Ann. Inst. Statist. Math.
Vol. 45, No. 1, 173–186 (1993)

CONVERGENCE OF THE GRAM-CHARIER EXPANSION AFTER THE NORMALIZING BOX-COX TRANSFORMATION

RYUEI NISHII

Department of Integrated Arts and Sciences, Faculty of Integrated Arts and Sciences, Hiroshima University, Naka-ku, Hiroshima 730, Japan

(Received July 19, 1991; revised April 17, 1992)

Abstract. Consider an exponential family such that the variance function is given by the power of the mean function. This family is denoted by $\mathrm{ED}^{(\alpha)}$ if the variance function is given by $\mu^{(2-\alpha)/(1-\alpha)}$, where μ is the mean function. When $0 \leq \alpha < 1$, it is known that the transformation of $\mathrm{ED}^{(\alpha)}$ to normality is given by the power transformation $x^{(1-2\alpha)/(3-3\alpha)}$, and conversely, the power transformation characterizes $\mathrm{ED}^{(\alpha)}$. Our principal concern will be to show that this power transformation has an another merit, i.e., the density of the transformed variate has an absolutely convergent Gram-Charier expansion.

Key words and phrases: Exponential dispersion model, exponential family, exponential tilting, power variance function, saddlepoint approximation, stable distribution.

1. Introduction

In statistical theory, the study of the exponential family has a long history. The family has nice statistical properties, and many important distributions are members of the family (Barndorff-Nielsen (1978) gave an excellent review on this family). In particular, an exponential family whose variance function is given by a power function of the mean is very interesting. This family is called the exponential family with power variance function (PVF).

All exponential families with PVF were found by Jørgensen (1987). He showed that these families are also exponential dispersion (ED) models. The exponential family with PVF is denoted by $\mathrm{ED}^{(\alpha)}$ when the variance function is given by $\mu^{(2-\alpha)/(1-\alpha)}$, where μ is the mean value. We may note that $\mathrm{ED}^{(\alpha)}$ coincides with Poisson, gamma, inverse Gaussian and normal distributions when $\alpha = -\infty$, 0, 1/2 and 2 respectively. When $\alpha \in (0,1) \cup (1,2)$, the density of $\mathrm{ED}^{(\alpha)}$ is given by exponential tilting of the stable distribution. The density, however, is developed by an infinite series except the case $\alpha = 1/2$.

It is well-known that the transformation of a chi-squared variate (or of a gamma variate) to normality is given by the cube root transformation. In general, suppose that a random variable X has a pdf in $\mathrm{ED}^{(\alpha)}$ with $0 \leq \alpha < 1$. Then Nishii (1990) proved that the transformation of X to normality is given by the power transformation $X^{(1-2\alpha)/(3-3\alpha)}$, and conversely, this transformation characterizes an exponential family. Our aim in this paper is to present one reason why the power transformation of $\mathrm{ED}^{(\alpha)}$ is effective. Suppose that β is a power in an interval around the normalizing power $(1-2\alpha)/(3-3\alpha)$. Then it will be proved that the density of the transformed variate X^{β} has a convergent Gram-Charier expansion.

In Section 2, we present a key theorem due to Cramér (1925), which gives two sufficient conditions such that a density of a random variable has a convergent Gram-Charier expansion. Also $\mathrm{ED}^{(\alpha)}$ is reviewed. In Section 3, it is proved that the cube root transformation of a gamma variate accelerates the convergence to normality. Also a similar result for the logarithmic transformation of an inverse Gaussian variate is obtained. In Section 4, we can generalize the result into the case $\mathrm{ED}^{(\alpha)}$ with $0 < \alpha < 1$. The error bound of the saddlepoint approximation of the extreme stable distribution is estimated. Using the result, we discuss the power transformation of the variate following $\mathrm{ED}^{(\alpha)}$. Section 5 gives a numerical example. Finally, we give the proofs of Theorem 4.1 and Corollary 4.1 in the Appendix.

2. Preliminary results

An exponential dispersion (ED) model is a family of probability density functions (pdf's) of the form $f(x) = a(x;\lambda)\exp[\lambda\{\theta x - \kappa(\theta)\}]$, where $a(x;\lambda)$ and $\kappa(\theta)$ are given functions, and (λ,θ) varies in a set $\Lambda \times \Theta$. Suppose that a random variable X has a pdf $f(x)$. Then $E(X) = \kappa'(\theta)$ and $V(X) = \kappa''(\theta)/\lambda$. An ED model satisfying $\kappa''(\theta) = \{\kappa'(\theta)\}^{(2-\alpha)/(1-\alpha)}$ is called an ED model with power variance function, and denoted by $\mathrm{ED}^{(\alpha)}$. Jørgensen (1987) showed that there exists $\mathrm{ED}^{(\alpha)}$ except the case $\alpha > 2$. Hougaard (1986) studied properties of the family with $0 < \alpha < 1$.

Hereafter, we consider the case $0 \leq \alpha < 1$. Then $\mathrm{ED}^{(\alpha)}$ is a family of densities of random variables taking positive values. The following theorem may be found in, e.g., Hougaard (1986).

THEOREM 2.1. *When $0 < \alpha < 1$, the density in $\mathrm{ED}^{(\alpha)}$ is given by*

$$f_\alpha(x) \equiv \gamma p(\gamma x;\alpha,1)\exp\left[\lambda\theta x + \lambda(1-\alpha)\alpha^{-1}\{-\theta/(1-\alpha)\}^{\alpha}\right] \quad \text{with} \quad \lambda > 0 \quad \text{and} \quad \theta < 0, \tag{2.1}$$

where

$$\gamma = \alpha^{1/\alpha}\{(1-\alpha)\lambda\}^{1-1/\alpha} \tag{2.2}$$

and $p(x;\alpha,1)$ is a density of the extreme stable distribution developed by the absolutely convergent series:

$$p(x;\alpha,1) = \begin{cases} -\dfrac{1}{\pi x}\displaystyle\sum_{k=1}^{\infty}\frac{\Gamma(\alpha k+1)}{k!}(-x^{-\alpha})^k\sin(\pi\alpha k) & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases} \tag{2.3}$$

This theorem asserts that the pdf in $\mathrm{ED}^{(\alpha)}$ is given by exponential tilting of the stable density (see Zolotarev (1986) for stable densities). The following theorem due to Cramér (1925) plays a key role in this article. See also pp. 173–174 of Kendall and Stuart (1977). Let $\phi(x) = (2\pi)^{-1/2}\exp(-x^2/2)$ and let $H_j(x) = \phi(x)^{-1}(-d/dx)^j\phi(x)$ be Hermite polynomials.

THEOREM 2.2. (Cramér) *Suppose that X is a random variable with a density $f(x)$, and let $\hat{f}(x)$ be its Gram-Charier expansion (G-C expansion):*

$$\hat{f}(x) = \sum_{j=0}^{\infty} c_j H_j(x)\phi(x)/j! \quad \textit{with} \quad c_j = \int_{-\infty}^{\infty} H_j(x)f(x)dx.$$

(i) *If $f(x)$ is a function having a continuous derivative such that $\int_{-\infty}^{\infty}\{f'(x)\}^2 \cdot \exp(x^2/2)dx$ exists and if $f'(x)$ tends to zero as $|x|$ tends to infinity, then $\hat{f}(x)$ converges absolutely and uniformly to $f(x)$ for $-\infty \le x \le \infty$.* (ii) *If $f(x)$ is of bounded variation in every finite interval and if $\int_{-\infty}^{\infty} f(x)\exp(x^2/4)dx$ exists, then $\hat{f}(x)$ converges everywhere to $\{f(x+0) + f(x-0)\}/2$. The convergence is uniform in every finite interval of continuity.*

This theorem gives sufficient conditions such that a random variable X is well-approximated by an expansion based on the standard normal distribution. Note that the assertion (i) is stronger than (ii). If one needs to get the G-C expansion of the pdf of X, one would standardize X by the mean and the variance to obtain faster convergence.

If X is a random variable on the positive axis, put $Y = (X - m)/s$ and $Z = X/s$ with positive constants m and s. Then the pdf's of Y and of Z are respectively given by $sf(sy + m)$ and $sf(sz)$. Immediately, it holds that

$$\int_{-m/s}^{\infty} \{s^2 f'(sy+m)\}^2 \exp(y^2/2)dy \le \exp\{m^2/(2s^2)\}\int_0^{\infty}\{s^2 f'(sz)\}^2\exp(z^2/2)dz \tag{2.4}$$

and

$$E\left[\exp(Y^2/4)\right] \le \exp\{m^2/(4s^2)\}E\left[\exp(Z^2/4)\right]. \tag{2.5}$$

This implies that the G-C expansion of the pdf of Y converges if Z fulfills the sufficient conditions.

3. Gamma distributions and inverse Gaussian distributions

Let X be a gamma variate whose pdf is given by

$$f_0(x) = \begin{cases} \Gamma(\lambda)^{-1}\lambda^{\lambda}\mu^{-\lambda}x^{\lambda-1}e^{-\lambda x/\mu} & \text{if } x > 0, \\ 0 & \text{if } x \le 0, \end{cases} \tag{3.1}$$

where λ and μ are positive parameters. Then it holds that $E(X) = \mu$ and $V(X) = \lambda^{-1}\mu^2$. Hence $f_0(x)$ is a member of $\mathrm{ED}^{(0)}$ because the variance function is given by $\mu^2 = \mu^{(2-\alpha)/(1-\alpha)}$ with $\alpha = 0$. Now consider the Box-Cox power transformation to X such as $X \to X^\beta$. The asymptotic mean and variance of X^β are respectively given by μ^β and $\lambda^{-1}\beta^2\mu^{2\beta}$ as λ tends to infinity. Thus we standardize X^β as

$$Y_\beta = \lambda^{1/2}\{(X/\mu)^\beta - 1\}/\beta. \tag{3.2}$$

THEOREM 3.1. *Let X follow a gamma distribution having the density $f_0(x)$ of* (3.1), *and let Y_β be its Box-Cox transformation defined by* (3.2). (i) *If $0 < \beta < 1/2$ and $\lambda > 2\beta$, then the G-C expansion of the density of Y_β converges absolutely and uniformly to the density of Y_β.* (ii) *If $0 < \beta < 1/2$ and $\beta < \lambda \le 2\beta$, then the G-C expansion of the density of Y_β converges to its density, and the convergence is uniform in every finite interval.*

PROOF. Suppose $0 < \beta < 1/2$. By the formulae (2.4) and (2.5), we shall check the sufficient conditions for the variate $Z_\beta \equiv \lambda^{1/2}X^\beta/(\beta\mu^\beta)$. The pdf of Z_β, denoted by $g_\beta(z)$, is easily derived as $g_\beta(z) = d_0 z^{\lambda/\beta-1}\exp\{-\beta^{1/\beta}\lambda^{1-1/(2\beta)}z^{1/\beta}\}$ with $d_0 = \beta^{\lambda/\beta-1}\lambda^{\lambda-\lambda/(2\beta)}/\Gamma(\lambda)$. Hence it holds that

$$g_\beta{}'(z) = d_0(d_1 - d_2 z^{1/\beta})z^{\lambda/\beta-2}\exp\{-\beta^{1/\beta}\lambda^{1-1/(2\beta)}z^{1/\beta}\}, \tag{3.3}$$

where $d_1 = \lambda/\beta - 1$ and $d_2 = \beta^{1/\beta-1}\lambda^{1-1/(2\beta)}$. (i) The assumption $\lambda/\beta - 2 > 0$ yields that $g_\beta{}'(z)$ is continuous since $\lim_{z\to+0} g_\beta{}'(z) = 0$. Further we have

$$\begin{aligned}\int_0^\infty &\{g_\beta{}'(z)\}^2\exp(z^2/2)dz \\ &= d_0^2\int_0^\infty (d_1 - d_2 z^{1/\beta})^2 z^{2\lambda/\beta-4}\exp\{z^2/2 - 2\beta^{1/\beta}\lambda^{1-1/(2\beta)}z^{1/\beta}\}dz.\end{aligned} \tag{3.4}$$

The integral (3.4) over the interval $[1,\infty)$ converges since $1/\beta > 2$, and the integral over (0,1] converges since $2\lambda/\beta - 4 > 0$. (ii) The assumption $\lambda > \beta$ implies that $d_1 = \lambda/\beta - 1$ is positive. Hence by (3.3), the equation $g_\beta{}'(z) = 0$ has the unique solution. Therefore, $g_\beta(z)$ is unimodal, which yields that $g_\beta(z)$ is of bounded variation. Further the integral

$$E\left[\exp(Z_\beta^2/4)\right] = d_0\int_0^\infty z^{\lambda/\beta-1}\exp\{z^2/4 - \beta^{1/\beta}\lambda^{1-1/(2\beta)}z^{1/\beta}\}dz$$

converges since $\lambda/\beta - 1 > 0$ and $1/\beta > 2$. The assertion (ii) is weaker than that of (i). This completes the proof.

Remark. The cube root transformation $Y_{1/3}$, which is a normalizing transformation of a gamma variate, has an absolutely and uniformly convergent G-C expansion if $\lambda > 2/3$. Next consider the variance-stabilizing transformation $Y_0 = \lambda^{1/2}\log(X/\mu)$. Then

$$E\left[\exp(Y_0^2/4)\right] = \text{const.}\int_0^\infty x^{\lambda-1}\exp\left[\lambda\{\log(x/\mu)\}^2/4 - \lambda\mu^{-1}x\right]dx$$

is divergent because $\lim_{x\to+0} x^{\lambda-1}\exp\left[\lambda\{\log(x/\mu)\}^2/4-\lambda\mu^{-1}x\right]=\infty$. Hence the variance-stabilizing transformation does not meet the sufficient condition. Actually, Y_0 has no G-C expansion because $E|H_j(Y_0)|=\infty$ if $j\geq 1$.

Next we treat an inverse Gaussian (IG) distribution. Let U be an IG variate whose pdf is given by

$$(3.5)\quad f_{1/2}(u)=\begin{cases}\{\lambda/(2\pi)\}^{1/2}u^{-3/2}e^{-\lambda/(2u)}\exp\{-\lambda u/(2\mu^2)+\lambda/\mu\} & \text{if } u>0,\\ 0 & \text{if } u\leq 0,\end{cases}$$

where λ and μ are positive parameters. Then it holds $E(U)=\mu$ and $V(U)=\lambda^{-1}\mu^3$. Hence $f_{1/2}(u)$ is a member of $\mathrm{ED}^{(1/2)}$ because the variance function is given by μ^3. Here we standardize the Box-Cox transformation as

$$(3.6)\qquad V_\beta=\begin{cases}(\lambda/\mu)^{1/2}\{(U/\mu)^\beta-1\}/\beta & \text{if } \beta\neq 0,\\ (\lambda/\mu)^{1/2}\log(U/\mu) & \text{if } \beta=0.\end{cases}$$

THEOREM 3.2. *Let U be an IG variate with the pdf $f_{1/2}(u)$ of* (3.5), *and let V_β be its Box-Cox transformation defined by* (3.6). *If $|\beta|<1/2$, then the G-C expansion of the density of V_β converges absolutely and uniformly to the density of V_β for any positive numbers λ and μ.*

PROOF. We shall prove that the sufficient condition (ii) of Theorem 2.2 is enjoyed. First, consider the case $\beta=0$, i.e., the logarithmic transformation. The pdf of $V_0=\rho^{-1}\log(U/\mu)$ is given by $\mu\rho e^{\rho v}f_{1/2}(\mu e^{\rho v})$ $(=q_0(v)$, say), where $\rho=(\mu/\lambda)^{1/2}$. Obviously,

$$q_0{}'(v)=(\mu/2)(1-\rho^2e^{\rho v}-e^{2\rho v})f_{1/2}(\mu e^{\rho v})$$

tends to zero as v tends to $\pm\infty$. Moreover, putting $v=\rho^{-1}\log(u/\mu)$, we know that

$$\begin{aligned}&\int_{-\infty}^{\infty}\{q_0{}'(v)\}^2\exp(v^2/2)dv\\ &\quad=(\mu^2/4)\int_{-\infty}^{\infty}(1-\rho^2e^{\rho v}-e^{2\rho v})^2f_{1/2}^2(\mu e^{\rho v})\exp(v^2/2)dv\\ &\quad=d_3\int_0^{\infty}(1-\rho^2u/\mu-u^2/\mu^2)^2u^{-4}\\ &\qquad\cdot\exp\left[\{\rho^{-1}\log(u/\mu)\}^2/2-\lambda/u-\lambda\mu^{-2}u\right]du\end{aligned}$$

is convergent, where $d_3=\lambda\mu^2e^{2\lambda/\mu}/(8\pi\rho)$. Second, in the case $0<\beta<1/2$, we only consider the variate $W_\beta=\rho^{-1}U^\beta/(\beta\mu^\beta)$ instead of V_β of (3.6). The pdf of W_β is given by $q_\beta(w)=\beta^{-1}\omega w^{1/\beta-1}f_{1/2}(\omega w^{1/\beta})$ with $\omega=(\beta\rho)^{1/\beta}\mu$. Accordingly,

$$q_\beta{}'(w)=(2\beta^2)^{-1}\{\lambda-(2\beta+1)\omega w^{1/\beta}-\lambda\mu^{-2}\omega^2w^{2/\beta}\}w^{-2}f_{1/2}(\omega w^{1/\beta})$$

satisfies $\lim_{w\to+0} q_\beta'(w) = \lim_{w\to\infty} q_\beta'(w) = 0$. Moreover, putting $w = (u/\omega)^\beta$, we get

$$\tag{3.7} \begin{aligned} &\int_0^\infty \{q_\beta'(w)\}^2 \exp(w^2/2)dw \\ &\quad = d_4 \int_0^\infty \{\lambda - (2\beta+1)u - \lambda\mu^{-2}u^2\}^2 u^{-3\beta-4} \\ &\qquad \cdot \exp\{u^{2\beta}/(2\omega^{2\beta}) - \lambda\mu^{-2}u - \lambda/u\}du, \end{aligned}$$

where $d_4 = \lambda\omega^{3\beta}e^{2\lambda/\mu}/(8\pi\beta^3)$. The integral (3.7) is convergent since $0 < 2\beta < 1$. In the case $-1 < 2\beta < 0$, we get a similar result. This completes the proof.

Remark. In this case, we note that the sufficient condition (i) is equivalent to (ii). The pdf of V_0, which is a normalizing transformation of an IG variate, has an absolutely and uniformly convergent G-C expansion. However, the pdf of $V_{-1/2}$ (variance-stabilizing transformation) does not meet the sufficient conditions.

4. Exponential dispersion model with power variance function

We shall consider the extreme stable density (2.3) with $0 < \alpha < 1$. The pdf $p(x;\alpha,1)$ is bell-shaped (see Gawronski (1984)). For large x the pdf is computed by the infinite series. For small x, however, the convergence of the series is slow. Near the origin, the saddlepoint approximation is available, see, e.g., Ibragimov and Linnik ((1971), pp. 62–69). That is

$$\tag{4.1} \hat{p}(x;\alpha,1) = \begin{cases} \{2\pi(1-\alpha)x^{(2-\alpha)/(1-\alpha)}\}^{-1/2} & \\ \quad \cdot \exp\left[-(1-\alpha)\left(\dfrac{\alpha}{x}\right)^{\alpha/(1-\alpha)}\right] & \text{if } x > 0, \\ 0 & \text{if } x \le 0. \end{cases}$$

The following theorem and corollary will be proved in the Appendix.

THEOREM 4.1. *When $0 < \alpha < 1$, let $p(x;\alpha,1)$ be the extreme stable density defined by* (2.3), *and let $\hat{p}(x;\alpha,1)$ be its saddlepoint approximation expressed as* (4.1). *Then the remainder term is estimated by*

$$\tag{4.2} \underline{R}(x;\alpha) \le p(x;\alpha,1) - \hat{p}(x;\alpha,1) \le \overline{R}(x;\alpha),$$

where the closed forms of $\overline{R}(x;\alpha)$ and of $\underline{R}(x;\alpha)$ are respectively given by (A.19) *and* (A.20) *in the Appendix.*

COROLLARY 4.1. *When x is sufficiently large, the upper bound of the remainder term is dominated by*

$$\overline{R}(x;\alpha) \le K_1\hat{p}(x;\alpha,1) + K_2.$$

When x is positive and sufficiently small, it holds that

$$\overline{R}(x;\alpha) \le K_3\hat{p}(x;\alpha,1) + K_4x^{-1}\exp\{-K_5x^{-\alpha/(1-\alpha)}\}.$$

Here

$$(4.3) \qquad K_2 = (\pi\alpha)^{-1}\{\cos(\pi\alpha/2)\}^{-1/\alpha}\Gamma(\alpha^{-1}), \qquad K_5 = \alpha^{\alpha/(1-\alpha)}\cos(\pi\alpha/2),$$

K_1, K_3 and K_4 are positive constants depending only on α.

Suppose a random variable X has a pdf (2.1) in $\mathrm{ED}^{(\alpha)}$ with $0 < \alpha < 1$, $\lambda > 0$ and $\theta < 0$. Then its mean and variance are respectively given by $E(X) = \{(1-\alpha)/(-\theta)\}^{1-\alpha}$ ($\equiv \mu$, say) and $V(X) = \lambda^{-1}\mu^{(2-\alpha)/(1-\alpha)}$. Using the mean parameter μ we rewrite the pdf as

$$(4.4) \qquad f_\alpha(x) = \gamma p(\gamma x; \alpha, 1)\exp\left[-(1-\alpha)\lambda\{\mu^{-1/(1-\alpha)}x - \alpha^{-1}\mu^{-\alpha/(1-\alpha)}\}\right],$$

where $\gamma = \alpha^{1/\alpha}\{(1-\alpha)\lambda\}^{-(1-\alpha)/\alpha}$. The saddlepoint approximation of $f_\alpha(x)$, say $\hat{f}_\alpha(x)$, is obtained by Hougaard (1986). It is expressed as

$$(4.5) \qquad \hat{f}_\alpha(x) = \gamma \hat{p}(\gamma x; \alpha, 1)\exp\left[-(1-\alpha)\lambda\{\mu^{-1/(1-\alpha)}x - \alpha^{-1}\mu^{-\alpha/(1-\alpha)}\}\right],$$

where $\hat{p}(x; \alpha, 1)$ is the saddlepoint approximation of $p(x; \alpha, 1)$ defined by (4.1). Jensen (1988) proved the validity of saddlepoint approximations under some conditions. His result is, however, not applicable to IG distributions ($\alpha = 1/2$), which he noted.

Nishii (1990) showed that the transformation of X to normality is given by the power transformation $X^{(1-2\alpha)/(3-3\alpha)}$ ($\alpha \neq 1/2$) or $\log X$ ($\alpha = 1/2$) when the concentration parameter λ is large. Using this theorem we give another reason why this power transformation is reasonable.

Here we define the Box-Cox transformation of X by

$$(4.6) \qquad h_\beta(X) = \begin{cases} \lambda^{1/2}\mu^{-\alpha/(2-2\alpha)}\{(X/\mu)^\beta - 1\}/\beta & \text{if } \beta \neq 0, \\ \lambda^{1/2}\mu^{-\alpha/(2-2\alpha)}\log(X/\mu) & \text{if } \beta = 0. \end{cases}$$

THEOREM 4.2. *Let X be a random variable having the pdf $f_\alpha(x)$ of (4.4) with $0 < \alpha < 1$. If $-\alpha/(2-2\alpha) < \beta < 1/2$, then the G-C expansion of the density of $h_\beta(X)$ converges to its density for any μ and λ, where $h_\beta(X)$ is defined in (4.6). The convergence is uniform in every finite interval.*

Remark. If one can obtain an approximation of ${f_\alpha}'(x)$ and its sharp error bounds, the sufficient condition (i) of Theorem 2.2 may be shown to be valid. The normalizing transformation of $\mathrm{ED}^{(\alpha)}$ is given by $h_{\beta*}(X)$ with $\beta* = (1-2\alpha)/(3-3\alpha)$. Obviously $-\alpha/(2-2\alpha) < \beta* < 1/2$. Hence the G-C expansion of the pdf of $h_{\beta*}(X)$ is convergent. On the other hand, the variance-stabilizing transformation is a power transformation with the power $-\alpha/(2-2\alpha)$. This power does not meet the sufficient conditions.

PROOF. We check the sufficient condition given at (ii) of Theorem 2.2. First, consider the case $\beta = 0$ (logarithmic transform). Put $\sigma = \lambda^{-1/2}\mu^{\alpha/(2-2\alpha)}$, $Y =$

$\sigma^{-1}\log(X/\mu)$ and let $g_0(y)$ be its density. Then $g_0(y) = \mu\sigma e^{\sigma y} f_\alpha(\mu e^{\sigma y})$ with $f_\alpha(x)$ of (4.4). The variation of $g_0(y)$ on the interval $[\log a, \log b]$ $(0 < a < b)$ is evaluated by

$$\int_{\log a}^{\log b} |g_0{}'(y)|dy \le \sigma \int_{\mu a^\sigma}^{\mu b^\sigma} \{x|f_\alpha{}'(x)| + f_\alpha(x)\}dx.$$

Here Hougaard (1986) showed that $f_\alpha(x)$ is unimodal. Hence $f_\alpha{}'(x)$ has one change of sign. Therefore $\int_{\mu a^\sigma}^{\mu b^\sigma} x|f_\alpha{}'(x)|dx$ is finite, i.e., $g_0(y)$ has finite variation over any finite interval. Further, putting $\eta = (1-\alpha)\lambda\mu^{-1/(1-\alpha)}$ we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} \exp(y^2/4) g_0(y)dy \\
&\quad = E\left[\exp\{(\log(X/\mu))^2/(4\sigma^2)\}\right] \\
&\quad = \text{const.} \int_0^\infty p(\gamma x; \alpha, 1) \exp\left[\{\log(x/\mu)\}^2/(4\sigma^2) - \eta x\right] dx \\
&\quad \le \text{const.} \int_0^\infty \{\hat{p}(\gamma x; \alpha, 1) + \overline{R}(\gamma x; \alpha)\} \exp\left[\{\log(x/\mu)\}^2/(4\sigma^2) - \eta x\right] dx,
\end{aligned}$$

where $\hat{p}(y;\alpha,1)$, $\overline{R}(y;\alpha)$ and γ are found in (4.1), (4.2) and (2.2) respectively. Now we get

$$\begin{aligned}
(4.7) \qquad &\int_0^\infty \hat{p}(\gamma x; \alpha, 1) \exp\left[\{\log(x/\mu)\}^2/(4\sigma^2) - \eta x\right] dx \\
&\quad = \text{const.} \int_0^\infty x^{-(2-\alpha)/(2-2\alpha)} \\
&\qquad \cdot \exp\left[\{\log(x/\mu)\}^2/(4\sigma^2) - \eta x - \tau x^{-\alpha/(1-\alpha)}\right] dx,
\end{aligned}$$

where $\tau = (1-\alpha)(\alpha/\gamma)^{\alpha/(1-\alpha)}$. The integral (4.7) is convergent since $\{\log(x/\mu)\}^2/(4\sigma^2) < \eta x/2$ when x is large, and since $\{\log(x/\mu)\}^2/(4\sigma^2) < \tau x^{-\alpha/(1-\alpha)}/2$ when x is sufficiently small. Further $\int_0^\infty \overline{R}(\gamma x;\alpha)\exp\left[\{\log(x/\mu)\}^2/4 - \eta x\right] dx$ is finite because $\overline{R}(x;\alpha)$ is estimated by Corollary 4.1.

Second, when $\beta \in (-\alpha/(2-2\alpha), 0) \cup (0, 1/2)$, put $Z = (X/\mu)^\beta/(|\beta|\sigma)$. Then the pdf of Z is given by $g_\beta(z) = \mu\sigma^{1/\beta}(|\beta|z)^{1/\beta-1} f_\alpha(\mu(|\beta|\sigma z)^{1/\beta})$, and the variation of $g_\beta(z)$ on a finite interval is finite because $f_\alpha(x)$ is unimodal. Further,

$$\begin{aligned}
&\int_0^\infty g_\beta(z)\exp(z^2/4)dz \\
&\quad = E\left[\exp\{(\beta\sigma\mu^\beta)^{-2}X^{2\beta}/4\}\right] \\
&\quad = \text{const.} \int_0^\infty p(\gamma x;\alpha,1)\exp\left[(\beta\sigma\mu^\beta)^{-2}x^{2\beta}/4 - \eta x\right] dx \\
&\quad \le \text{const.} \int_0^\infty \{\hat{p}(\gamma x;\alpha,1) + \overline{R}(\gamma x;\alpha)\}\exp\left[(\beta\sigma\mu^\beta)^{-2}x^{2\beta}/4 - \eta x\right] dx.
\end{aligned}$$

The integral based on the approximated pdf is:

$$\begin{aligned}(4.8)\qquad &\int_0^\infty \hat{p}(\gamma x;\alpha,1)\exp\left[(\beta\sigma\mu^\beta)^{-2}x^{2\beta}/4-\eta x\right]dx\\ &=\text{const.}\int_0^\infty x^{-(2-\alpha)/(2-2\alpha)}\\ &\cdot\exp\left[(\beta\sigma\mu^\beta)^{-2}x^{2\beta}/4-\eta x-\tau x^{-\alpha/(1-\alpha)}\right]dx,\end{aligned}$$

where η and τ are already defined in this proof. Then, the integral (4.8) over $[1,\infty)$ is finite since $2\beta<1$. On the other hand, the integral over (0,1] is finite since $2\beta>-\alpha/(1-\alpha)$. The existence of $\int_0^\infty \overline{R}(\gamma x;\alpha)\exp\left[(\beta\sigma\mu^\beta)^{-2}x^{2\beta}/4-\eta x\right]dx$ is similarly proved by using Corollary 4.1.

The referee raised the problem to which extent the theorem can be extended to cover the stable case $\theta=0$. Suppose X has a density of the form (2.3). Then, we can not account the mean and the variance of X since they are infinite. When x is small, $p(x;\alpha,1)$ is approximated by (4.1), and when x is large, it is $O(x^{-1-\alpha})$. Using these orders in the tail areas, we get the result that the G-C expansion of X^β is uniformly convergent to the density for every finite interval if $-\alpha/(2-2\alpha)<\beta<0$. However, $\log X$ does not meet the sufficient condition.

5. Numerical example

The accuracy of the saddlepoint approximation of $f_\alpha(x)$ is examined by a numerical example. We use the weight data of 98 newly-enrolled male students, which is analysed by Nishii (1990). The sample mean and variance are given

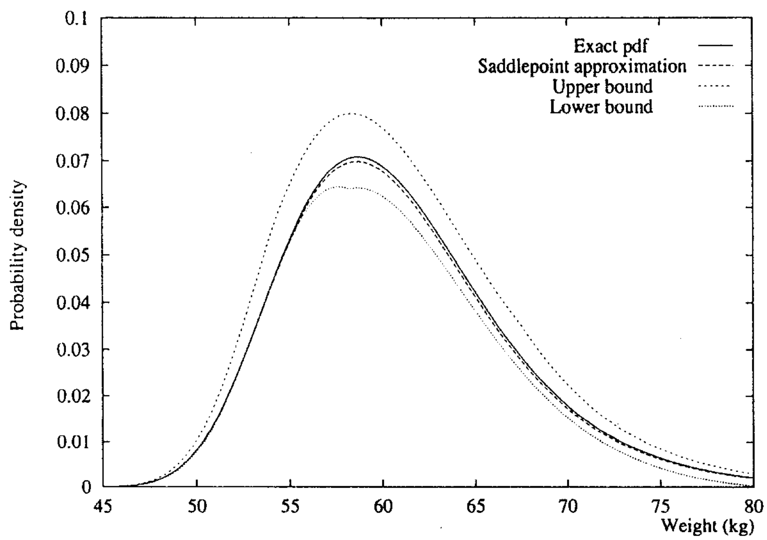

Fig. 1. Saddlepoint approximation of exponential dispersion model.

by 61.0(kg) and 38.8(kg^2), respectively. By means of the moment method, λ and θ are estimated by the solutions of $\{-\theta/(1-\alpha)\}^{\alpha-1} = 61.0$ and $\{-\theta/(1-\alpha)\}^{(\alpha-2)/(\alpha-1)} = 38.8$ for given α. We choose α which maximizes the likelihood. Thus we get $\hat{\alpha} = .878$, $\hat{\theta} = -.283 \times 10^{-15}$ and $\hat{\lambda} = .679 \times 10^{15}$.

The values of λ and θ may look extreme. But this comes from the way of the parametrization. Especially, when α is close to 1, $\hat{\theta}$ becomes small, and consequently $\hat{\lambda}$ becomes large. The density $f_{\hat{\alpha}}(x)$ of the form (4.4) is denoted by Exact pdf, and its saddlepoint approximation of the form (4.5) is denoted by Saddlepoint approximation in Fig. 1. The upper and lower bounds of the approximation are essentially based on (A.19) and (A.20). Both bounds are evaluated more accurately. These four curves are illustrated in Fig. 1.

Acknowledgement

I would like to express my hearty thanks to the referee whose helpful comments led to many improvements. His/her comments call my attention to the stable case in Theorem 4.2.

Appendix

PROOF OF THEOREM 4.1. Following Ibragimov and Linnik ((1971), pp. 62–69), we approximate $p(x;\alpha,1)$ with $0<\alpha<1$ based on the method of steepest descent. Recall that $p(x;\alpha,1)$ is derived by inverting the characteristic function. Dividing the integral region $(0,\infty)$ into three regions Π_0, Π_1 and Π_2 in the complex plane, we have

$$\begin{aligned}
\text{(A.1)} \qquad p(x;\alpha,1) &= (\pi x)^{-1}\mathrm{Re}\int_0^\infty \exp\{-is-(s/x)^\alpha e^{-i\pi\alpha/2}\}ds \\
&= (\pi x)^{-1}\mathrm{Re}\int_{\Pi_0} + (\pi x)^{-1}\mathrm{Re}\int_{\Pi_1} + (\pi x)^{-1}\mathrm{Re}\int_{\Pi_2} \\
&= I_0 + I_1 + I_2, \qquad \text{say,}
\end{aligned}$$

where $i=\sqrt{-1}$, $\Pi_0=(0,ir)$, Π_1 is the circular arc from ir to r, $\Pi_2=(r,\infty)$ and r is a saddlepoint

$$\text{(A.2)} \qquad r = \alpha^{1/(1-\alpha)}x^{-\alpha/(1-\alpha)}.$$

Obviously $I_0 = (\pi x)^{-1}\mathrm{Re}\left[i\int_0^r \exp(t-x^{-\alpha}t^\alpha)dt\right] = 0$. Next, I_2 of (A.1) is estimated as follows.

$$\begin{aligned}
\text{(A.3)} \quad |I_2| &= (\pi x)^{-1}\left|\int_r^\infty \exp\{-\cos(\pi\alpha/2)x^{-\alpha}s^\alpha\}\cos\{s-\sin(\pi\alpha/2)x^{-\alpha}s^\alpha\}ds\right| \\
&\le (\pi x)^{-1}\int_r^\infty \exp\{-\cos(\pi\alpha/2)x^{-\alpha}s^\alpha\}ds \\
&= \xi\int_{K_5x^{-\alpha/(1-\alpha)}}^\infty t^{1/\alpha-1}e^{-t}dt = \xi\left[\Gamma(1/\alpha)-\Gamma(1/\alpha, K_5x^{-\alpha/(1-\alpha)})\right],
\end{aligned}$$

where

$$\text{(A.4)} \qquad \xi = (\pi\alpha)^{-1}\{\cos(\pi\alpha/2)\}^{-1/\alpha} \quad \text{and} \quad K_5 = \alpha^{\alpha/(1-\alpha)}\cos(\pi\alpha/2).$$

Here in general $\Gamma(a, z)$ denotes an incomplete gamma function $\int_0^z s^{a-1}e^{-s}ds$.

On the other hand, the integral I_1 is given by the integral over a compact interval $[0, \pi/2]$.

$$\begin{aligned}
I_1 &= r(\pi x)^{-1}\mathrm{Re}\int_0^{\pi/2} \exp(re^{-it} - r\alpha^{-1}e^{-i\alpha t})e^{-it}dt \\
&= r(\pi x)^{-1}\int_0^{\pi/2} \exp(r\cos t - r\alpha^{-1}\cos\alpha t)\cos A(t)dt \\
\text{(A.5)} &\qquad (\text{where } A(t) \equiv t + r\sin t - r\alpha^{-1}\sin\alpha t) \\
&= r(\pi x)^{-1}\exp\{-(1-\alpha)r/\alpha\} \\
&\quad \cdot \int_0^{\pi/2} \exp\{-(1-\alpha)rt^2/2 + B(t)\}\cos A(t)dt \\
\text{(A.6)} &\qquad (\text{where } B(t) \equiv r\{\cos t - 1 + t^2/2 - \alpha^{-1}\cos\alpha t + \alpha^{-1} - \alpha t^2/2\}) \\
&= r^{1/2}(1-\alpha)^{-1/2}(\pi x)^{-1}\exp\{-(1-\alpha)r/\alpha\} \\
&\quad \cdot \int_0^c \exp\left[-v^2/2 + B(v/\sqrt{(1-\alpha)r})\right]\cos A(v/\sqrt{(1-\alpha)r})dv \\
\text{(A.7)} &\qquad (\text{where } c \equiv (\pi/2)\sqrt{(1-\alpha)r/2}, v \equiv \sqrt{(1-\alpha)r}t) \\
&= \hat{p}(x;\alpha,1)\sqrt{2/\pi}\left[\int_0^\infty e^{-v^2/2}dv - \int_c^\infty e^{-v^2/2}dv\right. \\
&\qquad \left. + \int_0^c e^{-v^2/2}\{e^{\overline{B}}\cos\overline{A} - 1\}dv\right], \\
\text{(A.8)} &\qquad (\text{where } \overline{A} \equiv A(v/\sqrt{(1-\alpha)r}), \overline{B} \equiv B(v/\sqrt{(1-\alpha)r}))
\end{aligned}$$

where $\hat{p}(x;\alpha,1)$ is defined by (4.1). By the relation $\cos\overline{A} = 1 - 2\sin^2(\overline{A}/2)$, it holds that

$$\begin{aligned}
I_1 &= \hat{p}(x;\alpha,1)\left[1 - 2\{1-\Phi(c)\} + \sqrt{2/\pi}\int_0^c e^{-v^2/2}(e^{\overline{B}} - 1)dv\right. \\
&\qquad \left. -2\sqrt{2/\pi}\int_0^c e^{-v^2/2}e^{\overline{B}}\sin^2(\overline{A}/2)dv\right] \\
\text{(A.9)} \quad &= \hat{p}(x;\alpha,1)\left[1 - 2\{1-\Phi(c)\} + \sqrt{2/\pi}J_{1,1} - 2\sqrt{2/\pi}J_{1,2}\right], \quad \text{say,}
\end{aligned}$$

where $\Phi(x)$ denotes the standard normal distribution function. Therefore,

$$\text{(A.10)} \quad p(x;\alpha,1) = \hat{p}(x;\alpha,1)\left[1 - 2\{1-\Phi(c)\} + \sqrt{2/\pi}J_{1,1} - 2\sqrt{2/\pi}J_{1,2}\right] + I_2.$$

The following identity on $\overline{B} \equiv B\left(v/\sqrt{(1-\alpha)r}\right)$ of (A.8) is shown by Ibragimov and Linnik ((1971), p. 64):

$$\text{(A.11)} \quad \overline{B} = B\left(v/\sqrt{(1-\alpha)r}\right) = v^2/2\left[1 - (\sin^2\varepsilon - \alpha^{-1}\sin^2\alpha\varepsilon)/\{(1-\alpha)\varepsilon^2\}\right],$$

where $0 \le \varepsilon = v/(2\sqrt{(1-\alpha)r}) \le \pi/4$. From a finite Taylor series with remainder we get $\sin^2\varepsilon - \alpha^{-1}\sin^2\alpha\varepsilon = (1-\alpha)\varepsilon^2 - (1-\alpha^3)\varepsilon^4/3 + 2(\cos\zeta - \alpha^5\cos\alpha\zeta)\varepsilon^6/45$ for $0 < \zeta < 1$. Thus $\overline{B} = v^2/2[(1+\alpha+\alpha^2)\varepsilon^2/3 - 2(\cos\zeta - \alpha^5\cos\alpha\zeta)\varepsilon^4/\{45(1-\alpha)\}]$. Taking the supremum of $\overline{B}$ w.r.t. ζ, we get

$$(A.12) \qquad 0 \le \overline{B} \le b_1 v^4/r + b_2 v^6/r^2 = \overline{B}_{\max}, \quad \text{say,}$$

where

$$(A.13) \quad b_1 = (1+\alpha+\alpha^2)/\{24(1-\alpha)\} \quad \text{and} \quad b_2 = \alpha^5\cos(\pi\alpha/2)/\{45\cdot 2^4(1-\alpha)^3\}.$$

Also (A.11) implies that $\overline{B}$ is dominated by $\delta v^2/2$ with

$$\delta = (1+\alpha+\alpha^2)\pi^2/48 + \alpha^5\pi^4\cos(\pi\alpha/2)/\{5760(1-\alpha)\} < 1$$

since $0 \le \varepsilon \le \pi/4$. Thus using $0 < \psi < 1$, it holds that

$$\begin{aligned}
(A.14) \quad 0 \le J_{1,1} &= \int_0^c e^{-v^2/2}(e^{\overline{B}} - 1)dv = \int_0^c e^{-v^2/2}(\overline{B} + \overline{B}^2 e^{\psi\overline{B}}/2)dv \\
&\le \int_0^c \left[\overline{B}_{\max} e^{-v^2/2} + \overline{B}_{\max}^2 \exp\{-v^2/2 + \overline{B}_{\max}\}/2\right] dv \\
&\le \int_0^c \overline{B}_{\max} e^{-v^2/2} dv + (1/2)\int_0^c \overline{B}_{\max}^2 \exp\{-(1-\delta)v^2/2\} dv.
\end{aligned}$$

Here $\overline{B}_{\max}$ of (A.12) is a polynomial of v. Consequently, the integrals in (A.14) can be expressed in terms of incomplete gamma functions as

$$\begin{aligned}
(A.15) \qquad 0 \le J_{1,1} \le\ & 2^{3/2}\Gamma(5/2, c^2/2) b_1 r^{-1} + 2^{5/2}\Gamma(7/2, c^2/2) b_2 r^{-2} \\
&+ 2^{5/2}\overline{\delta}^{-9/2}\Gamma(9/2, \overline{\delta}c^2/2) b_1^2 r^{-2} \\
&+ 2^{9/2}\overline{\delta}^{-11/2}\Gamma(11/2, \overline{\delta}c^2/2) b_1 b_2 r^{-3} \\
&+ 2^{9/2}\overline{\delta}^{-13/2}\Gamma(13/2, \overline{\delta}c^2/2) b_2^2 r^{-4} \qquad (\equiv \overline{J}_{1,1}, \ \text{say}),
\end{aligned}$$

where b_j are defined by (A.13), r is the saddlepoint as before, and $\overline{\delta} = 1 - \delta$. It is not difficult to show that $\overline{\delta}$ is positive for $0 < \alpha < 1$. On the other hand employing a Taylor expansion of $\overline{A}$ of (A.8), we get

$$(A.16) \qquad 0 \le \overline{A} \le a_1 v/\sqrt{r} - a_2 v^3/\sqrt{r} + a_3 v^4/r = \overline{A}_{\max}, \quad \text{say,}$$

where

$$(A.17) \qquad \begin{aligned} &a_1 = 1/\sqrt{1-\alpha}, \quad a_2 = (1+\alpha)/\{6\sqrt{1-\alpha}\} \quad \text{and} \\ &a_3 = 1/\{24(1-\alpha)^2\}. \end{aligned}$$

Hence we get $0 \le \sin^2(\overline{A}/2) \le \overline{A}^2/4 \le \overline{A}_{\max}^2/4$ and

$$0 \le 4J_{1,2} \le \int_0^c e^{-v^2/2}\overline{A}_{\max}^2 e^{\overline{B}} dv \le \int_0^c \overline{A}_{\max}^2 \exp\{-(1-\delta)v^2/2\} dv,$$

where c is defined in (A.7). Finally it holds that

$$\begin{aligned}(\text{A.18})\quad 0 \le 4J_{1,2} \le{}& \{2^{-1}(2/\overline{\delta})^{3/2}\Gamma(3/2,\overline{\delta}c^2/2)a_1^2 + 2^{-1}(2/\overline{\delta})^{7/2}\Gamma(7/2,\overline{\delta}c^2/2)a_2^2 \\ &- (2/\overline{\delta})^{5/2}\Gamma(5/2,\overline{\delta}c^2/2)a_1a_2\}r^{-1} + \{(2/\overline{\delta})^3\Gamma(3,\overline{\delta}c^2/2)a_1a_3 \\ &+ 2^{-1}(2/\overline{\delta})^{9/2}\Gamma(9/2,\overline{\delta}c^2/2)a_3^2r^{-2} \\ &- (2/\overline{\delta})^4\Gamma(4,\overline{\delta}c^2/2)a_2a_3\}r^{-3/2} \\ \equiv{}& 4\overline{J}_{1,2}, \qquad \text{say.}\end{aligned}$$

From (A.3) and (A.10), the upper and lower bounds of the residual $p(x;\alpha,1) - \hat{p}(x;\alpha,1)$ are obtained by

$$(\text{A.19})\qquad \begin{aligned}\overline{R}(x;\alpha) ={}& \hat{p}(x;\alpha,1)\left[-2\{1-\Phi(c)\} + \sqrt{2/\pi}\overline{J}_{1,1}\right] \\ &+ \xi\left[\Gamma(\alpha^{-1}) - \Gamma(\alpha^{-1}, K_5x^{-\alpha/(1-\alpha)})\right]\end{aligned}$$

and

$$(\text{A.20})\qquad \begin{aligned}\underline{R}(x;\alpha) ={}& \hat{p}(x;\alpha,1)[-2\{1-\Phi(c)\} - 2\sqrt{2/\pi}\overline{J}_{1,2}] \\ &- \xi\left[\Gamma(\alpha^{-1}) - \Gamma(\alpha^{-1}, K_5x^{-\alpha/(1-\alpha)})\right],\end{aligned}$$

where $c = 2^{-3/2}\pi(1-\alpha)^{1/2}\alpha^{1/(2-2\alpha)}x^{-\alpha/(2-2\alpha)}$, ξ, $\overline{J}_{1,1}$ and $\overline{J}_{1,2}$ are respectively defined by (A.4), (A.15) and (A.18).

Proof of Corollary 4.1. By the formula (A.19), we have

$$(\text{A.21})\qquad \overline{R}(x;\alpha) \le \sqrt{2/\pi}\overline{J}_{1,1}\hat{p}(x;\alpha,1) + \xi\{\Gamma(\alpha^{-1}) - \Gamma(\alpha^{-1}, K_5x^{-\alpha/(1-\alpha)})\},$$

where K_5 and $\overline{J}_{1,1}$ are defined in (4.3) and (A.15), respectively. Employing the following inequality on the incomplete gamma function:

$$\Gamma(a,x) \equiv \int_0^x s^{a-1}e^{-s}ds < \min\{\Gamma(a), a^{-1}x^a\} \qquad (a>0,\ x>0),$$

we evaluate the first term (omitting the multiplier) of $\overline{J}_{1,1}$ as

$$\begin{aligned}\Gamma(5/2, c^2/2)r^{-1} \le \min\Big\{&\Gamma(5/2)\alpha^{-1/(1-\alpha)}x^{\alpha/(1-\alpha)}, \\ &2^{-9}5^{-1}\pi^5(1-\alpha)^{5/2}\alpha^{3/(2-2\alpha)}x^{-3\alpha/(2-2\alpha)}\Big\}\end{aligned}$$

since $c = 2^{-3/2}\pi(1-\alpha)^{1/2}\alpha^{1/(2-2\alpha)}x^{-\alpha/(2-2\alpha)}$ and $r = \alpha^{1/(1-\alpha)}x^{-\alpha/(1-\alpha)}$. Other terms of $\overline{J}_{1,1}$ are similarly evaluated. Concerning the second term of the right hand side of (A.21), the following inequality on the complementary incomplete gamma function is available.

$$\begin{aligned}\int_x^\infty s^{a-1}e^{-s}ds \le \min[\Gamma(a), e^{-x}x^{a-1}\{&1 + (a-1)x^{-1} + (a-1)(a-2)x^{-2} \\ &+ \cdots + (a-1)(a-2)\cdots(a-[a])x^{-[a]}\}],\end{aligned}$$

where $[a]$ denotes the maximum integer not exceeding a. Using this relation, we have $\Gamma(\alpha^{-1}) - \Gamma(\alpha^{-1}, K_5 x^{-\alpha/(1-\alpha)}) \leq K x^{-1} \exp\{-K_5 x^{-\alpha/(1-\alpha)}\}$ for a positive but small x, where K is a positive constant depending only on α. This completes the proof.

References

Barndorff-Nielsen, O. E. (1978). *Information and Exponential Families in Statistical Theory*, Wiley, New York.

Cramér, H. (1925). On some classes of series used in mathematical statistics, *Skandinaviske Matematikercongres*, Copenhagen, 399–425.

Gawronski, W. (1984). On the bell-shape of stable densities, *Ann. Probab.*, **12**, 230–242.

Hougaard, P. (1986). Survival models for heterogeneous populations derived from stable distributions, *Biometrika*, **73**, 387–396 (Correction: *ibid.* (1988). **75**, p. 395).

Ibragimov, I. A. and Linnik, Y. V. (1971). *Independent and Stationary Sequences of Random Variables*, Wolters-Noordhoff Publishing Groningen, Netherland.

Jensen, J. L. (1988). Uniform saddlepoint approximations, *Adv. in Appl. Probab.*, **20**, 622–634.

Jørgensen, B. (1987). Exponential dispersion models, *J. Roy. Statist. Soc. Ser. B*, **49**, 127–162.

Kendall, M. and Stuart, A. (1977). *The Advanced Theory of Statistics*, Vol. 1, Fourth ed., Charles Griffin, London.

Nishii, R. (1990). The relation between the normalizing Box-Cox transformations and the exponential dispersion models with power variance functions, Tech. Report #279, Statistical Research Group, Hiroshima University.

Zolotarev, V. M. (1986). One-dimensional Stable Distributions, *Transl. Math. Monographs*, **65**.