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ESTIMATION OF SYSTEM RELIABILITY IN BROWNIAN STRESS-STRENGTH MODELS BASED ON SAMPLE PATHS*

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Abstract. Reliability of many stochastic systems depends on uncertain stress and strength patterns that are time dependent. In this paper, we consider the problem of estimating the reliability of a system when both $X(t)$ and $Y(t)$ are assumed to be independent Brownian motion processes, where $X(t)$ is the system stress, and $Y(t)$ is the system strength, at time t.

Key words and phrases: Stress-strength model, stopped process, maximum likelihood estimator, binary performance process, Brownian motion, homogeneous Markov process, nonhomogeneous Markov process, first passage time, inverse Gaussian distribution.

1. Introduction

The traditional approach to estimation of the reliability of a stochastic system in the context of uncertain stress and strength patterns may be described as follows. Let X and Y be two independent random variables with cumulative distribution functions F and G respectively. Suppose Y is the strength of a system that is subjected to a stress X. Then the reliability, R, of the system is defined as

$$R = P(Y > X) = \int_0^\infty \bar{G}(x)dF(x),$$

where $\bar{G}(x) = 1 - G(x)$. Assuming the parameter R is unknown, n identical systems are put on test, the time to failures $T_1, \dots, T_n$ are observed, and appropriate inference procedures are derived based on this data. For a bibliography of available results, see Ebrahimi (1982) and Johnson (1988).

We believe that, in many important applications involving catastrophic situations at the time of system failure, it is most prudent that we estimate the

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reliability of the system without necessarily waiting to observe the system failure. Such estimation may be achieved by monitoring the behavior of stress and strength over time. Motivated by this, Basu and Ebrahimi (1983) introduced a dynamic approach to modeling system reliability in the presence of processes $X(t)$ and $Y(t)$, respectively denoting the stress that the system is experiencing at time t and strength of the system at time t. They defined the random life-time, T, of the system as

$$T = \inf\{t : t \geq 0, Z(t) \leq 0\}, \tag{1.1}$$

where $Z(t) = Y(t) - X(t)$.

Frequently, engineers are interested in the reliability of the system over a specific time period, say $(0, t_0]$. It follows from (1.1) that the reliability of the system is given by $R(t_0)$ where, for $t_0 > 0$,

$$R(t_0) = P(T > t_0) = P\left(\inf_{0<t\leq t_0} Z(t) > 0\right). \tag{1.2}$$

Since the strength of the system changes very little over small time intervals and since, for a given t, the Brownian motion process is normally distributed we believe that in many practical situations $\{Y(t) : t \geq 0\}$ can be assumed to be a Brownian motion. Also, there are situations in which $X(t)$ can be assumed to be a Markov process with continuous sample paths such as a diffusion processes. For example, the earthquake after-shocks can be modeled by a Markov process (see Vere-Jones (1970)). Since every diffusion process can be transferred to a Brownian motion using appropriate transformations (see Cinlar (1979)), $X(t)$ can be assumed to be a Brownian motion (at least approximately). One context in which such modeling is reasonable is when $X(t)$ ($Y(t)$) represents tensile stress (strength). For more details and further examples, see Whitmore (1990). Consequently, throughout this paper, we will assume that $X(t)$ and $Y(t)$ are independent Brownian motions so that $Z(t)$ is also a Brownian motion.

In Section 2, we derive distributional properties of the hitting time T in (1.1). Two different schemes to sample data from $Z(t)$ process are distinguished in that section. For one of the schemes, maximum likelihood estimation of $R(t_0)$ in (1.2) is treated in Section 3. In Section 4, we show that the likelihood-based inference in the other scheme can be done using estimation of parameters of the inverse-Gaussian distribution.

2. The structure of Brownian stress-strength models

We shall now delineate two situations in which inference questions concerning properties of T will be dealt with. At the outset, we should note that, once the system fails, $Z(t) \leq 0$, it stays inoperative thereafter, $Z(s) \leq 0$, for $s \geq t$. Thus, data pertaining to the stress and strength patterns of the system may be collected as follows:

Situation A. The stopped processes $X(t \wedge T)$, $Y(t \wedge T)$ and $Z(t \wedge T)$ can be measured over time, where, for any process $\Lambda(t)$, with $t \wedge T = \min(t, T)$, we let

$$\Lambda(t \wedge T) = \begin{cases} \Lambda(t) & \text{if } t < T \\ \Lambda(T) & \text{if } t \geq T. \end{cases} \tag{2.1}$$

Situation B. Often, we can observe not the actual stress and strength of the system over time but only whether or not the system is operating. Equivalently, we can only detect whether the strength of the system is more than the stress or not. Thus we observe the stopped process $U(t \wedge T)$, where $U(t)$ is the binary performance process given by

$$U(t) = \begin{cases} 1 & \text{if } Z(t) > 0 \\ 0 & \text{if } Z(t) \leq 0. \end{cases}$$

We now introduce some notations that will hence-forth be used. For simplicity, we let $Z^*(t) = Z(t \wedge T)$ and, in view of (1.1) and (2.1),

$$U^*(t) = U(t \wedge T) = \begin{cases} 1 & \text{if } Z^*(t) > 0 \\ 0 & \text{if } Z^*(t) = 0. \end{cases} \tag{2.2}$$

We shall assume that the system is started at known strength and stress levels so that, unless stated otherwise, for a fixed but otherwise arbitrary $x > 0$, $P(Z(0) \equiv x) = 1$. Throughout this paper, depending on the need to emphasize the effect of x on probabilities and expectations, we write P_x and E_x respectively. From the well-known facts about the moments of the Brownian motions $X(t)$, $Y(t)$, $Z(t)$, we obtain, for some constants μ_1, μ_2, σ_1^2, σ_2^2 that are presumed to be unknown, $E(X(t)) = \mu_1 t$, $E(Y(t)) = \mu_2 t$, $E(Z(t)) = \mu t$, $\text{Var}(X(t)) = \sigma_1^2 t$, $\text{Var}(Y(t)) = \sigma_2^2 t$, $\text{Var}(Z(t)) = \sigma^2 t$, where $\mu = \mu_2 - \mu_1$ and $\sigma^2 = \sigma_2^2 + \sigma_1^2$ are respectively called the drift and variance coefficients of the $Z(t)$ process.

For all $s < t$, we let

$$P_x(s,t) = P_x(U^*(t) = 1 \mid U^*(s) = 1), \tag{2.3a}$$

$$Q_x(s,t) = 1 - P_x(s,t), \tag{2.3b}$$

$$q_x(s,t) = \frac{\partial Q_x(s,b)}{\partial b} \Big|_{b=t} . \tag{2.3c}$$

For any $a > 0$, $t \geq 0$, and the parameters $\boldsymbol{\nu} = (\mu, \sigma)'$, let

$$H_a(t) = H_a(t; \boldsymbol{\nu}) = \Phi\left(\frac{a + \mu t}{\sigma\sqrt{t}}\right) - \left\{\Phi\left(\frac{-a + \mu t}{\sigma\sqrt{t}}\right) \exp\left(-\frac{2\mu a}{\sigma^2}\right)\right\}, \tag{2.3d}$$

where $\Phi(u) = \int_{-\infty}^{u} \phi(y)dy$ is the cumulative distribution function of the standard normal density $\phi(y)$. Also, for $t \geq 0$, $a > 0$, $b > 0$, let

$$\psi_t(a, b; \boldsymbol{\nu}) = \frac{1}{\sigma\sqrt{t}} \left[\phi\left(\frac{-b + a + \mu t}{\sigma\sqrt{t}}\right) - \phi\left(\frac{-b - a + \mu t}{\sigma\sqrt{t}}\right) \exp\left(\frac{-2\mu a}{\sigma^2}\right)\right]. \tag{2.3e}$$

Before we state our first result, we record two simple facts that readily follow from the path continuity of $Z(t)$ and the property that the interval $(-\infty, 0]$ is an absorbing barrier of the $Z(t)$ process.

$$P_x(Z^*(t) > 0 \mid Z^*(s) = 0) = 0, \qquad \forall s < t \tag{2.4a}$$

$$P_x(U^*(t) = 1 \mid U^*(s) = 0) = 0, \qquad \forall s < t. \tag{2.4b}$$

THEOREM 2.1. (i) *$\{Z^*(t), t \geq 0\}$ is a time-homogeneous Markov process with a transition kernel whose discrete and continuous parts are defined as follows: For $a > 0$, $z \geq 0$, $t \geq 0$,*

$$P(Z^*(t) > z \mid Z^*(0) = a) = \int_z^\infty \psi_t(a, b)db, \tag{2.5}$$

$$P(Z^*(t) = 0 \mid Z^*(0) = a) = 1 - H_a(t) \tag{2.6}$$

where $\psi_t(a, b)$ and $H_a(t)$ are given by (2.3e) *and* (2.3b) *respectively.*

(ii) *For $a > 0$, $0 < t < \infty$, let*

$$h_a(t) = \frac{a}{\sigma t^{3/2}\sqrt{2\pi}} \exp\left\{-\frac{(a + \mu t)^2}{2t\sigma^2}\right\}. \tag{2.7}$$

Then, for $\mu > 0$, T in (1.1) *is a defective randon variable with pdf $h_x(t)$ given by* (2.7) *in the sense that*

$$P_x(T < \infty) = \exp\left(-\frac{2\mu x}{\sigma^2}\right).$$

However, for $\mu \leq 0$, T has the inverse-Gaussian density function $h_x(t)$.

PROOF. Equations (1) and (2) of Harrison (1985, p. 46) easily yield part (i). Now, letting $a = x$ in (2.6), we note that (2.6) can be rewritten as

$$P_x(T \leq t) = P_x(Z^*(t) = 0) = 1 - H_x(t). \tag{2.8}$$

Since $h_x(t) = -(\partial/\partial t)H_x(t)$, part (ii) now follows from equation (2.8).

THEOREM 2.2. *The process $\{U^*(t), t \geq 0\}$ is a continuous-time non-stationary Markov process with state space $\{0, 1\}$ and transition probability matrix*

$$\begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 1 & 0 \\ Q_x(s,t) & P_x(s,t) \end{bmatrix} \end{array}, \qquad \forall s < t,$$

where $P_x(s,t)$ is given by (2.9) *below and* (2.3b) *provides $Q_x(s,t)$.*

PROOF. In view of (2.4b), $U^*(t)$ is a binary process with zero as an absorbing state. It is now easy to show that such a process has the Markov property. Furthermore, it follows from (2.3a), (2.2) and (2.8)

$$P_x(s,t) = P_x(Z^*(t) > 0)/P_x(Z^*(s) > 0) = H_x(t)/H_x(s). \tag{2.9}$$

Remark 1. The conditional density of the time to failure of the system, given that $Z^*(s) = a$, for any positive number a, is given by $h_a(t)$ in (2.7). However, given only the information that the system was operating at time s, that is $Z^*(s) > 0$, the conditional density of the time to failure of the system is given by

$$q_x(s,t) = h_x(t)/H_x(s) \tag{2.10}$$

where $q_x(s,t)$ was defined in (2.3c).

3. Reliability estimation in Situation A

In this section, we give the maximum likelihood estimation of the vector parameter $\boldsymbol{\nu} = (\mu, \sigma)$ with the assumption that the stopped process $Z^*(t)$ can be measured over time independently for each of n systems. It should be noted here that a number of fine works, including Kutoyants (1984), address maximum likelihood estimation of the parameters of a diffusion process. However, our sampling designs to collect data from the underlying process are different from the ones contained in these works. In order to facilitate our proof, later in this section, of the uniqueness of the maximum likelihood estimator (m.l.e), we shall now reparametrize $\boldsymbol{\nu}$ as $\boldsymbol{\theta} = (\theta_1, \theta_2)$, where

$$\theta_1 = \frac{\mu}{\sigma} \quad \text{and} \quad \theta_2 = \frac{1}{\sigma}. \tag{3.1}$$

In view of (2.3d) and (3.1), once an estimate $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ is obtained, we can estimate the reliability

$$R_x(t_0; \boldsymbol{\theta}) = P_x(Z^*(t_0) > 0) = H_x\left(t_0; \frac{\theta_1}{\theta_2}, \frac{1}{\theta_2}\right) \tag{3.2}$$

as $R_x(t_0; \hat{\boldsymbol{\theta}})$.

We let subscript $i = 1, \ldots, n$ refer to the label of the system and subscript $j = 0, \ldots, M_i$ denote the label of the proposed observations of $Z^*(t)$ over time for the i-th system. For the i-th system, let t_{ij} be the time from the initial observation ($t_{i0} = 0$) and $Z_i^*(t_{ij})$ be the observation at time t_{ij}, $i = 1, \ldots, n$; $j = 0, \ldots, M_i$. Furthermore, for some known $x > 0$, let $Z^*(t_{i0}) \equiv x$, $i = 1, 2, \ldots, n$. It is important to note that, if $Z_i^*(t_{ij}) > 0$, the system is operating at time t_{ij}. However, if $Z_i^*(t_{ij-1}) > 0$ and $Z_i^*(t_{ij}) = 0$, then we stop gathering data from the i-th system and record the actual time to failure of the i-th system, T_i, where $t_{ij-1} < T_i \leq t_{ij}$. In addition, we relabel j as m_i and the corresponding proposed observation time, t_{im_i}, is relabeled as T_i. It should be noted that $Z_i^*(t_{ik}) = 0$, $k = m_i, \ldots, M_i$. Thus, for $i = 1, \ldots, n$, we observe $Z_i^*(t_{ik})$, $k = 0, 1, \ldots, M_i$, and $V_i = T_i \wedge t_{iM_i}$. In order to write the likelihood function of this data set in the light of (2.4a), (2.5) and (2.6), we introduce the function

$$g_t(a,b) = \begin{cases} \psi_t(a,b) & \text{if } a > 0 \text{ and } b > 0, \\ h_a(t) & \text{if } a > 0 \text{ and } b = 0, \\ 1 & \text{if } a = 0 \text{ and } b = 0, \end{cases} \tag{3.3}$$

where $\psi_t(a,b)$ and $h_a(t)$ are given by (2.3e) and (2.7) respectively. We can now write the log-likelihood of the data from the n systems as

$$l(\boldsymbol{\theta}) = \sum_{i=1}^{n}\sum_{j=1}^{M_i} \log g_{\Delta_{ij}}(z_i^*(t_{i,j-1}), z_i^*(t_{ij})), \tag{3.4}$$

where $\Delta_{ij} = t_{ij} - t_{ij-1}$.

For simplicity, we shall assume, during the remainder of this section, that $M_1 = M_2 = \cdots = M_n = M$, say, and that the data are equally spaced, that is $\Delta_{ij} \equiv \Delta > 0$, for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, M$. A similar approach can be used if the observations are not taken according to this simplified scheme. To maximize the log-likelihood, we consider two equations given by $\dot{\boldsymbol{l}}(\boldsymbol{\theta}) = \mathbf{0}$, where $\dot{\boldsymbol{l}}(\boldsymbol{\theta}) = \partial l(\boldsymbol{\theta})/\partial\boldsymbol{\theta}$ is the vector of partial derivatives. This score vector and the 2×2 matrix of second-order partial derivatives, $\ddot{l}(\boldsymbol{\theta})$, can be computed using (3.4) and the partial derivatives of the function $g_t(a,b)$ in (3.3) that are provided in the Appendix. The roots of the likelihood equations will be denoted by $\hat{\boldsymbol{\theta}}_A = (\hat{\theta}_{1A}, \hat{\theta}_{2A})$.

An iterative scheme such as the Newton-Raphson may be employed here to obtain $\hat{\boldsymbol{\theta}}_A$. The initializing values of θ_1 and θ_2 in such a scheme may be obtained by pretending that the portion of our observables, $Z_i^*(t_{ij})$, $j = 1, 2, \ldots, M$, $i = 1, 2, \ldots, n$ are values of the original process $Z(t)$. From the well-known transition function of the Markov process $Z(t)$ (see Karlin and Taylor (1975), p. 356), we can write the joint likelihood of the data under this pretension as

$$\left(\frac{1}{\sqrt{2\pi\sigma^2\Delta}}\right)^N \exp\left\{-\sum_{i=1}^{n}\sum_{k=1}^{M}(y_{ik} - \mu\Delta)^2/2\sigma^2\Delta\right\}, \tag{3.5}$$

where $y_{ik} = z_i^*(t_{ik}) - z_i^*(t_{i,k-1})$, $k = 1, 2, \ldots, M$, $i = 1, 2, \ldots, n$ and $N = nM$ is the total number of proposed snapshots from the n stopped processes $Z_i^*(t)$, $i = 1, 2, \ldots, n$ that are available for inference. Maximizing (3.5) with respect to μ and σ we obtain the initial values of θ_1 and θ_2 as $\hat{\theta}_{10} = \hat{\mu}_0/\hat{\sigma}_0$ and $\hat{\theta}_{20} = 1/\hat{\sigma}_0$, where

$$\hat{\mu}_0 = \sum_{i=1}^{n}\sum_{k=1}^{M} y_{ik}/N\Delta, \qquad \hat{\sigma}_0^2 = \sum_{i=1}^{n}\sum_{k=1}^{M}(y_{ik} - \hat{\mu}_0\Delta)^2/N\Delta.$$

We now discuss the asymptotic properties of the m.l.e. $\hat{\boldsymbol{\theta}}_A$.

THEOREM 3.1. *If $n \to \infty$, then we have* (a) $\hat{\boldsymbol{\theta}}_A$ *converges in probability to* $\boldsymbol{\theta}$, *that is* $\hat{\boldsymbol{\theta}}_A \xrightarrow{\text{pr}} \boldsymbol{\theta}$. (b) $\sqrt{n}(\hat{\boldsymbol{\theta}}_A - \boldsymbol{\theta})$ *converges weakly to a bivariate normal distribution with mean vector* $\mathbf{0}$ *and covariance-matrix* $\Gamma_A(\boldsymbol{\theta})$ *given in* (3.10) *below, that is* $\sqrt{n}(\hat{\boldsymbol{\theta}}_A - \boldsymbol{\theta}) \xrightarrow{L} N(\mathbf{0}, \Gamma_A(\boldsymbol{\theta}))$.

PROOF. It is clear that the likelihood in (3.4) is based on n i.i.d copies, $\boldsymbol{W}_1, \boldsymbol{W}_2, \ldots, \boldsymbol{W}_n$, of $\boldsymbol{W}$, where $\boldsymbol{W} = (Z^*(\Delta), \ldots, Z^*(M\Delta), V)$. Furthermore, with the function g defined in (3.3), the log-density of $\boldsymbol{W}$ is seen to be

$$\log f(\boldsymbol{w};\boldsymbol{\theta}) = \sum_{j=1}^{M} \log g_{\Delta}(Z^*((j-1)\Delta), Z^*(j\Delta)). \tag{3.6}$$

We first argue that, as a function of θ_1 and θ_2, the log-likelihood $l(\boldsymbol{\theta})$ in (3.4) provides a unique maximum likelihood estimate. We shall do so by verifying the following sufficient conditions (see the remarks above Theorem 2.6 in Makelainen *et al.* (1981)).

(i) $\mathrm{Lim}_{\boldsymbol{\theta}\to\partial\Theta} l(\boldsymbol{\theta}) = -\infty$, where $\partial\Theta$ is the boundary of the parameter space $\Theta = \{(\theta_1,\theta_2) : -\infty < \theta_1 < \infty, \theta_2 > 0\}$, and

(ii) The Hessian matrix, $\ddot{l}(\boldsymbol{\theta})$, of second partial derivatives of $l(\boldsymbol{\theta})$ is negative definite at every point $\boldsymbol{\theta} \in \Theta$.

Starting with (i), we let $r = r_n = \sum_{i=1}^{n} I(V_i < M\Delta)$ denote the count of those systems that fail during our proposed observation scheme. When the Binomial random variable $r > 0$, there are contributions to $l(\boldsymbol{\theta})$, due to the r failures, which we shall denote as $\log h_{a_i}(s_i)$, $i = 1,2,\ldots,r$. Letting

$$S_1(\boldsymbol{\theta}) = \sum_{i=1}^{r} [a_i\theta_2 + s_i\theta_1]^2/2s_i, \tag{3.7}$$

we obtain

$$\prod_{i=1}^{r} h_{a_i}(s_i) \simeq (\theta_2)^r \exp\{-S_1(\boldsymbol{\theta})\}, \tag{3.8}$$

where $\simeq$ means that the left-side is proportional to the right-side. It is easy to show that, if $r = 1$, or $s_i = ca_i$, $i = 1,2,\ldots,n$, for some constant c, then the right-side of (3.8) gets unbounded as $\boldsymbol{\theta} \to \partial\Theta$ and that, outside of these events, $S_1(\boldsymbol{\theta})$ in (3.7) is a positive definite quadratic form in $\boldsymbol{\theta}$. However, as $n \to \infty$, the probabilities of these events tend to zero, so that such cases will not matter in the large-sample properties of $\hat{\boldsymbol{\theta}}_A$. We conclude from (3.8) that terms in $l(\boldsymbol{\theta})$ of the type $\log h_a(t)$ collectively tend to $-\infty$, as $\boldsymbol{\theta} \to \partial\Theta$.

Letting $S_2(\boldsymbol{\theta}) = [\Delta^2\theta_1^2 + (a-b)^2\theta_2^2 + 2\theta_1\theta_2\Delta(a-b)]/2\Delta$ and $S_3(\boldsymbol{\theta}) = [\Delta^2\theta_1^2 + (a+b)^2\theta_2^2 + 2\theta_1\theta_2\Delta(a-b)]/2\Delta$, we obtain from equation (A.1) of the Appendix that $\psi_\Delta(a,b) \simeq \theta_2[\exp\{-S_2(\boldsymbol{\theta})\} - \exp\{-S_3(\boldsymbol{\theta})\}]$. Consequently, as $\boldsymbol{\theta} \to \partial\Theta$, $\psi_\Delta(a,b) \to 0$ and any term of the type $\log\psi_\Delta(a,b)$ in $l(\boldsymbol{\theta})$ tends to $-\infty$. We have therefore verified that condition (i) above holds.

To show (ii), we first note that it is sufficient to argue that, as functions of $\boldsymbol{\theta}$, $\log\psi_\Delta(a,b)$ and $\log h_a(t)$ have Hessian matrices that are negative definite (see, for e.g., p. 152 of Apostol (1973) for conditions guaranteeing negative definiteness). From equations (A.4) to (A.6) of the Appendix, it readily follows that

$$\frac{\partial^2 \log\psi_\Delta(a,b)}{\partial\theta_1^2} < 0,$$

and that

$$\left(\frac{\partial^2 \log \psi_\Delta(a,b)}{\partial \theta_1^2}\right)\left(\frac{\partial^2 \log \psi_\Delta(a,b)}{\partial \theta_2^2}\right) - \left(\frac{\partial^2 \log \psi_\Delta(a,b)}{\partial \theta_1 \partial \theta_2}\right)^2 = \left[\frac{\Delta}{\theta_2^2}\right] + 2ab[S_4], \tag{3.9}$$

where, letting $y = (ab/\Delta)\theta_2^2$, $S_4 = 1 + [2y/(\sinh(y))^2] - \operatorname{ctnh}(y)$. In view of the inequality $1 - e^{-y} < y$, $y > 0$, we obtain $S_4 > 0$ so that the right-hand side of (3.9) is positive.

Also, equations (A.9) to (A.11) of the Appendix yield

$$\frac{\partial^2 \log h_a(t)}{\partial \theta_1^2} < 0,$$

and

$$\left(\frac{\partial^2 \log h_a(t)}{\partial \theta_1^2}\right)\left(\frac{\partial^2 \log h_a(t)}{\partial \theta_2^2}\right) - \left(\frac{\partial^2 \log h_a(t)}{\partial \theta_1 \partial \theta_2}\right)^2 > 0.$$

We have therefore shown that condition (ii) above also holds. It follows that there is a unique maximum likelihood estimate $\hat{\boldsymbol{\theta}}_A$, provided by $l(\boldsymbol{\theta})$ in (3.4), unless our data belong to certain events whose probabilities tend to zero as $n \to \infty$.

Since the regularity conditions of Theorem 4.1 on p. 429 of Lehmann (1983) can be easily verified, we conclude that $\hat{\boldsymbol{\theta}}_A$ is globally consistent for $\boldsymbol{\theta}$. Defining the matrix $\Gamma_A(\boldsymbol{\theta})$ by

$$\Gamma_A(\boldsymbol{\theta}) = \begin{bmatrix} \gamma_{11}(\boldsymbol{\theta}) & \gamma_{12}(\boldsymbol{\theta}) \\ \gamma_{12}(\boldsymbol{\theta}) & \gamma_{22}(\boldsymbol{\theta}) \end{bmatrix}^{-1} \tag{3.10}$$

where, using (3.6),

$$\gamma_{11}(\boldsymbol{\theta}) = -E\left[\frac{\partial^2}{\partial \theta_1^2} \log f(\boldsymbol{W}, \boldsymbol{\theta})\right],$$

$$\gamma_{12}(\boldsymbol{\theta}) = -E\left[\frac{\partial^2}{\partial \theta_1 \partial \theta_2} \log f(\boldsymbol{W}, \boldsymbol{\theta})\right]$$

and

$$\gamma_{22}(\boldsymbol{\theta}) = -E\left[\frac{\partial^2}{\partial \theta_2^2} \log f(\boldsymbol{W}, \boldsymbol{\theta})\right],$$

we obtain part (b) of the theorem.

Remark 2. Since the expectations in (3.10) are difficult to obtain in closed-form, usual approximations to the Fisher-information matrix $\Gamma_A(\boldsymbol{\theta})$, including those that use the observed information matrix, may, in practice, be used.

COROLLARY 3.1. *If $n \to \infty$, then the following hold for the reliability estimate $R_x(t_0; \hat{\boldsymbol{\theta}}_A)$, which is defined through (3.2), where $\hat{\boldsymbol{\theta}}_A$ is the m.l.e.*

(i) $R_x(t_0; \hat{\boldsymbol{\theta}}_A) \xrightarrow{\text{pr}} R_x(t_0; \boldsymbol{\theta})$.

(ii) $\sqrt{n}(R_x(t_0; \hat{\boldsymbol{\theta}}_A) - R_x(t_0; \boldsymbol{\theta})) \xrightarrow{L} N(0, \tau_A^2)$, *where* $\tau_A^2 = (\boldsymbol{\eta}')\Gamma_A(\boldsymbol{\theta})(\boldsymbol{\eta})$, $\boldsymbol{\eta}' = (H_x^{(\theta_1)}(t_0), H_x^{(\theta_2)}(t_0))$ *being the vector of partial derivatives given by equations* (A.12) *and* (A.13) *of the Appendix.*

PROOF. Part (i) is straightforward and we obtain part (ii) by applying the δ-method (in view of (3.2)) to the asymptotic distribution in Theorem 3.1(b).

4. Reliability estimation in Situation B

In this section, we discuss the estimation of the vector parameter $\boldsymbol{\nu} = (\mu, \sigma)$ with the assumption that the stopped process $U^*(t)$ can be measured over time independently for each of n systems. We let subscript $i = 1, \dots, n$ refer to the label of the system and subscript $j = 0, \dots, M_i$ denote the label of the proposed observations of $U^*(t)$ over time for the i-th system. For the i-th system, let $U_i^*(t_{ij})$ be the observation at time t_{ij}, $i = 1, \dots, n$; $j = 0, \dots, M_i$. It is important to note that, if $U_i^*(t_{ij}) = 1$, the system is operating at time t_{ij}. However, if $U_i^*(t_{ij-1}) = 1$ and $U_i^*(t_{ij}) = 0$, then we stop gathering data from the i-th system and record the actual time to failure of the i-th system, T_i, where $t_{ij-1} < T_i \le t_{ij}$. Furthermore, we relabel j as m_i and the corresponding proposed observation time, t_{im_i}, is relabeled as T_i. Thus, for $i = 1, \dots, n$, we observe $U_i^*(t_{ik})$, $k = 0, 1, \dots, M_i$ and $V_i = T_i \wedge t_{iM_i}$.

In view of (2.4b), (2.9) and (2.10), assuming that all the n systems are initially working so that $P(U_i^*(0) = 1) = 1$, $i = 1, 2, \dots, n$, we note that the log-likelihood of the data is given by

$$
(4.1) \qquad l(\boldsymbol{\nu}) = \sum_{i=1}^{n} \sum_{j=1}^{M_i} [u_i^*(t_{i,j-1}) u_i^*(t_{ij}) \log P_x(t_{i,j-1}, t_{ij}) + u_i^*(t_{i,j-1})(1 - u_i^*(t_{ij})) \log q_x(t_{i,j-1}, t_{i,j})].
$$

However, we can simplify $l(\boldsymbol{\nu})$ further using the fact that the data from the i-th system is of the type $(1, 1, \dots, 1)$ or $(1, 1, \dots, 1, 0)$, depending on whether $V_i = t_{iM_i}$ or $V_i < t_{iM_i}$ respectively. Indeed, assuming without loss of generality that the failed systems, if any, are the first r systems with failure times $s_1, \dots, s_r$, respectively, we can collapse (4.1) to

$$
(4.2) \qquad l(\boldsymbol{\nu}) = \begin{cases} \left[\sum_{i=1}^{r} \log h_x(s_i)\right] + \sum_{i=r+1}^{n} \log H_x(t_{iM_i}), & \text{if } 0 < r \le n, \\ \sum_{i=1}^{n} \log H_x(t_{iM_i}), & \text{if } r = 0. \end{cases}
$$

It should be noted that (4.2) is the log-likelihood of n i.i.d. observations from the distribution $h_x(t)$, in (2.7), that are censored at the known times t_{iM_i}, $i = 1, 2, \dots, n$. In fact, upon suitable reparametrization of $\boldsymbol{\nu}$, (4.2) is a special case of the likelihood considered by Whitmore (1983). Thus, once an estimate $\hat{\nu}$ of $\boldsymbol{\nu}$ is obtained using the procedure of Whitmore (1983), we can estimate the system reliability during the period $[0, t_0]$ through (2.3d) as $H_x(t_0, \hat{\nu})$.

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Appendix

In the following, we have used notations for hyperbolic functions that are given in Peirce and Foster (1956). Letting $K = ab/t$ and rewriting $\psi_t(a,b)$ in (2.3e), after the reparametrization (3.1), as

$$\psi_t(a,b;\boldsymbol{\theta}) = \theta_2\sqrt{\frac{2}{\pi t}}\exp\left\{-\frac{1}{2t}S(\boldsymbol{\theta})\right\}\sinh(K\theta_2^2), \tag{A.1}$$

where $S(\boldsymbol{\theta})$ is given by $S(\boldsymbol{\theta}) = t^2\theta_1^2 + (a^2+b^2)\theta_2^2 + 2t(a-b)\theta_1\theta_2$, we obtain the following derivatives, which can be used to write down the likelihood equations relating to (3.4), and compute the asymptotic variance-covariance matrix, $\Gamma_A(\boldsymbol{\theta})$ in (3.10).

$$\frac{\partial \log\psi_t}{\partial\theta_1} = (b-a)\theta_2 - t\theta_1, \tag{A.2}$$

$$\frac{\partial \log\psi_t}{\partial\theta_2} = \frac{1}{\theta_2} + \frac{1}{t}[t(b-a)\theta_1 - (a^2+b^2)\theta_2] + 2K\theta_2(\operatorname{ctnh}(K\theta_2^2)), \tag{A.3}$$

$$\frac{\partial^2 \log\psi_t}{\partial\theta_1^2} = -t, \tag{A.4}$$

$$\begin{aligned}\frac{\partial^2 \log\psi_t}{\partial\theta_2^2} = &-\left[\frac{1}{\theta_2^2} + \frac{1}{t}(a^2+b^2)\right]\\ &+ 2K[\operatorname{ctnh}(K\theta_2^2) - 2K\theta_2^2(\operatorname{csch}(K\theta_2^2))^2],\end{aligned} \tag{A.5}$$

$$\frac{\partial^2 \log\psi_t}{\partial\theta_1\partial\theta_2} = b-a, \tag{A.6}$$

$$\frac{\partial \log h_a(t)}{\partial\theta_1} = -(a\theta_2 + t\theta_1), \tag{A.7}$$

$$\frac{\partial \log h_a(t)}{\partial\theta_2} = \frac{1}{\theta_2} - \frac{a}{t}(a\theta_2 + t\theta_1), \tag{A.8}$$

$$\frac{\partial^2 \log h_a(t)}{\partial\theta_1^2} = -t, \tag{A.9}$$

$$\frac{\partial^2 \log h_a(t)}{\partial\theta_2^2} = -\frac{1}{\theta_2^2} - \frac{a^2}{t}, \tag{A.10}$$

$$\frac{\partial^2 \log h_a(t)}{\partial\theta_1\partial\theta_2} = -a. \tag{A.11}$$

Finally, denoting the partial derivatives of $H_x(t;\boldsymbol{\theta})$, given by (2.3d) and (3.1), as $H_x^{(\theta_i)}(t)$, $i = 1,2$, and using (2.3e) we obtain

$$H_x^{(\theta_1)}(t) = \frac{t}{\theta_2}\psi_t\left(x, 0; \frac{\theta_1}{\theta_2}, \frac{1}{\theta_2}\right) \tag{A.12}$$

$$+\left\{2\theta_2 x\Phi\left(\frac{-x\theta_2}{\sqrt{t}}+\theta_1\sqrt{t}\right)\exp[-2\theta_1\theta_2 x]\right\},$$

and

$$H_x^{(\theta_2)}(t) = \frac{x}{\theta_2}\psi_t\left(x, 0; \frac{\theta_1}{\theta_2}, \frac{1}{\theta_2}\right) + \left\{2\theta_1 x\Phi\left(\frac{-x\theta_2}{\sqrt{t}}+\theta_1\sqrt{t}\right)\exp[-2\theta_1\theta_2 x]\right\}, \tag{A.13}$$

References

Apostol, T. M. (1973). *Mathematical Analysis*, 6th ed., Addison-Wesley, Reading, Massachusetts.

Basu, A. P. and Ebrahimi, N. (1983). On the reliability of stochastic systems, *Statist. Probab. Lett.*, **1**, 265–267.

Cinlar, E. (1979). On increasing continuous processes, *Stochastic Process. Appl.*, **9**, 147–154.

Ebrahimi, N. (1982). Estimation of reliability for a series stress-strength system, *IEEE Transactions on Reliability*, **31**, 202–205.

Harrison, M. J. (1985). *Brownian Motion and Stochastic Flow Systems*, Wiley, New York.

Johnson, R. (1988). Stress-strength models for reliability, *Handbook of Statistics*, Vol. 7 (eds. P. R. Krishnaiah and C. R. Rao), 27–54, Elsevier Science, Amsterdam.

Karlin, S. and Taylor, H. M. (1975). *A First Course in Stochastic Processes*, Academic Press, New York.

Kutoyants, Yu. A. (1984). *Parameter Estimation for Stochastic Processes* (translated and edited by B. L. S. Prakasa Rao), Heldermann Verlag, Berlin.

Lehmann, E. L. (1983). *Theory of Point Estimation*, Wiley, New York.

Mäkeläinen, T., Schmidt, K. and Styan, G. P. H. (1981). On the existence and uniqueness of the maximum likelihood estimate of a vector-valued parameter in fixed-size samples, *Ann. Statist.*, **9**, 758–767.

Peirce, B. O. and Foster, R. M. (1956). *A Short Table of Integrals*, 4th ed., Ginn and Co., New York.

Vere-Jones, D. (1970). Stochastic models for earthquake occurrence, *J. Roy. Statist. Soc. Ser. B*, **32**, 1–62.

Whitmore, G. A. (1983). A regression method for censored inverse-Gaussian data, *Canad. J. Statist.*, **11**, 305–315.

Whitmore, G. A. (1990). On the reliability of stochastic systems: comment, *Statist. Probab. Lett.*, **10**, 65–67.