and f' are bounded, Taylor's theorem and integration by parts yield

$$\begin{aligned}\int_a^{a+\epsilon_n} (x-a)^2 f(x)dx &= f(a+\epsilon_n)\epsilon_n^3/3 - (1/3)\int_a^{a+\epsilon_n}(x-a)^3 f'(x)dx \\ &= f(a+)\epsilon_n^3/3 + O(\epsilon_n^4),\end{aligned}$$

and for any integer $k \ge 1$

$$\begin{aligned}&\int_a^{a+\epsilon_n} (x-a)(f(x))^k dx \\ &\quad = (f(a+\epsilon_n))^k\epsilon_n^2/2 - (k/2)\int_a^{a+\epsilon_n}(x-a)^2(f(x))^{k-1}f'(x)dx \\ &\quad = (f(a+))^k\epsilon_n^2/2 + O(\epsilon_n^3).\end{aligned}$$

Also, it is easy to see that

$$\int_a^{a+\epsilon_n} (f(x))^k dx = (f(a+))^k \epsilon_n + O(\epsilon_n^2).$$

Hence, from (2.13) and the above expressions, we obtain after some algebra

$$\begin{aligned}
&\int_a^{a+\epsilon_n} (l_n(x))^2 f(x) dx \\
&\quad = (f(a+)\epsilon_n)^{2m-1}\{(m-1)^2/3 + m^2/4 + (m-2)^2/4 + m(m-1)/2 \\
&\qquad\qquad - (m-1)(m-2)/2 - (m-2)m/2\} + O(\epsilon_n^{2m}) \\
&\quad = (m^2+m+1)\epsilon_n^{2m-1}(f(a+))^{2m-1}/3 + O(\epsilon_n^{2m}),
\end{aligned}$$

which completes the proof of (2.17).

Finally, we claim that as $n \to \infty$

$$\text{(2.23)} \quad \mu_n = \epsilon_n^{m-1}/\gamma(m-1)! - \epsilon_n^m (m-1)\{(f(a+))^m + (f(b-))^m\}/2 + O(\epsilon_n^{m+1})$$

which can be proved as follows. Applying result (2.3.3) of David (1981), some Taylor expansions, (1.7) and the boundedness of f' and f'', we find that

$$\begin{aligned}
\mu_n &= m \int_a^{b-\epsilon_n} [F(y+\epsilon_n) - F(y)]^{m-1} f(y) dy + [1 - F(b-\epsilon_n)]^m \\
&= m\epsilon_n^{m-1} \int_a^{b-\epsilon_n} [f(y) + f'(y)\epsilon_n/2 + f''(y+\theta_1\epsilon_n)\epsilon_n^2/6]^{m-1} f(y) dy \\
&\quad + [f(b-)\epsilon_n - f'(b-\theta_2\epsilon_n)\epsilon_n^2/2]^m \\
&= m\epsilon_n^{m-1} \int_a^{b-\epsilon_n} [(f(y))^{m-1} + (m-1)(f(y))^{m-2} f'(y)\epsilon_n/2] f(y) dy \\
&\quad + \epsilon_n^m (f(b-))^m + O(\epsilon_n^{m+1}) \\
&= m\epsilon_n^{m-1} \left\{ \int_a^b (f(y))^m dy - (f(b-))^m \epsilon_n \right\} \\
&\quad + \epsilon_n^m (m-1)\{(f(b-))^m - (f(a+))^m\}/2 + \epsilon_n^m (f(b-))^m + O(\epsilon_n^{m+1}) \\
&= \epsilon_n^{m-1}/\gamma(m-1)! - \epsilon_n^m (m-1)\{(f(a+))^m + (f(b-))^m\}/2 + O(\epsilon_n^{m+1}).
\end{aligned}$$

Note that the $O(\epsilon_n^{m+1})$-term in (2.23) is zero if f is uniform on (a,b). The proofs of (2.6) and (2.8) now follow from (2.4) and (2.17)–(2.23). The proofs of (2.7) and (2.9) follow in exactly the same way and will be omitted. This completes the proof of the lemma. □

Next, let us consider two sets $\{i(1), \ldots, i(m)\}$ and $\{j(1), \ldots, j(m)\}$ of m distinct integers from $\{1, \ldots, n\}$ and let c be the number of integers common to the two sets. Let

$$\zeta_c := E\{\tilde{f}_n(X_{i(1)}, \ldots, X_{i(m)}) \tilde{f}_n(X_{j(1)}, \ldots, X_{j(m)})\}$$

where $\tilde{f}_n(x_1, \ldots, x_m) := f_n(x_1, \ldots, x_m) - \mu_n$. Note that $\zeta_1 = Eg_n^2(X_1)$. We now obtain the following lemma

LEMMA 2.2. *Under* (1.2)–(1.6), *we have as* $n \to \infty$

$$\zeta_c = O(\epsilon_n^{2m-c-1})$$

for $2 \le c \le m$.

PROOF. Let

$$Y_1 := \min(X_1, \ldots, X_c), \qquad Y_2 := \max(X_1, \ldots, X_c),$$
$$Z_1 := \min(X_{c+1}, \ldots, X_m), \qquad Z_2 := \max(X_{c+1}, \ldots, X_m).$$

It follows by symmetry of $\tilde{f}_n$ and by independence of $\{X_1, \ldots, X_n)$ that

$$\zeta_c = \int_a^b \int_{y_1}^b \{P(\max(y_2, Z_2) - \min(y_1, Z_1) \le \epsilon_n)\}^2 p(y_1, y_2) dy_2 dy_1 - \mu_n^2 \tag{2.24}$$

where $p(y_1, y_2)$ denotes the joint probability density function of Y_1 and Y_2. Let $A_n := \{\max(y_2, Z_2) - \min(y_1, Z_1) \le \epsilon_n\}$ then

$$\begin{aligned} P(A_n) = {} & P(A_n, y_1 \le Z_1, y_2 \ge Z_2) + P(A_n, y_1 \ge Z_1, y_2 \le Z_2) \\ & + P(A_n, y_1 \le Z_1, y_2 \le Z_2) + P(A_n, y_1 \ge Z_1, y_2 \ge Z_2). \end{aligned} \tag{2.25}$$

Note that by the mean-value theorem we obtain

$$\begin{aligned} & \int_a^b \int_{y_1}^b \{P(A_n, y_1 \le Z_1, y_2 \ge Z_2)\}^2 p(y_1, y_2) dy_2 dy_1 \\ & = \int_a^b \int_{y_1}^{y_1+\epsilon_n} \{P(y_1 \le Z_1, y_2 \ge Z_2)\}^2 p(y_1, y_2) dy_2 dy_1 \\ & = \epsilon_n \int_a^b \{P(y_1 \le Z_1, y_1 + \theta\epsilon_n \ge Z_2)\}^2 p(y_1, y_1 + \theta\epsilon_n) dy_2 dy_1 \end{aligned} \tag{2.26}$$

for some $0 < \theta < 1$. Since f is bounded we also conclude that

$$\begin{aligned} P(y_1 \le Z_1, y_1 + \theta\epsilon_n \ge Z_2) &= \{F(y_1 + \theta\epsilon_n) - F(y_1)\}^{m-c} \\ &= O(\epsilon_n^{m-c}) \end{aligned} \tag{2.27}$$

and using the well-known expression for $p(y_1, y_2)$ (e.g., see David (1981), p. 10) we find that

$$p(y_1, y_1 + \theta\epsilon_n) = f(y_1) O(\epsilon_n^{c-2}). \tag{2.28}$$

Substituting (2.27) and (2.28) into (2.26) it follows that the right-hand side of (2.26) is $O(\epsilon_n^{2m-c-1})$. Similar arguments can be used to show that all the terms (quadratic and cross-product) obtained when we substitute (2.25) into the integrand in (2.24) are $O(\epsilon_n^{2m-c-1})$. This together with (2.23) prove the lemma. □

PROOF OF THEOREM 1.1. For the proof of the theorem we need some further notation. Set

$$\gamma_n := \gamma + d(n) \tag{2.29}$$

and for any finite constant t put

$$\epsilon_n := t/(a(n)b(n)) + \gamma_n/b(n). \tag{2.30}$$

Furthermore, let

$$\sigma_n^2 := n\left\{\binom{n-1}{m-1}\right\}^2 Eg_n^2(X_1) \sim n^{2m-1}Eg_n^2(X_1)/((m-1)!)^2 \tag{2.31}$$

as $n \to \infty$. Finally, define for each $n \geq m$

$$U_n := \sum_c f_n(X_{i(1)}, \ldots, X_{i(m)}) \tag{2.32}$$

where $\sum_c$ denotes summation over the $\binom{n}{m}$ combinations of m distinct elements $\{i(1), \ldots, i(m)\}$ from $\{1, \ldots, n\}$.

Hence, using (1.1), (2.2), (2.30) and (2.32) we arrive at

$$\begin{aligned} H_n(t) &:= P(T_n \leq t) \\ &= P\left(\left(U_n - \binom{n}{m}\mu_n\right)\Big/\sigma_n \geq \left(c(n) - \binom{n}{m}\mu_n\right)\Big/\sigma_n\right). \end{aligned} \tag{2.33}$$

We will now first show that as $n \to \infty$

$$\left(U_n - \binom{n}{m}\mu_n\right)\Big/\sigma_n \to_d N(0,1). \tag{2.34}$$

Define the projection of the U-statistic U_n by

$$\hat{U}_n := \sum_{i=1}^{n} E(U_n \mid X_i) - (n-1)E(U_n).$$

Note that (see, e.g. Serfling (1980), p. 188) $\hat{U}_n$ can also be written as

$$\hat{U}_n - \binom{n}{m}\mu_n = \binom{n-1}{m-1}\sum_{i=1}^{n} g_n(X_i) \tag{2.35}$$

and the variance of $\hat{U}_n$ is given by $\mathrm{Var}(\hat{U}_n) = \sigma_n^2$, as defined in (2.31). Also, the variance of U_n is given by (Serfling (1980), p. 183)

$$\mathrm{Var}(U_n) = \binom{n}{m}\sum_{c=1}^{m}\binom{m}{c}\binom{n-m}{m-c}\zeta_c. \tag{2.36}$$

Hence, from (2.6), (2.8), Lemma 2.2, (2.31), (2.36) and the conditions imposed on the sequence $\{\beta_n\}$ we find that $\operatorname{Var}(U_n)/\operatorname{Var}(\hat{U}_n) \to 1$ as $n \to \infty$ and since $E(U_n - \hat{U}_n)^2 = \operatorname{Var}(U_n) - \operatorname{Var}(\hat{U}_n)$, see Serfling ((1980) p. 300), we also have that

$$E(U_n - \hat{U}_n)^2/\operatorname{Var}(\hat{U}_n) \to 0 \tag{2.37}$$

as $n \to \infty$. The proof of (2.34) will now follow from (2.37) and Chebishev's inequality provided that as $n \to \infty$

$$\left(\hat{U}_n - \binom{n}{m}\mu_n\right)\Big/\sigma_n \to_d N(0,1). \tag{2.38}$$

From (2.35) it is clear that standard central limit theory can be applied to prove (2.38). It suffices to verify the Lindeberg condition (see, e.g. Billingsley (1968), p. 44), i.e.

$$\nu_n^{-4}\sum_{k=1}^{n} E(g_n(X_k))^4 \to 0$$

as $n \to \infty$, where $\nu_n = (nEg_n^2(X_1))^{1/2}$, and this follows directly from Lemma 2.1.

We will now show that as $n \to \infty$

$$\left(\binom{n}{m}\mu_n - c(n)\right)\Big/\sigma_n \to t. \tag{2.39}$$

To prove this, first consider the case when f is a non-uniform density. Let $G_m := -m!(m-1)\{(f(a+))^m + (f(b-))^m\}/2$. Using (1.5), (1.9), (2.23), (2.30) and some finite Taylor series expansions, it follows fairly easily that

$$\begin{aligned}\binom{n}{m}\mu_n - c(n) = \binom{n}{m}[&(1/\gamma)(t/a(n)b(n))(\gamma_n/b(n))^{m-2}/(m-2)!\\ &+ (1/\gamma)(\gamma_n/b(n))^{m-1}/(m-1)!\\ &+ O(\{(a(n))^2(b(n))^{m-1}\}^{-1})\\ &+ (G_m/m!)(\gamma_n/b(n))^m + O(\{a(n)(b(n))^m\}^{-1})\\ &+ O(1/(b(n))^{m+1}) - 1/((m-1)!\beta_n)].\end{aligned} \tag{2.40}$$

Consequently, we derive from (1.8), (1.10), (1.12), (2.6), (2.29), (2.30), (2.31) and (2.40) that as $n \to \infty$

$$\begin{aligned}&\left(\binom{n}{m}\mu_n - c(n)\right)\Big/\sigma_n\\ &\quad\sim t + Ca(n)(b(n))^{m-1}[(1/\gamma)(\gamma_n/b(n))^{m-1}/(m-1)!\\ &\qquad\qquad + (G_m/m!)(\gamma_n/b(n))^m - 1/((m-1)!\beta_n)]\\ &\quad=: t + D_n\end{aligned} \tag{2.41}$$

for some finite constant C. Finally, applying Taylor's theorem once again, together with (1.8), (1.10), (1.12), (2.29) and the assumptions imposed on the sequence

$\{\beta_n\}$, we deduce that as $n \to \infty$

$$\begin{aligned}(2.42)\quad D_n = Ca(n)(b(n))^{m-1}[&(1/\gamma)(d(n)/b(n))(\gamma/b(n))^{m-2}/(m-2)!\\ &+ (1/\gamma)(\gamma/b(n))^{m-1}/(m-1)!\\ &+ (G_m/m!)(\gamma/b(n))^m - 1/((m-1)!\beta_n)\\ &+ O((d(n))^2/(b(n))^{m-1}) + O(d(n)/(b(n))^m)]\\ \to 0.&\end{aligned}$$

The proof of (2.39) now follows from (2.41) and (2.42). The proof of (2.39) for the uniform density f proceeds exactly as above and will be omitted. The proof of the theorem therefore follows from (2.33), (2.34) and (2.39). □

Acknowledgements

The author wishes to thank P. Janssen for his very helpful discussions and the referee for his constructive criticisms which led to several improvements in the paper.

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