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OPTIMAL TESTS FOR NO CONTAMINATION IN SYMMETRIC MULTIVARIATE NORMAL MIXTURES

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Abstract. SenGupta and Pal (1991, *J. Statist. Plann. Inference*, **29**, 145–155) have recently obtained the locally optimal test for zero intraclass correlation coefficient in symmetric multivariate normal mixtures, with known mixing proportion, for the case when the common mean, m, and the common variance, σ^2, are known. Here, we establish that even under the general situation, when some or none of m and σ^2 are known, simple optimal tests can be derived, which are locally most powerful similar, whose exact cut-off points are already available and which retain all the previous optimality properties, e.g. unbiasedness, monotonicity and consistency. Some power tables are presented to demonstrate the favorable performances of these tests.

Key words and phrases: Intraclass correlation coefficient, locally most powerful similar test, mixture distribution.

1. Introduction

Little seems to be known about any general method of construction of optimal tests for no mixture against mixture models. We note here that in a symmetric multivariate normal (SMN) (Rao ((1973), p. 196), Johnson and Wichern ((1982), p. 373)) mixture population, the test for no mixture (contamination) reduces to the test for zero intraclass correlation coefficient, ρ, when the mixing proportion, p, is known. Situations where distributions with known p are used are quite common in practice, e.g., bidirectional ($p = 1/2$) circular normal distribution (Bartels (1984)) in directional data analysis; mixture of standard SMN (SSMN) distributions (Titterington, *et al.* ((1985), p. 68), $p = 1/2$) and mixture of SSMN and SMN distributions (Henze and Zirkler ((1990), p. 3610)) in test for multivariate normality; mixture of SSMN distributions (Kocherlakota and Kocherlakota (1981)) and mixture of SSMN and SMN distributions (Srivastava and Lee (1984)) in robustness studies of estimators and tests etc. Also "... another general area

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where mixtures of distributions are important" is that of reliability studies for "the overall failure distribution" of a multi-component item (Everitt ((1985), p. 560)). Intraclass correlation structure is also quite popular in "... many areas of applications, particularly population genetics ..., reliability studies (products from the same machine ...) ... and survey sampling ..." (Koch ((1983), p. 212)). Here again, in many applied areas ρ, though unknown, is usually positive, e.g., in split-plot experimental designs (Koch *et al.* ((1988), p. 48)), finite population regression models in sample survey (Brewer and Tam ((1990), p. 423)), multivariate linear models (Bai *et al.* ((1990), p. 515)), efficient combination of experiments (Verrill *et al.* (1990)), etc. The SMN mixture model with $\rho > 0$ thus applies also, when a product, in batches, is acquired from two different suppliers (machines) in a known (or approximately known) proportion according to say, possibly time and cost considerations. Further applications of this model may thus be envisaged from the above examples as particular cases, e.g. tests for multivariate normality as in Titterington *et al.* (1985) when $\rho \geq 0$, etc.; or as generalizations, e.g., models for robustness studies—taking possibly dependent ($\rho \geq 0$), rather than independent, homoscedastic variables for the second component in expression (1) of Srivastava and Lee (1984), or SMN components with $\rho_1 = 0$, $\rho_2 \geq 0$ rather than SSMN components in expression (1.1) of Kocherlakota and Kocherlakota (1981), or taking $\rho_1 = 0$, $\delta = 0$, $\rho_2 \geq 0$ in the model of Henze and Zirkler (1990). In a recent paper, SenGupta and Pal (1991) have considered the locally most powerful (LMP) test for $\rho = 0$ in a SSMN mixture population with known p. For further motivations and applications, the reader is referred to that paper. Here we consider the extension from the SSMN to the SMN mixture population. Our introduction of the common mean m and the common variance σ^2, possibly both unknown, for the marginals of each of the two SMN components of the mixture, constitutes the natural and practical generalizations of the situations considered above. We establish the appealing simplicity of the LMP similar tests derived here, ease of construction and of availability of cut-off points, monotonicity of power functions and hence unbiasedness of the tests, consistency of the tests and finally asymptotic normality of the test statistics under both the null and alternative hypotheses.

Let $g(\boldsymbol{x} \mid m, \sigma^2, \rho)$ denote the k-variate SMN density, $N_k(\boldsymbol{M}, \sigma^2\Sigma_\rho)$, with mean vector $\boldsymbol{M} = (m, \ldots, m)'$, $\sigma^2 > 0$, $\Sigma_\rho = ((\rho + (1-\rho)\delta_{ij}))$, δ_{ij} being the Kronecker delta and $-(k-1)^{-1} < \rho < 1$. This density can also be regarded as a density of k exchangeable normal random variables with the same marginal parameters m and σ^2. Let $g_{(p)}(\boldsymbol{x} \mid m, \sigma^2, \rho)$ be the density obtained as a "p-mixture" of $g(\boldsymbol{x} \mid m, \sigma^2, 0)$ and $g(\boldsymbol{x} \mid m, \sigma^2, \rho)$, that is

$$
\begin{aligned}
(1.1) \qquad g_{(p)}(\boldsymbol{x} \mid m, \sigma^2, \rho) = {} & p(2\pi\sigma^2)^{-k/2} \exp\{-(\boldsymbol{x} - \boldsymbol{M})'(\boldsymbol{x} - \boldsymbol{M})/2\sigma^2\} \\
& + q(2\pi\sigma^2)^{-k/2}(\det \Sigma_\rho)^{-1/2} \\
& \cdot \exp\{-(\boldsymbol{x} - \boldsymbol{M})'\Sigma_\rho^{-1}(\boldsymbol{x} - \boldsymbol{M})/2\sigma^2\}
\end{aligned}
$$

where $0 < p < 1$, and $q = 1 - p$. For brevity, write $g_{(p)}(\boldsymbol{x} \mid m, \sigma^2, \rho)$, $g(\boldsymbol{x} \mid m, \sigma^2, \rho)$ and $g(\boldsymbol{x} \mid m, \sigma^2, 0)$ as $g_{(p)}$, g_ρ and g_0 respectively. Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ be a random sample from (1.1). Here we derive LMP similar tests of $H_0 : \rho = 0$ against the one sided alternatives $H_1 : \rho > 0$. Note that for large k, ρ should be nonnegative. Usual reversal of inequalities in the definitions of the critical regions yields analogous tests for the alternatives $\rho < 0$.

2. LMP similar tests

Suppose we want to test $H_0 : \theta = \theta_0$ against $H_1 : \theta > \theta_0$, $\theta \in \Omega \subset R^1$, in the presence of a nuisance parameter $\eta \in \mathcal{N} \subset R^k$, $k \geq 1$. Let D_α be the class of all similar level α tests of H_0 against H_1 and let $\beta_\varphi(\theta, \eta)$ be the power function of a test $\varphi \in D_\alpha$. Spjøtvoll (1968) presented the form of the LMP similar test in some generality. For the existence of the test, it was assumed that $\partial\beta_\varphi(\theta, \eta)/\partial\theta$ exists for all $\theta \in \Omega$ and $\eta \in \mathcal{N}$ for every $\varphi \in D_\alpha$. However, it was observed later (Durairajan and Kale (1982)) that this condition alone does not suffice. The dominance of the test over a local interval with its one appropriate end-point at θ_0 may be destroyed, since such an interval in general will depend on the nuisance parameter η. A second condition that for every $\varphi \in D_\alpha$, the family $\{\partial\beta_\varphi(\theta, \eta)/\partial\theta; \eta \in \mathcal{N}\}$ is equicontinuous at $\theta = \theta_0$ suffices. This is satisfied, e.g., if $\partial^2\beta_\varphi(\theta, \eta)/\partial\theta^2 \leq M < \infty$, $\theta \in \Omega$. For further details on equicontinuity see, e.g., Dugundji ((1975), p. 266). For the construction of the test the regularity conditions assumed were that the underlying distribution admits a probability density function (p.d.f.) with support independent of the parameters, the partial derivative in $\partial\beta_\varphi(\theta, \eta)/\partial\theta_0$ can be passed inside the integral arising for the power function and the family of p.d.f.s possesses a boundedly complete sufficient statistic under H_0.

The structure of the LMP similar test for the mixture population in each of the cases in the previous section is based on the inequality

$$\sum_{s=1}^{n} \left[\frac{\partial}{\partial\rho} \ln g_{(p)}(\boldsymbol{x}_s \mid m, \sigma^2, \rho) \right] |_{\rho=0} > c(t), \tag{2.1}$$

where $c(t)$ generically denotes a constant depending on a fixed value t of the sufficient statistic under H_0. Verification of regularity conditions for our problem is straightforward though tedious. It needs to be pointed out that for each of the three cases considered below, the LMP similar test statistic is invariant with respect to the corresponding nuisance (location-scale) parameter(s) involved under the respective group of affine transformations $\boldsymbol{X} \to \boldsymbol{Y}$, $\boldsymbol{Y} = a\boldsymbol{X} + b\boldsymbol{1}$, $a > 0$, $b \in R^1$. Hence $\beta_\varphi(\theta, \eta) = \beta_\varphi(\theta)$ is free of η. Thus the condition of equicontinuity becomes superfluous—the usual condition (Ferguson ((1967), p. 235)) of mere continuity of $\partial\beta_\varphi(\theta)/\partial\theta$ at $\theta = \theta_0$ suffices.

Since p is known, (2.1) reduces to a similar expression with $g_{(p)}$ being replaced by g_ρ, i.e., the test statistics coincide with those for the SMN distribution. For ease of reference, the test statistics are quoted below from Gokhale and SenGupta (1986) wherefrom their exact null distributions and the corresponding cut-off points are also available. Thus a tremendous gain is achieved here both from the theoretical and the computational aspects.

To complete the notation, let $\boldsymbol{X}_s = (X_{s1}, X_{s2}, \ldots, X_{sk})'$, X_{si} denoting the i-th component of the vector $\boldsymbol{X}_s$, $\bar{\boldsymbol{X}}$ denote the sample mean vector; $\bar{\bar{X}} =$

$(\sum_{s=1}^n \sum_{i=1}^k X_{si})/nk$, $\bar{X}_s = (\sum_{i=1}^k X_{si})/k$, $s = 1, 2, \ldots, n$, $W = \sum_{s=1}^n \sum_{i=1}^k (X_{si} - \bar{X}_s)^2$, $B = k\sum_{s=1}^n (\bar{X}_s - \bar{\bar{X}})^2$ and $T = \sum_{s=1}^n \sum_{i=1}^k (X_{si} - \bar{\bar{X}})^2 = B + W$. Under g_ρ, $W/(1-\rho)\sigma^2$ is distributed as $\chi^2_{n(k-1)}$, $B/\{1+(k-1)\rho\}\sigma^2$ is distributed as χ^2_{n-1} and W and B are independent (Rao (1973)).

For the cases when the parameters m and σ^2 are known, without loss of generality we assume $m = 0$ and $\sigma^2 = 1$. In each of the four cases, the critical region reduces to the form: Reject H_0 if $T_i > c$, so that it suffices to present the test statistics, T_i only.

(2.2) *Case* 1. m and σ^2 both known: $T_1 = \sum_{s=1}^{n} \sum_{i \neq j=1}^{k} X_{si}X_{sj}$,

(2.3) *Case* 2. m known, σ^2 unknown: $T_2 = \sum_s \left(\sum_i X_{si}\right)^2 \Big/ \sum_s \sum_i X_{si}^2$,

(2.4) *Case* 3. m and σ^2 both unknown: $T_3 = B/T$,

(2.5) *Case* 4. m unknown, σ^2 known: $T_4 = (k-1)B - W$.

[There is a misprint for this Case 3 in Gokhale and SenGupta (1986). For case (iii) there replace $\bar{x}_s$ by $\bar{\bar{x}}$ in the expression for T, p. 267, l. 8]. The problems under H_0 associated with Case 1 for the non-mixture situation deserve special mention and interested readers are referred to SenGupta (1982, 1987). SenGupta and Pal (1991) have established monotonicity, unbiasedness, consistency and asymptotic normality related to the special Case 1 of the mixture situation. We establish below these properties for the general Cases 2 through 4 also.

3. Monotonicity of the power functions

Case 2. m known, σ^2 unknown: The critical region can be written as

$$T_2' = \sum_{s=1}^n Y_s' > 0 \quad \text{where} \quad Y_s' = \left(\sum_i X_{si}\right)^2 - c\sum_i X_{si}^2.$$

Y_s', $s = 1, 2, \ldots, n$ are i.i.d. and each Y_s' has the distribution given below:

$$\sigma^{-2}Y_s' \stackrel{\mathrm{d}}{=} \begin{cases} k[1+(k-1)\rho]\chi_1^2 & \\ \quad - c[(1-\rho)\chi_{k-1}^2 + \{1+(k-1)\rho\}\chi_1^2] & \text{under } g_\rho \\ k\chi_1^2 - c[\chi_{k-1}^2 + \chi_1^2] & \text{under } g_0, \end{cases} \tag{3.1}$$

and

$$\begin{aligned} P(T_2 > c \mid \rho) &= 1 - P(T_2' \leq 0 \mid \rho) \\ &= 1 - \sum_{s=0}^n \binom{n}{s} p^s (1-p)^{n-s} \left[G_0^{s^*} \star G_\rho^{(n-s)^*}(0)\right] \end{aligned} \tag{3.2}$$

where $G_0(x)$ and $G_\rho(x)$ are respectively the c.d.f.s of Y'_s with distributions as specified in (3.1), and "$\star$" denotes convolution. The expression in the RHS of (3.2) follows directly since T'_2 is a symmetric statistic of Y'_s's.

Note that the last expression in the RHS of (3.1) is an increasing function of ρ for $\rho > 0$. This implies that for $0 \leq \rho_1 < \rho_2 \leq 1$, Y'_s is (stochastically) larger under g_{ρ_2} than under g_{ρ_1}. Hence, $G_{\rho_2}(x) < G_{\rho_1}(x)$ for all x belonging to the set of continuity points of Y'_s. Consequently for any positive integer r, $G^{r^\star}_{\rho_2}(x) < G^{r^\star}_{\rho_1}(x)$. From (3.2) we then get,

$$\begin{aligned} P(T_2 > c \mid \rho_2) &= 1 - \sum_{s=0}^{n} \binom{n}{s} p^s (1-p)^{n-s} \left[G_0^{s^\star} \star G_{\rho_2}^{(n-s)^\star}(0)\right] \\ &> 1 - \sum_{s=0}^{n} \binom{n}{s} p^s (1-p)^{n-s} \left[G_0^{s^\star} \star G_{\rho_1}^{(n-s)^\star}(0)\right] \\ &= 1 - P(T'_2 \leq 0 \mid \rho_1) = P(T_2 > c \mid \rho_1). \end{aligned}$$

The power function is thus monotonically increasing and hence the test is unbiased.

Case 3. Both m, σ^2 unknown: The critical region can be equivalently written as

$$T'_3 = k(1-c)\sum_s (\bar{X}_s - \bar{\bar{X}})^2 - c\sum_s\sum_i (X_{si} - \bar{X}_s)^2 > 0. \tag{3.3}$$

Let us first regroup (Behboodian (1972)) the n observation vectors $\boldsymbol{X}_1$, $\boldsymbol{X}_2$, ..., $\boldsymbol{X}_n$ into two groups: one of independent vectors $\boldsymbol{X}_{u1}, \boldsymbol{X}_{u2}, \ldots, \boldsymbol{X}_{uu}$ with common density g_0 and the other of independent vectors $\boldsymbol{X}_{uu+1}, \boldsymbol{X}_{uu+2}, \ldots, \boldsymbol{X}_{un}$ with common density g_ρ. Let $T_{u,n} = k(1-c)\sum_s (\bar{X}_{us} - \bar{\bar{X}}_u)^2 - c\sum_s \sum_i (X_{usi} - \bar{X}_{us})^2$ where,

$$\boldsymbol{X}_{us} = (X_{us1}, X_{us2}, \ldots, X_{usk})', \qquad \bar{X}_{us} = \sum_{i=1}^{k} X_{usi}/k \quad \text{and} \quad \bar{\bar{X}}_u = \sum_{s=1}^{n} \bar{X}_{us}/n.$$

Let,

$$\begin{aligned} \bar{\bar{X}}_{u1} &= \sum_{s=1}^{u} \bar{X}_{us}/u, \qquad T^2_{u1} = \sum_{s=1}^{u} (\bar{X}_{us} - \bar{\bar{X}}_{u1})^2/u, \\ \bar{\bar{X}}_{u2} &= \sum_{s=u+1}^{n} \bar{X}_{us}/(n-u), \qquad T^2_{u2} = \sum_{s=u+1}^{n} (\bar{X}_{us} - \bar{\bar{X}}_{u2})^2/(n-u), \\ Z^2_{1u} &= \sum_{s=1}^{u}\sum_{i=1}^{k} (X_{usi} - \bar{X}_{us})^2 \quad \text{and} \quad Z^2_{2u} = \sum_{s=u+1}^{n}\sum_{i=1}^{k} (X_{usi} - \bar{X}_{us})^2. \end{aligned}$$

Then,

$$\begin{aligned} T_{u,n} &= ku(1-c)T^2_{u1} + k(n-u)(1-c)T^2_{u2} \\ &\quad + (ku(n-u)(1-c)/n)(\bar{\bar{X}}_{u1} - \bar{\bar{X}}_{u2})^2 - c(Z^2_{1u} + Z^2_{2u}). \end{aligned}$$

Under g_ρ, $uT_{u1}^2 \sim (\sigma^2/k)\chi^2_{u-1}$, $(n-u)T_{u2}^2 \sim [1+(k-1)\rho](\sigma^2/k)\chi^2_{n-u-1}$,

$$(\overline{\overline{X}}_{u1} - \overline{\overline{X}}_{u2})^2 \sim [(1/u) + \{(1+(k-1)\rho)/(n-u)\}](\sigma^2/k)\chi_1^2,$$
$$Z_{1u}^2 \sim \sigma^2\chi^2_{u(k-1)} \quad \text{and} \quad Z_{2u}^2 \sim (1-\rho)\sigma^2\chi^2_{(n-u)(k-1)}.$$

So,

$$\begin{aligned}\sigma^{-2}T_{u,n} \stackrel{\mathrm{d}}{=} & (1-c)\chi^2_{u-1} + (1-c)\{1+(k-1)\rho\}\chi^2_{n-u-1} \\ & + (ku(n-u)(1-c)/n) \\ & \quad \cdot [(1/u) + \{(1+(k-1)\rho)/(n-u)\}](1/k)\chi_1^2 \\ & - c[\chi^2_{u(k-1)} + (1-\rho)\chi^2_{(n-u)(k-1)}].\end{aligned} \tag{3.4}$$

The distribution of T_u under g_0 is the same as in (3.4) with $\rho = 0$. Moreover, under both g_0 and g_ρ, all the χ^2 variables occurring in (3.4) are mutually independent. We also note that the expression in the RHS of (3.4) is an increasing function of ρ for $\rho > 0$. Hence, for any two values ρ_1 and ρ_2, $0 \le \rho_1 < \rho_2$, of ρ we have,

$$\begin{aligned}P(T_3' > 0 \mid \rho_2) &= 1 - P(T_3' \le 0 \mid \rho_2) \\ &= 1 - \sum_{u=0}^{n} \binom{n}{u} p^u(1-p)^{n-u} P(T_{u,n} \le 0 \mid \rho_2) \\ &> 1 - \sum_{u=0}^{n} \binom{n}{u} p^u(1-p)^{n-u} P(T_{u,n} \le 0 \mid \rho_1) = P(T_3' > 0 \mid \rho_1).\end{aligned}$$

This implies that the power function is monotonically increasing in ρ and hence the test is unbiased.

Case 4. m unknown, σ^2 known: In this case the critical region comes out in a similar form as in (3.3). Hence the monotonicity and unbiasedness follow by arguments similar to those in Case 4.

We obtain the numerical values for the powers with $n = 10$ thru simulation. The cut-off points at $\alpha = .05$ are tabulated in Gokhale and SenGupta (1986). For each n, p and ρ, we generate 1000 T_i values by generating 1000 values for each of the χ^2-variables involved, with necessary modifications for $u = 0$ and $u = n$. Power is obtained, then, thru the empirical c.d.f. of T_i. So, for example, to obtain Table 1, the simulation loop was executed $p{\cdot}\rho{\cdot}(n{+}1){\cdot}1000.6 = 9{\cdot}10{\cdot}26{\cdot}1000{\cdot}6 = 14,040,000$ times. From Tables 1, 2 and 3 we note that the performance of the test is quite good, for small p, even for as small a sample size as 10. The power increases rapidly with ρ. The monotonicity of the power function is clearly exhibited. It can be shown that, as in Case 1 (SenGupta and Pal (1991)), here also the power increases rapidly, for small p, with increase in n. For large p, the power is not high. This is to be expected. For large p, the distribution $g_{(p)}$ even under the alternative hypothesis $H_1 : \rho > 0$ approaches that under the null hypothesis $H_0 : \rho = 0$, and hence any reasonable test will have to suffer the consequences.

Table 1. Power of T_2-test.

$\rho\backslash p$	.1	.2	.3	.4	.5	.6	.7	.8	.9
.1	.2426	.2276	.2281	.2158	.2012	.1889	.1794	.1697	.1645
.2	.3490	.3202	.2941	.2857	.2475	.2399	.2100	.1946	.1663
.3	.4251	.3986	.3726	.3304	.3186	.2774	.2400	.2173	.1872
.4	.5109	.4851	.4456	.3864	.3505	.3186	.2791	.2275	.1968
.5	.5893	.5479	.4943	.4620	.3997	.3428	.2951	.2526	.2046
.6	.5407	.6121	.5565	.5104	.4579	.4090	.3381	.2798	.2084
.7	.6885	.6392	.6059	.5359	.4835	.4318	.3804	.2878	.2203
.8	.7314	.6879	.6487	.5802	.5283	.4455	.4050	.3148	.2487
.9	.7603	.7265	.6816	.6213	.5593	.4867	.4194	.3474	.2528

Table 2. Power of T_3-test.

$\rho\backslash p$	.1	.2	.3	.4	.5	.6	.7	.8	.9
.1	.1394	.1157	.1074	.1015	.0956	.0835	.0868	.0620	.0620
.2	.2414	.2227	.1924	.1621	.1452	.1188	.0998	.0837	.0669
.3	.1845	.3441	.2837	.2502	.2104	.1733	.1424	.1134	.0789
.4	.5482	.4825	.4201	.3538	.2820	.2312	.1709	.1319	.0829
.5	.0996	.6388	.5506	.4390	.3663	.2914	.2108	.1560	.0999
.6	.8275	.7415	.6608	.5485	.4529	.3500	.2646	.1747	.1133
.7	.9108	.8317	.7417	.6473	.5255	.4257	.3055	.2025	.1228
.8	.9621	.8984	.8160	.7193	.6104	.4934	.3539	.2337	.1299
.9	.9811	.9431	.8764	.7741	.6639	.5496	.3999	.2688	.1499

Table 3. Power of T_4-test.

$\rho\backslash p$	.1	.2	.3	.4	.5	.6	.7	.8	.9
.1	.1231	.1190	.1095	.0855	.0894	.0667	.0743	.0615	.0530
.2	.2290	.2102	.1842	.1625	.1285	.1097	.0924	.0866	.0637
.3	.3677	.3137	.2705	.2446	.2021	.1633	.1252	.0958	.0726
.4	.5005	.4370	.3793	.3178	.2651	.2146	.1727	.1207	.0769
.5	.6036	.5346	.4683	.3983	.3286	.2752	.2119	.1401	.0983
.6	.7140	.6426	.5533	.4865	.4064	.3178	.2431	.1648	.0986
.7	.7939	.7299	.6449	.5581	.4689	.3783	.2853	.1977	.1229
.8	.8552	.7836	.7191	.6277	.5304	.4290	.3302	.2257	.1288
.9	.9062	.8528	.7775	.6795	.5949	.4691	.3652	.2403	.1317

4. Asymptotic normality and consistency

Case 2. m known, σ^2 unknown: Rewrite T_2 as $T_2 = \bar{U}/\bar{V}$ where $\bar{U} = \sum_s U_s/n$, $U_s = (\sum_i X_{si})^2$, $\bar{V} = \sum_s V_s/n$, $V_s = \sum_i X_{is}^2$. Then $\boldsymbol{Q}_s = (U_s, V_s)'$

for $s = 1, 2, \ldots, n$ are i.i.d. random vector variables. Under H_1, $E(U_s) = [1 + q(k-1)\rho]k\sigma^2 = \xi_1$, say; $E(V_s) = k\sigma^2 = \xi_2$, say; and Disp.$(\boldsymbol{Q}_s) = (\sigma_{ij}) = \Sigma$, say, where σ_{ij}' s may be obtained by tedious calculations. By the multivariate Central Limit Theorem (CLT), $\sqrt{n}(\bar{\boldsymbol{Q}} - \boldsymbol{\xi}) \xrightarrow{\mathrm{L}} N_2(\mathbf{0}, \Sigma)$, where $\bar{\boldsymbol{Q}} = (\bar{U}, \bar{V})'$ and $\boldsymbol{\xi} = (\xi_1, \xi_2)'$.

Thus under H_1, $\sqrt{n}[T_2 - \{1 + q(k-1)\rho\}]$ is asymptotically distributed as a normal variable with mean 0 and variance η_ρ^2, say, where $\eta_\rho^2 > 0$. Note that the function $h(\bar{U}, \bar{V}) = \bar{U}/\bar{V} \equiv T_2$ is totally differentiable. So, the explicit form for η_ρ^2, though not needed here, may be obtained by the Delta method (Rao ((1973), p. 388)) if one is still interested. Under H_0, the relevant quantities are obtained by substituting $\rho = 0$ in the corresponding above expressions. Let α henceforth denote the size of the test.

$$\alpha = P[(\bar{U}/\bar{V}) > c \mid H_0] \to 1 - \Phi[(c-1)/(\eta_0/\sqrt{n})] \Rightarrow c \doteq 1 + (\tau_\alpha \eta_0/\sqrt{n}),$$

τ_α being the upper α point of the standard normal variable. For large n, the power of the test under $\rho > 0$ can be written as

$$\begin{aligned} 1 - \Phi &\left[[1 + (\eta_0\tau_\alpha/\sqrt{n}) - \{1 + q(k-1)\rho\}]/(\eta_\rho/\sqrt{n})\right] \\ &= 1 - \Phi[\{\eta_0\tau_\alpha - \sqrt{n}q(k-1)\rho\}/\eta_\rho] \to 1 \quad \text{as} \quad n \to \infty \end{aligned}$$

since $q(k-1)\rho > 0$ for $\rho > 0$.

Case 3. m, σ^2 both unknown: Rewrite the critical region given by (2.4) as $T_3'' > c'$, where $T_3'' = (B/n)/(W/n) = [k\sum_s(\bar{X}_s - \bar{\bar{X}})^2/n]/[\sum_s\sum_i(X_{si} - \bar{X}_s)^2/n]$ $= (k\sum_s Y_s/n)/(\sum_s Z_s/n) = k\bar{Y}/\bar{Z}$, say. Note that, under both H_0 and H_1, $\bar{\bar{X}} \xrightarrow{\mathrm{P}} m$ and $E(\bar{X}_s) = m$. Also under H_1, $V(\bar{X}_s) = [p\sigma^2 + q\{1 + (k-1)\rho\}\sigma^2]/k = \mu_\rho/k$, say. So, $\bar{Y} \xrightarrow{\mathrm{P}} E(\bar{X}_s - m)^2 = \mu_\rho/k$, say. Further, Z_s, $s = 1, 2, \ldots, n$ are i.i.d. random variables. Under H_1, $Z_s \sim p\sigma^2\chi^2_{(1)} + q(1-\rho)\sigma^2\chi^2_{(2)}$, where $\chi^2_{(1)}$ and $\chi^2_{(2)}$ are χ^2 variables with d.f. $(k-1)$ each, so that $E(Z_s) = (1 - q\rho)(k-1)\sigma^2 = \theta_\rho$, say, and $\operatorname{var}(Z_s) = \sigma^4(k-1)[(k+1)\{p + q(1-\rho)^2\} - (k-1)(1-q\rho)^2] = \beta_\rho^2$, say.

Then by CLT, $\sqrt{n}(\bar{Z} - \theta_\rho) \xrightarrow{\mathrm{L}} N(0, \beta_\rho^2)$. Let $h(\bar{Z}) = 1/\bar{Z}$. Then $\sqrt{n}(1/\bar{Z} - 1/\theta_\rho) \xrightarrow{\mathrm{L}} N(0, \gamma_\rho^2)$, where γ_ρ^2 is obtained by the Delta method as $\gamma_\rho^2 = \beta_\rho^2/\theta_\rho^4$. Hence by Slutsky's theorem, under H_1, $\sqrt{n}[T_3'' - \mu_\rho/\theta_\rho]$ is asymptotically distributed as a normal variable with mean 0 and variance $\tilde{\eta}_\rho^2 = \gamma_\rho^2\mu_\rho^2$. Under H_0, the relevant quantities are obtained by substituting $\rho = 0$. Then,

$$\alpha = P[T_3'' > c' \mid H_0] \to 1 - \Phi[\sqrt{n}(c' - \mu_0/\theta_0)/\tilde{\eta}_0] \Rightarrow c' \doteq (\mu_0/\theta_0) + \tau_\alpha\tilde{\eta}_0/\sqrt{n}.$$

For large n, the power of the test under $\rho > 0$ can be written as

$$1 - \Phi[\{\sqrt{n}(\mu_0/\theta_0 - \mu_\rho/\theta_\rho) + \tau_\alpha\tilde{\eta}_0\}/\tilde{\eta}_\rho] \to 1 \qquad \text{as} \qquad n \to \infty,$$

since $\mu_0/\theta_0 - \mu_\rho/\theta_\rho = -qk\rho/[(k-1)(1-q\rho)] < 0$ for $\rho > 0$.

Case 4. m unknown, σ^2 known: Rewrite the critical region given by (2.5) as $T_4' > c/n$, where $T_4' = T_4/n = k(k-1)\bar{Y} - \bar{Z}$, $\bar{Y}$ and $\bar{Z}$ being as defined in

Case 3 above. Let μ'_ρ, θ'_ρ and β'^2_ρ denote the value of μ_ρ, θ_ρ and β^2_ρ respectively, of Case 3 above with $\sigma^2 = 1$. Also let $\delta'_\rho = (k-1)\mu'_\rho - \theta'_\rho$. Hence by CLT and using Slutsky's theorem , under $H_1, \sqrt{n}\left[T'_4 - \delta'_\rho\right]$ is asymptotically distributed as a normal variable with mean 0 and variance β'^2_ρ.

Then,

$$\alpha = P[T'_4 > c/n \mid H_0] \to 1 - \Phi[\sqrt{n}\{c/n - \delta'_0\}/\beta'_0] \Rightarrow c \doteq \sqrt{n}\tau_\alpha \beta'_0 + n\delta'_0.$$

For large n, the power of the test under $\rho > 0$ can be written as

$$1 - \Phi[\{\tau_\alpha \beta'_0 - \sqrt{n}(\delta'_\rho - \delta'_0)\}/\beta'_\rho] \to 1 \quad \text{as} \quad n \to \infty,$$

since $d\delta'_\rho / d\rho = k(k-1)q > 0$.

5. Comments

As discussed in SenGupta and Pal (1991), one would probably have knowledge about p more often than about ρ and hence we have considered here tests for ρ. However, in case ρ is known, tests for p, $0 \le p \le 1$, may be needed i.e. for H_0 : $p = 1$. Then, one can obtain LMP similar tests for p (Durairajan and Kale (1982)) following (2.1) and considering derivatives of p from the left at $p = 1$. Further, if both p and ρ need to be tested simultaneously, one may use the multiparameter locally most mean powerful similar test of SenGupta and Vermeire (1986). Finally when either or both ρ and q may assume zero values, the test may be based on an appropriate Pivotal Parametric Product (P^3) (SenGupta (1991)). Details may be pursued in the lines of Rachev and SenGupta (1992) and SenGupta and Pal (1993).

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