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SOME EXACT EXPRESSIONS FOR THE MEAN AND HIGHER MOMENTS OF FUNCTIONS OF SAMPLE MOMENTS*

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Abstract. Examples of exact expressions for the moments (mainly of the mean) of functions of sample moments are given. These provide checks on alternative developments such as asymptotic series for $n \to \infty$, and simulation processes. Exact expressions are given for the mean of the square of the sample coefficient of variation, particularly in uniform sampling; Frullani integrals studied by G. H. Hardy arise. It should be kept in mind that exact results for (joint) moment generating functions (mgfs) are of interest as they produce a means of obtaining exact results for (cross) moments—including moments with negative indices. Thus an exact expression for the joint mgf of the 1st two noncentral moments can be used to obtain the mean of the $(c.v.)^2$ (but not for the mean of the $c.v.$). A general expression is given for the moment generating function of the sample variance. The limitations of Fisher's symbolic formula for the characteristic function of sample moments (or more general statistics) are noted.

Key words and phrases: Coefficient of variation, Frullani integrals, moment series, sample variance, symbolic characteristic function.

1. Introduction

We may consider the three classes of moments of sample moments.

(i) Moments of positive powers of non-central and central sample moments (m'_s, m_s for a sample of size n) are finite polynomials in n^{-1}: the higher the power and sample moment order s the greater the complication, except for rare special cases such as moments relating to a normal density, or χ^2 for example. If there is interest in the algebraic structure, then a computer approach would apply (REDUCE, MAPLE, etc.); recursive schemes are available for numerical approaches and loss of accuracy has to be kept in mind.

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(ii) As soon as we consider fractional powers of moments (such as the sample standard deviation), the series in n^{-1} becomes infinite and convergence problems emerge; here "convergence" covers the behavior of summation processes and other convergence acceleration devices.

(iii) If we continue to structure a hierarchy of difficulty for sample moments then our next candidate would relate to ratios of moments. Common examples are the sample skewness, sample kurtosis, and coefficient of variation. Unless there is independence of some sort for ratios such as $m_3/m_2^{3/2}$, m_4/m_2^2 under normality, or $m_2/m_1'^2$, $m_3/m_1'^3$ for exponential families, then exact closed forms are few. From a mathematical point of view, some very interesting results for characteristic functions of functions have been given by Good (1968*a*, 1968*b*).

Why should there be interest in this problem? First, if for a statistic t, there is the infinite series

$$E(t) \sim \tau_0 + \tau_1/n + \cdots,$$

no approach has been found (in the general case) to allow the investigation of properties of the *general coefficients* τ_i in terms of a given set of parameters and an arbitrary underlying distribution. Suppose then we can determine $\tau_{20}, \tau_{21}, \ldots, \tau_{50}$ using a computer. Apart from simulation with increased precision, with other safeguards to check the program, etc., we are devoid of scientific checks. Thus the more examples for different statistics and different underlying distributions we can construct providing numerical assessments for $\tau_1, \tau_2, \ldots$, and usable closed forms, the better. The analysis brings together statistical theory, elementary mathematical analysis, and computerized implementation. It is a field in which there is some satisfaction in seeing numerical consistency at work. An alternative approach to the present problem is to use computer algebra; for example, REDUCE, MACSYMA, and MAPLE. References maybe made, for example, to Draper and Tierney (1973), Niki and Konishi (1986) and Niki (1987, 1989). However there are two points to consider. First the expressions may become extremely cumbersome and involve large coefficients (usually produced in exact rational fraction form), so that insights are blurred. Second, if the expressions are converted to numerical form, there may be a convergence problem in the sense that successive approximants do not stabilize.

We give several new examples of moments of ratios of sample moments under uniform sampling and touch on symbolic approaches, including one due to R. A. Fisher. There is also an example giving the mean of the sample standard deviation when the population is a modified normal density.

2. A fundamental formulae

For sampling from the uniform density $U(0,1)$ we have the moment generating function

$$E(e^{-\alpha m_1' - \beta m_2'}) = \left\{ \int_0^1 e^{(-\alpha x - \beta x^2)/n} \right\}^n dx = \Phi^n(\alpha, \beta).$$

for the first two non-central sample moments m_1' and m_2'. By differentiation with respect to β and an integration with respect to α, we have

$$\begin{aligned} E(m_2' e^{-\alpha m_1'}) &= n\Phi^{n-1}(\alpha, \beta)\frac{d\Phi(\alpha,\beta)}{d\beta}\bigg|_{\beta=0} \\ &= \left\{\frac{n}{\alpha}(1-e^{-\alpha/n})\right\}^{n-1}\left\{\frac{2n^3}{\alpha^3} - e^{-\alpha/n}\left(\frac{2n^3}{\alpha^3}+\frac{2n^2}{\alpha^2}+\frac{n}{\alpha}\right)\right\}, \\ &\qquad (n = 1, 2, \ldots) \end{aligned}$$

so that, using

$$\int_0^\infty \alpha^s e^{-m_1'\alpha} d\alpha = s!/m_1'^{s+1},$$

we have for $n = 2, 3, \ldots,$

$$E\left(\frac{m_2'}{m_1'^s}\right) = \frac{n^s}{(s-1)!}\int_0^\infty \left(\frac{1-e^{-x}}{x}\right)^{n-1}\left\{\frac{2-e^{-x}(x^2+2x+2)}{x^{4-s}}\right\}dx, \tag{2.1}$$
$$(n > s-2;\;\; s = 1, 2, \ldots)$$

whereas for $n = 1$, expectation and integral take the values $1/2$ for $s = 1$, 1 for $s = 2$ and ∞ for $s > 2$.

Higher moments are, though more complicated, readily set up; for example, the variance would be derived from the integral form $E(m_2'^2 e^{-\alpha m_1'})$ followed by an adjustment for the square of the mean (see Appendix).

3. The coefficient of variation (c.v.)

3.1 *Exact formula*

For the $(c.v.)^2$ we have for the mean

$$E(m_2/m_1'^2) = E(m_2'/m_1'^2) - 1,$$

and from (2.1)

$$E\left(\frac{m_2'}{m_1'^2}\right) = n^2\int_0^\infty \left(\frac{1-e^{-x}}{x}\right)^{n-1}\left\{\frac{2-e^{-x}(x^2+2x+2)}{x^2}\right\}dx.$$

Now

$$\frac{d}{dx}\left(\frac{1-e^{-x}}{x}\right) = \frac{xe^{-x}-(1-e^{-x})}{x^2} = \frac{e^{-x}(x+1)-1}{x^2}.$$

Hence

$$\begin{aligned} E\left(\frac{m_2'}{m_1'^2}\right) &= -2n^2\int_0^\infty \left(\frac{1-e^{-x}}{x}\right)^{n-1}\left\{-d\left(\frac{1-e^{-x}}{x}\right)\right\} \\ &\quad - n^2\int_0^\infty e^{-x}\left(\frac{1-e^{-x}}{x}\right)^{n-1}dx \\ &= 2n - n^2\int_0^\infty e^{-x}\left(\frac{1-e^{-x}}{x}\right)^{n-1}dx. \end{aligned} \tag{3.1}$$

But

$$\begin{aligned}
&\int_0^\infty e^{-x}\left(\frac{1-e^{-x}}{x}\right)^{n-1} dx \\
&= -\frac{1}{n-2}\int_0^\infty e^{-x}(1-e^{-x})^{n-1} d\frac{1}{x^{n-2}} \qquad (n>2) \\
&= \frac{1}{n-2}\int_0^\infty \frac{D_x\{e^{-x}(1-e^{-x})^{n-1}\}}{x^{n-2}} dx \\
&= \frac{1}{(n-2)!}\int_0^\infty \frac{D_x^{n-2}\{e^{-x}(1-e^{-x})^{n-1}\}}{x} dx \qquad (n\geq 2).
\end{aligned}$$

Now from Hardy (1901) there is the Frullani integral

$$\int_0^\infty \frac{\sum Ae^{-ax}}{x} dx = -\sum A\ln a \tag{3.2a}$$

where $\sum A = 0$, and the real parts of the a's are positive (see also Bromwich (1926)). Using (3.2a) we finally find the exact formula

$$\begin{aligned}
v_n^* = E\left(\frac{m_2'}{m_1'^2}\right) &= 2n + \frac{(-1)^n n^2}{(n-2)!}\sum_{s=0}^{n-1}(-1)^s\binom{n-1}{s}(s+1)^{n-2}\ln(s+1) \\
&\qquad\qquad (n = 2, 3, \ldots) \\
&\to 4/3 \qquad (n\to\infty).
\end{aligned} \tag{3.2b}$$

$$\begin{aligned}
n=2;&\quad v_2^* = 4-4\ln 2 = 1.2274, \\
n=3;&\quad v_3^* = 6+9(4\ln 2-3\ln 3) = 1.2908, \\
n=4;&\quad v_4^* = 8-8(12\ln 2-27\ln 3+16\ln 4) = 1.3124, \\
n=5;&\quad v_5^* = 10+\frac{25}{6}(32\ln 2-162\ln 3+256\ln 4-125\ln 5) = 1.3214. \\
n=7;&\quad v_7^* = 1.3281, \qquad n=10;\quad v_{10}^* = 1.3310, \qquad n=15;\quad v_{15}^* = 1.3324.
\end{aligned}$$

In passing we note that there appears to be no simple approach to the exact moments of the *c.v.* itself, which involves $\sqrt{m_2}$.

3.2 $E(m_2'/m_1')$

Other values of s in (2.1) can be evaluated. A general approach is indicated for the case $s = 1$, for which

$$\begin{aligned}
E\left(\frac{m_2'}{m_1'}\right) & \\
&= n\int_0^\infty \frac{(1-e^{-x})^{n-1}g(x)}{x^{n+2}} dx \qquad (g(x) = 2-e^{-x}(x^2+2x+2);\ n\geq 2) \\
&= \frac{n}{(n+1)!}\int_0^\infty \frac{\{D_x^{n+1}(1-e^{-x})^{n-1}g(x)\}}{x} dx
\end{aligned} \tag{3.3}$$

$$= \frac{n}{(n+1)!}$$
$$\times \int_0^\infty \frac{\{D_x^{n+1} e^{-x}(1-e^{-x})^{n-1}(g(x)-2) + 2D_x^{n+1}(1-e^{-x})^{n-1}\}}{x} dx$$
$$= \frac{n}{(n+1)!}(-J_n + H_n), \qquad \text{say.}$$

The integral in (3.3) is readily seen to equal 1/2 when $n = 1$. Now for the time being ignoring a potential singularity of the integrand at $x = 0$, we have

$$\begin{aligned} J_n &= \int_0^\infty \frac{D_x^{n+1} e^{-x}(1-e^{-x})^{n-1}(x^2+2x+2)}{x} dx \\ &= \int_0^\infty \bigg\{ (x^2+2x+2) D_x^{n+1} e^{-x}(1-e^{-x})^{n-1} \\ &\qquad + (n+1)(2x+2) D_x^n e^{-x}(1-e^{-x})^{n-1} \\ &\qquad + \binom{n+1}{2} 2 D_x^{n-1} e^{-x}(1-e^{-x})^{n-1} \bigg\} / x \, dx \\ &= \int_0^\infty x D_x^{n+1} e^{-x}(1-e^{-x})^{n-1} dx \\ &\quad + \int_0^\infty \frac{(2x+2)}{x} D_x^n \{e^{-x}(1-e^{-x})^{n-1} + (n-1)e^{-x}(1-e^{-x})^{n-2}\} dx \\ &\quad + (n+1)n \int_0^\infty \frac{D_x^{n-1} e^{-x}(1-e^{-x})^{n-1}}{x} dx \end{aligned}$$

and

$$H_n = \int_0^\infty \frac{2D_x^{n+1}(1-e^{-x})^{n-1}}{x} dx.$$

The contributing components for J_n are:

(i) $\displaystyle \int_0^\infty x D_x^{n+1} e^{-x}(1-e^{-x})^{n-1} dx = (n-1)!,$

(ii) $\displaystyle 2\int_0^\infty D_x^n e^{-x}(1-e^{-x})^{n-1} dx = -2(n-1)!,$

(iii) $\displaystyle 2(n-1)\int_0^\infty D_x^n e^{-x}(1-e^{-x})^{n-2} dx = n(n-1)(n-1)!,$

(iv) $\displaystyle (n+1)n \int_0^\infty \frac{D_x^{n-1} e^{-x}(1-e^{-x})^{n-1}}{x} dx,$

(v) $\displaystyle 2\int_0^\infty \frac{D_x^n\{e^{-x}(1-e^{-x})^{n-1} + (n-1)e^{-x}(1-e^{-x})^{n-2}\}}{x} dx.$

As for the singularity at the origin we have for the numerators of the integrand in $J_n - H_n$, when $x \to 0$, the contributions

$$\begin{aligned} n![-(n+1) + \{(n-1)^2 + (n^2-3n+2)(3n-5)/12\} \\ +(n+1) - (n^2-1)(3n-2)/12] = 0. \end{aligned}$$

Using these and H_n along with (3.2a), we finally find

$$E\left(\frac{m_2'}{m_1'}\right) = -\frac{(n^2-n-1)}{n+1} + \frac{(-1)^{n+1}n}{(n+1)!}\sum_{s=0}^{n-1}(-1)^s\binom{n-1}{s} \times (n^2+n-2s-2)(s+1)^{n-1}\ln(s+1) \quad (n=1,2,\ldots) \tag{3.4}$$

$$\to 2/3 \quad (n\to\infty).$$

For example,

$$\begin{aligned}
n=1;&\quad E(m_2'/m_1') = 1/2 = 0.5,\\
n=2;&\quad E(m_2'/m_1') = (4\ln 2 - 1)/3 = 0.5909,\\
n=3;&\quad E(m_2'/m_1') = (-5 - 32\ln 2 + 27\ln 3)/4 = 0.6205,\\
n=4;&\quad E(m_2'/m_1') = (-11 + 64\ln 2 - 189\ln 3 + 128\ln 4)/5 = 0.6339,\\
n=5;&\quad E(m_2'/m_1') = (-114 - 416\ln 2 + 2916\ln 3 - 5632\ln 4 + 3125\ln 5)/36.
\end{aligned}$$

3.3 $E(m_2'/m_1'^s)$, $s=3$, $s=4$

Without going into details we have by similar methods

$$E\left(\frac{m_2'}{m_1'^3}\right) = \frac{(-1)^n n^3}{2!(n-1)!}\left\{\left(n-\frac{3}{2}\right)(-1)^{n-1}\delta_{n,2} + \sum_{s=0}^{n-1}(-1)^{s-1}\binom{n-1}{s} \times (n^2-3n+2s+4)(s+1)^{n-3}\ln(s+1)\right\} \quad (n=2,3,\ldots) \tag{3.5}$$

$$\to 8/3 \quad (n\to\infty).$$

$$E\left(\frac{m_2'}{m_1'^4}\right) = \frac{n^4}{3!(n-2)!}\left\{-\frac{1}{3}\delta_{n,3} + (-1)^{n-2}\sum_{s=0}^{n-1}(-1)^s\binom{n-1}{s} \times (n^2-5n+4s+10)(s+1)^{n-4}\ln(s+1)\right\} \quad (n=3,4,\ldots) \tag{3.6}$$

$$\to 16/3 \quad (n\to\infty).$$

Here $\delta_{n,r}$ is the kronecker delta function. Formulas (3.3), (3.4), (3.5) and (3.6) are useful for small to moderate values of n. We next consider the case n large, and develop series in descending powers of n.

4. c.v. for large n in uniform sampling

From (3.1)

$$E\left(\frac{m_2'}{m_1'^2}\right) = 2n - n\int_0^\infty \frac{1}{x^{n-1}} d(1-e^{-x})^n$$
$$= 2n - n(n-1)\int_0^\infty \left(\frac{1-e^{-x}}{x}\right)^n dx.$$

In particular consider

$$J_n = \int_0^\infty \left(\frac{1-e^{-x}}{x}\right)^n dx \quad (n = 2, 3, \ldots). \tag{4.1}$$

Now from the g.f. for Bernoulli numbers

$$\frac{x}{e^x - 1} = 1 + \frac{B_1 x}{1!} + \frac{B_2 x^2}{2!} + \cdots \quad (B_{2s+1} = 0)$$

with $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$, etc., we have formally

$$J_n = \int_0^\infty \exp\left(-\frac{nx}{2} + \frac{nb_2x^2}{2!} + \frac{nb_4x^4}{4!} + \cdots\right) dx \quad (b_s = B_s/s)$$
$$= \frac{2}{n}\int_0^\infty \exp\left(-t + \frac{K_2t^2}{2!} + \frac{K_4t^4}{4!} + \cdots\right) dt$$

where

$$K_s = 2^s b_s / n^{s-1} \quad (n = 2, 3, \ldots;\ s = 2, 3, \ldots).$$

Hence from the cumulant-moment conversion formula (Kendall and Stuart, Vol. 1 (1958), p. 70)

$$J_n = \frac{2}{n}(1 + \hat\mu_2 + \hat\mu_4 + \cdots),$$

where

$$\hat\mu_2 = 2^2 b_2/n = K_2,$$
$$\hat\mu_4 = \frac{3(2^2b_2)^2}{n^2} + \frac{2^4 b_4}{n^3} = 3K_2^2 + K_4,$$
$$\hat\mu_6 = \frac{15(2^2b_2)^3}{n^3} + \frac{15\cdot 2^6 b_4 b_2}{n^4} + \frac{2^6 b_6}{n^5} = 15K_2^3 + 15K_4K_2 + K_6,$$
$$\hat\mu_8 = 2^8\left(\frac{105b_2^4}{n^4} + \frac{210b_4b_2^2}{n^5} + \frac{28b_6b_2 + 35b_4^2}{n^6} + \frac{b_8}{n^7}\right)$$
$$= 105K_2^4 + 210K_4K_2^2 + 28K_6K_2 + 35K_4^2 + K_8,$$
$$\hat\mu_{10} = 2^{10}\left(\frac{945b_2^5}{n^5} + \frac{3150b_4b_2^3}{n^6}\right.$$
$$\left. + \frac{630b_6b_2^2 + 1575b_4^2b_2}{n^7} + \frac{45b_8b_2 + 210b_6b_4}{n^8} + \frac{b_{10}}{n^9}\right)$$
$$= 945K_2^5 + 3150K_4K_2^3 + 630K_6K_2^2$$
$$+ 1575K_4^2K_2 + 45K_8K_2 + 210K_6K_4 + K_{10},$$

and so on. The usual partition formula is readily recognized. Coefficients of powers of n^{-1} may now be collected. Numerically, we prefer the recursive scheme

$$\hat{\mu}_{2s} = \sum_{r=1}^{s} \binom{2s-1}{2r-1} K_{2r} \hat{\mu}_{2s-2r} \qquad (\hat{\mu}_0 = 1)$$

and this leads to the series in powers of n^{-1} for

$$\text{(4.2)} \qquad \frac{n}{2} J_n = \frac{n}{2} \int_0^\infty \left(\frac{1-e^{-x}}{x} \right)^n dx \sim 1 + \hat{\mu}_2 + \hat{\mu}_4 + \cdots$$
$$\sim 1 + \frac{a_1}{n} + \frac{a^2}{n^2} + \cdots .$$

Table 1 gives $a_1, a_2, \ldots, a_{16}$.

Table 1. Values of a_s.

s	a_s	s	a_s
1	0.3333333333333333D+00	2	0.3333333333333333D+00
3	0.4222222222222222D+00	4	0.6296296296296294D+00
5	0.1031746031746031D+01	6	0.1696296296296293D+01
7	0.2279012345678993D+01	8	-0.1604938271606215D+00
9	-0.2118645716423543D+02	10	-0.1292597687634734D+03
11	-0.5763552148810995D+03	12	-0.1991344764517038D+04
13	-0.3483610577651300D+04	14	0.2387028998628259D+05
15	0.3625193832972050D+06	16	0.3089531708503723D+07

Series for

$$E\left(\frac{m_2'}{m_1'^2}\right) = 2n - n(n-1)J_n \quad \text{and} \quad E\left(\frac{m_2}{m_1'^2}\right) = 2n - n(n-1)J_n - 1$$

are readily set up. Using the Computer Oriented Extended Taylor Series (COETS) algorithm (Bowman and Shenton (1989)), we give the first 31 terms in Table 2.

In particular

$$E\left(\frac{m_2'}{m_1'^2}\right) \sim \frac{4}{3} - \frac{8/45}{n^2} - \frac{56/135}{n^3} - \frac{152/189}{n^4} + \cdots \qquad (n \to \infty)$$

and terms to the coefficient of n^{-30} are available from Table 2. Several problems now become obvious

(a) Are the coefficients derived from the computer oriented algorithm for moments of sample moments (COETS) reasonably accurate? There is almost certain to be some loss of accuracy especially for the higher coefficients. As far as we are aware there is no alternate scheme to check so many terms.

Table 2. COETS series for $(n/2)\int_0^\infty ((1-e^{-x})/x)^n dx$.

s	a_s	s	a_s
1	0.3333333333333334D+00	2	0.3333333333333334D+00
3	0.4222222222222222D+00	4	0.6296296296296294D+00
5	0.1031746031746031D+01	6	0.1696296296296295D+01
7	0.2279012345679012D+01	8	-0.1604938271604834D+00
9	-0.2118645716423501D+02	10	-0.1292597687634692D+03
11	-0.5763552148811218D+03	12	-0.1991344764517476D+04
13	-0.3483610577654536D+04	14	0.2387028998629938D+05
15	0.3625193832976514D+06	16	0.3089531708510458D+07
17	0.2053816997826113D+08	18	0.1040457541926382D+09
19	0.2349105178666393D+09	20	-0.3198213561876769D+10
21	-0.6357113282323224D+11	22	-0.7425101947496909D+12
23	-0.6737623308322023D+13	24	-0.4635173657612202D+14
25	-0.1480955497222340D+15	26	0.2296049254802481D+16
27	0.6145083428086140D+17	28	0.9335122325719160D+18
29	0.1091018294933994D+20	30	0.9688935218900340D+20
31	0.4363522376894407D+21		

(b) Questions of assessing a *value* from the series especially if there is apparent divergence, lead to the use of summation procedures (usually Padé rational fraction sequences in n). This approach is at its worst when there are series sequences of one-signed terms, as in Table 2. Direct summation sometimes works well.

(c) Assessments of the moments, such as $E(m_2'/m_1'^s)$ in general will depend on the success of (b) in the sense of stability of successive approximants and comparison with other procedures—a last resort is to use simulation with some attempt here at error assessment.

(d) In a few cases we may attempt to derive moment values for moderate size samples of n by a resort to quadrature in n dimensions. Note that the COETS approach should improve for moderate to large n whereas quadrature possibly deteriorates in this case or maybe economically expensive.

(e) *Since there are few cases of exact closed forms for moments of sample moments, it is clearly important to record a few which are non-trivial* and relevant. To highlight some of these aspects, see the numerical comparison in Table 3.

This is a truly amazing series, with a sign pattern of periodicity twelve (there are 8 positive terms in the first cycle). Moreover, it is most gratifying to see that to 17 terms, there is agreement to at least six significant digits with the completely independent approach using the asymptotic development of the integral form in (4.1); incidentally this *does not* imply that there is disagreement for higher order terms—we have merely not extended the formula for enough terms.

Note also that we have *not* proved that the series continues to have a periodicity of 12, with increasing positive and negative signs; it is a conjecture from the calculations as they now stand. It appears to be extremely difficult to settle the problem concerning the sign of the coefficient of n^{-s}. One may consider, for example, the much simpler problem of the sign of B_{2s}, a Bernoulli number.

Table 3. $E(m'_2/m'^2_1)$: Comparisons of exact values, direct sum to n^{-8}, and Padé approximants.

n	Exact	Direct sum	Padé Approx.
2	1.2274113	1.1638916	
3	1.2907667	1.2834017	
4	1.3124468	1.3113096	1.3015984
5	1.3214017	1.3211787	1.3225076
7	1.3280928	1.3280778	1.3281027
10	1.3310471	1.3310463	1.3310472
15	1.3324023	1.3324026	1.3324026

5. Symbolic approaches

There appears to be no solution in general sampling to the moment problem (i.e. the expectation of a function of sample moments) for forms in which there is a ratio with m_s $(s = 2, 3, \ldots)$ in the denominator. For examples, we have the sample skewness $(\sqrt{b_1} = m_3/m_2^{3/2})$ and kurtosis (m_4/m_2^2); these of course can be treated under normality because of independence with m_2. We note however one symbolic approach.

Fisher (1930) stated that if $M(t_1, t_2, \ldots)$ is the c.f. of the simultaneous distribution of $x_1, x_2, \ldots$ and if $M'(\tau_1, \tau_2, \ldots)$ is the c.f. of the variates $\zeta_1, \zeta_2, \ldots$ with

$$\zeta_1 = f_1(x_1, x_2, \ldots), \quad \zeta_2 = f_2(x_1, x_2, \ldots), \quad \ldots,$$

then

$$M'(\tau_1, \tau_2) = e^{\tau_1 f_1 + \tau_2 f_2 + \cdots} M(t_1, t_2, \ldots) \,|_{\underline{t}=0}, \tag{5.1}$$

where f_p in the index represents

$$f_p(d_1, d_2, \ldots) \qquad \left(d_1 = \frac{d}{dt_1}, d_2 = \frac{d}{dt_2}, \ldots\right).$$

(Note that Fisher's c.f. drops the usual $i = \sqrt{-1}$.)

The result in (5.1) is not particularly useful except for structures involving the normal, for the succession of derivatives accumulates algebraic complications. However, there is an interesting application in the construction of the c.f. for a set of central sample moments. Whereas for non-central sample moments

$$E(e^{\alpha_1 m'_1 + \alpha_2 m'_2 + \cdots}) = \left[E\left\{\int e^{(\alpha_1 x_1 + \alpha_2 x^2 + \cdots)/n} d\sigma(x)\right\}\right]^n$$

$\sigma(\cdot)$ being a distribution function, this breaks down for $E(e^{\alpha_1 m'_1 + \alpha_2 m_2 + \alpha_3 m_3 + \cdots})$, because of the powers and products involved, such as m'^2_1, m'_1, m'_2, etc. We do have an answer from (4.2). Let

$$\begin{aligned} x_1 &= m'_1 - \mu'_1, \\ x_2 &= m'_2 - 2m'_1\mu'_1 + \mu'^2_1, \\ x_3 &= m'_3 - 3m'_2\mu'_1 + 3m'_1\mu'^2_1 - \mu'^3_1 \end{aligned}$$

etc., and for example

$$\begin{aligned}
\zeta_1 &= m_1',\\
\zeta_2 &= m_2' - m_1'^2 = m_2,\\
\zeta_3 &= m_3' - 3m_2'm_1' + 2m_1'^3 = m_3,\\
\zeta_4 &= m_4' - 4m_3'm_1' + 6m_2'm_1'^2 - 3m_1'^4 = m_4.
\end{aligned}$$

Then

$$\begin{aligned}
(5.2)\qquad & E(e^{\tau_1\zeta_1+\tau_2\zeta_2+\tau_3\zeta_3+\tau_4\zeta_4})\\
&= e^{\tau_1 f_1+\tau_2 f_2+\tau_3 f_3+\tau_4 f_4} E(e^{t_1x_1+t_2x_2+t_3x_3+t_4x_4})\,|_{\underline{t}=0}\\
&= e^{\sum_1^4 \tau_s f_s}[E\{e^{(t_1X+t_2X^2+t_3X^3+t_4X^4)/n}\}]^n
\end{aligned}$$

where $X = x - \mu_1'$, and

$$\begin{aligned}
f_1 &= d_1,\\
f_2 &= d_2 - d_1^2,\\
f_3 &= d_3 - 3d_2d_1 + 2d_1^3,\\
f_4 &= d_4 - 4d_3d_1 + 6d_2d_1^2 - 3d_1^4.
\end{aligned}$$

Fisher apparently only considered normal populations, but no doubt he was aware of the serious limitations inherent in the general formula (5.2). For example, we have to find

$$e^{\sum_1^4 \tau_s f_s}\{\Phi(t_1, t_2, t_3, t_4)\}^n$$

which involves

(a) a closed form $\Phi(\underline{t})$,

and

(b) Faà di Bruno's (1876) formula for derivatives of a function of a function.

Note that in (5.2) only central moments of the population are involved. As a particular case, we have for the second sample moment m_2

$$\begin{aligned}
(5.3)\quad & E(e^{\tau_2 m_2})\\
&= e^{\tau_2(d_2-d_1^2)}\left\{\int e^{(t_1X+t_2X^2)/n}d\sigma(x)\right\}^n\Bigg|_{\underline{t}=0} \qquad (X = x - \mu_1')\\
&= e^{\tau_2 d_2-\tau_2 d_1^2}\left\{\Phi_0\left(\frac{t_2}{n}\right) + \frac{t_1}{n}\Phi_1\left(\frac{t_2}{n}\right) + \frac{t_1^2}{2!n^2}\Phi_2\left(\frac{t_2}{n}\right) + \cdots\right\}^n\Bigg|_{\underline{t}=0}\\
&= e^{-\tau_2 d_1^2}\left\{\Phi_0\left(\frac{\tau_2}{n}\right) + \frac{t_1}{n}\Phi_1\left(\frac{\tau_2}{n}\right) + \frac{t_1^2}{2!n^2}\Phi_2\left(\frac{\tau_2}{n}\right) + \cdots\right\}^n\Bigg|_{t_1=0}
\end{aligned}$$

where

$$\begin{aligned}
\Phi_0\left(\frac{\tau_2}{n}\right) &= 1 + \frac{\tau_2}{n}\mu_2 + \frac{\tau_2^2}{2!n^2}\mu_4 + \cdots,\\
\Phi_1\left(\frac{\tau_2}{n}\right) &= 0 + \frac{\tau_2}{n}\mu_3 + \frac{\tau_2^2}{2!n^2}\mu_5 + \cdots,\\
\Phi_2\left(\frac{\tau_2}{n}\right) &= \mu_2 + \frac{\tau_2}{1!n}\mu_4 + \frac{\tau_2^2}{2!n^2}\mu_6 + \cdots, \qquad \text{etc.}
\end{aligned}$$

Introducing the expansion of the multinomial in (5.3), we have

(5.4) $E(e^{\tau_2 m_2})$

$$= \sum_{r=0}^{\infty} \frac{(-\tau_2)^r}{r! n^{2r}} \sum_{(r)} \left(\frac{\Phi_1(\tau_2/n)}{1!} \right)^{\pi_1} \left(\frac{\Phi(\tau_2/n)}{2!} \right)^{\pi_2} \cdots \left(\frac{\Phi_n(\tau_n/n)}{m!} \right)^{\pi_m}$$
$$\times \frac{(2r)!}{\pi_1! \pi_2! \cdots \pi_m!} n^{(\sum \pi)} \{\Phi_0(\tau_2/n)\}^{n-\sum \pi},$$

the inner summation being over (r)

$$p_1\pi_1 + p_2\pi_2 + \cdots + p_m\pi_m = 2r \qquad (\pi \text{ \& } p \text{ non-negative integers}).$$

Thus

$$E(e^{\tau_2 m_2}) = \sum_{0}^{\infty} \frac{(-\tau_2)^r}{r!} H_r\left(\frac{\tau_2}{n}\right),$$

where

(i) $H_0 = \Phi_0^n(\tau_2/n)$,

(ii) $H_1 = \{(n-1)\Phi_1^2\Phi_0^{n-2} + \Phi_2\Phi_0^{n-1}\}/n$,

(iii) $H_2 = \{(n-1)^{(3)}\Phi_1^4\Phi_0^{n-4} + 6(n-1)^{(2)}\Phi_2\Phi_1^2\Phi_0^{n-3} + 3(n-1)\Phi_2^2\Phi_0^{n-2} + 4(n-1)\Phi_3\Phi_1\Phi_0^{n-2} + \Phi_4\Phi_0^{n-1}\}/n^3$

and so on. There is an isomorphism with the formulas for non-central moments in terms of cumulants. The combinatorial aspect of the partitions occurring in (5.4) can be solved and the scheme implemented by computer algebra.

6. Integration and a symbolic method

The derivative approach hits a problem when fractional moments are involved. Fisher remarked on this in considering the simultaneous distribution of the cumulant ratios $\gamma = k_3/k_2^{3/2}$, $\delta = k_4/k_2^2$, $\mu = m_6/k_2^3, \ldots$, under normality, but was able to avoid negative indices (related to $k_2^{-3/2}$, $k_2^{-2}, \ldots$) from the independence of k_2 and $\gamma, \delta, \ldots$. Even so his operator contained terms such as $D_3 D_2^{3/2}$, $D_6 D_2^{5/2}$. How these were to be interpreted he did not explain. Take a simple case.

Now consider

$$E(e^{-\alpha m_2' + \beta m_1'}) = \left\{ \int_{-\infty}^{\infty} e^{(\beta x - \alpha x^2)/n} d\sigma(x) \right\}^n$$

so that using integration on β we have

$$\int_{-\infty}^{\infty} e^{-k\beta^2} E(e^{-\alpha m_2' + \beta m_1'}) d\beta = \sqrt{\pi/k} E(e^{-\alpha m_2' + m_1'^2/4k}) = 2\sqrt{\pi\alpha} E(e^{-\alpha m_2})$$

with $k = 1/(4\alpha)$.

Thus, formally,

$$E(e^{-\alpha m_2}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2} \left\{ \int_{-\infty}^{\infty} e^{(-\alpha x^2 + xt\sqrt{2\alpha})/n} d\sigma(x) \right\}^n dt.$$

Now there will be some interest in setting up an expression for the mean value of the standard deviation (or $\sqrt{m_2}$ as the definition). This will require a derivative-integration operator. Thus

$$E(\sqrt{m_2}\,) = E\left\{ \int_0^{\infty} \frac{m_2 e^{-\alpha m_2}}{\sqrt{\pi\alpha}} \right\} d\alpha$$

and

$$\begin{aligned}(6.1)\quad E(m_2 e^{-\alpha m_2}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2} &\left\{ \int_{-\infty}^{\infty} e^{(-\alpha x^2 + xt\sqrt{2\alpha})/n} d\sigma(x) \right\}^{(n-1)} \\ \times &\int_{-\infty}^{\infty} \left(x^2 - \frac{xt}{\sqrt{2\alpha}} \right) e^{(-\alpha x^2 + xt\sqrt{2\alpha})/n} d\sigma(x) dt\end{aligned}$$

provided the integrals converge and $E(\sqrt{m_2}\,)$ exists.

Example. Take the standard deviation and the modified normal

$$f(x) = x^2 e^{-x^2/2}/\sqrt{2\pi} \qquad (-\infty < x < \infty)$$

with central moments

$$\begin{aligned}\mu_{2s+1} &= 0, \\ \mu_{2s} &= 1 \cdot 3 \cdot 5 \cdots (2s+1).\end{aligned}$$

Formula (6.1) gives the correct answer for the normal population $N(0,1)$, and this modified normal happens to work out in closed form; the majority of cases investigated raise problems.

Now for $E(\sqrt{m_2}\,)$ in sampling from $N(0,1)$, with sample size n^*

$$\begin{aligned}E(\sqrt{m_2}\,) &= (n^* - 1)\Gamma(n^*/2)/\left\{ \sqrt{2n^*}\,\Gamma\left(\frac{n^*+1}{2}\right) \right\} \\ &\sim a_0 + a_1/n^* + a_2/n^{*2} + \cdots \qquad (n^* \to \infty) \\ &= Z(n^*) \qquad \text{say},\end{aligned}$$

where

$$(6.2)\qquad \begin{aligned}a_0 &= 1.0, \\ a_1 &= -0.75, \\ a_2 &= -0.21875, \\ a_3 &= -0.0703125, \\ a_4 &= 0.02880859375,\end{aligned}$$

etc. (see Bowman and Shenton (1988), p. 38). The modified normal uses a modified form of this series. Omitting details, we find the exact result

$$E(\sqrt{m_2}\,) = \frac{\sqrt{n}}{2}\frac{\Gamma(3n/2)}{\Gamma(3n/2+1/2)}\sum_{s=0}^{n}\left(\frac{(2s)!}{2^s s!}\right)^2\frac{1}{(2n)^s}\binom{n}{s}\frac{\Gamma(3n/2-s)}{\Gamma(3n/2)}A_s(n)$$

where

$$A_s(n) = 3 - \frac{3(2s+1)}{n} + \frac{2s(2s+1)}{n^2}.$$

Table 4. Series for the moments of $E\{\sqrt{m_2/\mu_2}\}$.

s	μ_1'	μ_2	μ_3	μ_4
1	-5.833333333333334D-01	1.666666666666667D-01	0.000000000000000D+00	0.000000000000000D+00
2	-3.298611111111111D-01	3.194444444444444D-01	2.777777777777777D-02	8.333333333333333D-02
3	-4.898726851851850D-01	5.949074074074070D-01	-5.393518518518521D-01	3.194444444444444D-01
4	-1.213089554398148D+00	1.745852623456790D+00	-1.640142746913579D+00	2.236689814814814D+00
5	-4.181090555073300D+00	6.623730066872426D+00	-6.996117862654319D+00	8.798900462962957D+00
6	-1.814983592929494D+01	3.038145549339824D+01	-3.504435673466439D+01	4.412225919495928D+01
7	-9.332867566471882D+01	1.615356655181900D+02	-1.980130349409097D+02	2.532488412530088D+02
8	-5.426539134798546D+02	9.588825315439080D+02	-1.226949374638573D+03	1.602392530820614D+03
9	-3.408313372843368D+03	6.094033219706652D+03	-8.052894321512260D+03	1.079852797502544D+04
10	-2.169801765530414D+04	3.890871425434631D+04	-5.284583746696139D+04	7.371105405524946D+04
11	-1.209600165851249D+05	2.134472722083956D+05	-2.997071512374554D+05	4.557306557989062D+05
12	-2.096737849023051D+05	2.581471833573875D+05	-4.789612983999342D+05	1.477645165888082D+06
13	1.156794357432186D+07	-2.349775985396583D+07	3.095114027674524D+07	-2.899293355996856D+07
14	3.387434700043956D+08	-6.643573948211530D+08	9.057570711999676D+08	-1.025258968948889D+09
15	7.397013084643852D+09	-1.439209917202119D+10	1.991045223529389D+10	-2.375602046372978D+10
16	1.499883569410377D+11	-2.911152934686309D+11	4.065704796359189D+11	-4.979196673139020D+11
17	2.991950474008752D+12	-5.803688506041115D+12	8.163206939879479D+12	-1.016113985418120D+13
18	5.999677363970388D+13	-1.163958873755611D+14	1.646561408020631D+14	-2.073272328338878D+14
19	1.220027026165194D+15	-2.367916129820232D+15	3.365826749091160D+15	-4.275861149418266D+15
20	2.522838874276106D+16	-4.899046200924357D+16	6.992651235806511D+16	-8.948474276530232D+16
21	5.301427747803464D+17	-1.029979919520193D+18	1.475581061374459D+18	-1.900407130947212D+18
22	1.128267210833465D+19	-2.192882787821210D+19	3.152204978850890D+19	-4.083936874590458D+19
23	2.416552582124434D+20	-4.697693800744645D+20	6.774341542269927D+20	-8.828950590706916D+20
24	5.153283361327697D+21	-1.001659317032084D+22	1.449022647166871D+22	-1.900809191129418D+22
25	1.073487283717417D+23	-2.085119266319057D+23	3.026687009128129D+23	-4.002594145344365D+23
26	2.101375438734438D+24	-4.073763042318933D+24	5.938844427789093D+24	-7.949840888974391D+24
27	3.488408606235305D+25	-6.723821854703140D+25	9.876975029813245D+25	-1.356420101894680D+26
28	2.865335923323225D+26	-5.305788708392090D+26	8.101267066266283D+26	-1.273488613682888D+27
29	-1.351239552903729D+28	2.738443883711636D+28	-3.861006517320831D+28	4.369333734170663D+28
30	-1.098182922360105D+30	2.179200250012949D+30	-3.137123162914755D+30	3.892230954257207D+30

Asymptotically,

$$E(\sqrt{m_2}\,) \sim Z_0(N)\left\{A_0(n)B_0(n) + \frac{A_1(n)B_1(n)}{n} + \frac{A_2(n)B_2(n)}{n^2} + \cdots\right\} \qquad (n \to \infty),$$

where

$$\begin{aligned}
&\text{(a)} \quad B_0(n) = 1,\\
&\text{(b)} \quad B_s(n) = \frac{(2s-1)^2}{3s}\left\{\frac{1-(s-1)/n}{1-2s/n}\right\} B_{s-1}(n) \qquad (s = 1, 2, \ldots).\\
&\text{(c)} \quad Z_0(n) = \frac{1}{\sqrt{3}}\sum_{s=0}^{\infty}\left\{\frac{1}{(3n)^s}\sum_{r=0}^{s} a_r\right\},
\end{aligned}$$

and $a_0, a_1, \ldots,$ are given in (6.2).

Series for the first four moments of $\sqrt{m_2}$ derived by COETS, are displayed in Table 4.

For the mean, for example, the modulus of successive coefficients increases showing moderate divergence, whereas the sign pattern exhibits unusual subsets of like-signs, making summation techniques fragile unless n exceeds 10 or so. For the variance the sign pattern appears to be indicating a periodicity of 24. This aspect will receive further study.

7. General case

In response to a referee we have reconsidered the general case of (2.1), namely

$$E(m'_r/m_1'^s) = H(r, s, n) \qquad (U(0,1) \text{ sampling}),$$

r and s being positive integers. For the integral representation use the r-th derivative of the mgf, to derive

$$H(r,s,n) = \frac{r!n^s}{(s-1)!}\int_0^\infty \left(\frac{1-e^{-x}}{x}\right)^{n-1}\left(1 - e^{-x}\sum_{k=0}^{r}\frac{x^k}{k!}\right)\frac{dx}{x^{r-s+2}}$$
$$(n - s + r > 0;\ \ n, s = 1, 2, \ldots).$$

Following the approach of Subsection 3.1, we find by successive integration by parts

$$\text{(7.1)} \quad H(r,s,n) = K(r,s,n)\int_0^\infty\left\{\frac{D_x^R(1-e^{-x})^{n-1}\left(1 - e^{-x}\sum_{k=0}^{r}\frac{x^k}{k!}\right)}{x}\right\}dx$$
$$(R = n - s + r,\ \ K(r,s,n) = r!n^s/\{(s-1)!R!\}).$$

The integrand is now seen to consist of two components (see (3.3) for example), the second leading to

$$J_n(r,s) = -K(r,s,n)\int_0^\infty \left\{\frac{D_x^R \sum_{k=1}^r \frac{x^k}{k!}e^{-x}(1-e^{-x})^{n-1}}{x}\right\} dx.$$

In this expression there is a Frullani component (involving logarithmic terms) and a non-Frullani component; for the Frullani parts we must incorporate one term from (7.1), i.e.

$$K(r,s,n)\int_0^\infty \frac{D_x^R(1-e^{-x})^{n-1}dx}{x} \qquad (R>0).$$

The non-Frullani integral arises from

$$\begin{aligned}\Bigg\{\frac{x^{r-1}}{r!}D_x^R &+ \frac{x^{r-2}}{(r-1)!}\left(D_x^R+\binom{R}{1}D_x^{R-1}\right)\\ &+\frac{x^{r-3}}{(r-2)!}\left(D_x^R+\binom{R}{1}D_x^{R-1}+\binom{R}{2}D_x^{R-2}\right)\\ &+\cdots+\frac{x}{2!}\left(D_x^R+\binom{R}{1}D_x^{R-1}+\cdots+\binom{R}{r-1}\right)\Bigg\}e^{-x}(1-e^{-s})^{n-1}.\end{aligned}$$

Here it seems simplest to expand $e^{-x}(1-e^{-x})^{n-1}$ and collect terms. Finally then, we have

$$\begin{aligned}(7.2)\quad &\frac{H(r,s,n)}{K(r,s.n)}\\ &= (-1)^R\sum_{\mu=1}^n(-1)^{\mu+1}\Bigg\{\binom{n-1}{\mu}\mu^R\\ &\qquad\qquad + \binom{n-1}{\mu-1}\sum_{\lambda=0}^r(-1)^\lambda\binom{R}{\lambda}\mu^{R-\lambda}\Bigg\}\ln\mu\\ &+(-1)^R\sum_{\lambda=1}^r\sum_{\mu=0}^{\lambda-1}\sum_{\nu=0}^{n-1}\frac{(-1)^{\nu-\mu-1}\binom{R}{\mu}\binom{n-1}{\nu}(\nu+1)^{n-\mu+\lambda-s-1}}{(r-\lambda+1)}\\ &\qquad\qquad (R>0, n>0, s=1,2,\ldots).\end{aligned}$$

The special cases given in (3.2b), (3.4), (3.5) and (3.6) agree with (7.2); the triple summatory term in certain cases may be simplified using binomial sums as

$$1-\binom{n-1}{1}2^{n-2}+\binom{n-1}{2}3^{n-2}+\cdots=\delta_{n,1}.$$

$$1-\binom{n-1}{1}2^{n-1}+\binom{n-1}{2}3^{n-1}+\cdots=(-1)^{n-1}(n-1)!,$$

$$1-\binom{n-1}{1}2^{n}+\binom{n-1}{2}3^{n}+\cdots=\frac{1}{2}(-1)^{n+1}(n+1)!,$$

derived from derivatives at $x = 0$ of

$$\left(x - \frac{x^2}{2!} + \frac{x^3}{3!} - \cdots\right)^{n-1}.$$

For $r = 2$, there is the general result for $s = 3, 4, \ldots$,

$$E\left(\frac{m_2'}{m_1'^s}\right) = -\frac{1}{2}\frac{\delta_{n,s-1}}{(s-1)} + \frac{(-1)^{n-s}n^s}{(n-s+2)!(s-1)!} \times \sum_{\lambda=0}^{n-1}(-1)^{n-1}(-1)^{\lambda}\binom{n-1}{\lambda}(\lambda+1)^{n-s}A(n,s;\lambda)\ln(\lambda+1)$$

where $A(n,s;\lambda) = n^2 - (2s-3)n + (s-2)(s+1) + 2(s-2)\lambda \; (n > s-2)$.

Appendix

Higher moments of m_2'/m_1' and a uniform distribution $U(0,1)$
Let

$$H^*(\alpha,\beta) = E(e^{-\alpha m_1' + \beta m_2'}) = H^n(\alpha,\beta)$$

where

$$H(\alpha,\beta) = \int_0^1 e^{(-\alpha x + \beta x^2)/n}dx.$$

Define

$$H(\alpha,0) = I(\alpha) = (1 - e^{\alpha/n})n/\alpha,$$
$$A_{2s}(\alpha) = \int_0^1 x^{2s}e^{-\alpha x/n}dx/n^s,$$
$$M_r(\alpha) = E(m_2'^r e^{-\alpha m_1'}),$$

where

$$A_{2s} = \frac{n^{s+1}}{\alpha^{2s+1}}\left\{(2s)! - e^{-\alpha/n}\sum_{r=0}^{2s}\binom{2s}{r}(2s-r)!\left(\frac{\alpha}{n}\right)^r\right\}.$$

Moreover

$$M_1(\alpha) = nI^{n-1}(\alpha)A_2(\alpha),$$
$$M_2(\alpha) = n^{(2)}I^{n-2}(\alpha)A_2^2(\alpha) + nI^{n-1}(\alpha)A_4(\alpha),$$
$$M_3(\alpha) = n^{(3)}I^{n-3}(\alpha)A_2^3(\alpha) + 3n^{(2)}I^{n-2}A_2(\alpha)A_4(\alpha) + nI^{n-1}(\alpha)A_6(\alpha),$$
$$M_4(\alpha) = n^{(4)}I^{n-4}(\alpha)A_2^4(\alpha) + 6n^{(3)}I^{n-3}(\alpha)A_2^2(\alpha)A_4(\alpha) + 4n^{(2)}I^{n-2}(\alpha)A_2(\alpha)A_6(\alpha) + 3n^{(2)}I^{n-2}(\alpha)A_4^2(\alpha) + nI^{n-1}(\alpha)A_8(\alpha),$$

and these are examples of Faà di Bruno's (1876) formula for the s-th derivative of a function of a function (note that this formula is isomorphic to the s-th derivative of $\exp(f(x))$, $f(x)$ at least being differentiable). In general these formulas up to M_4 will provide basic expressions occurring in the first four non-central moments of m_2'/m_1'.

Finally the required moments are derived by evaluating terms

$$\{1/\Gamma(s)\} \int_0^\infty \alpha^{s-1} M_s(\alpha) d\alpha \qquad (s = 1, 2, 3, 4)$$

which are assumed to exist for specified values of n, the sample size.

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