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ON BILINEAR FORMS IN NORMAL VARIABLES

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Abstract. Bilinear forms in normal variables when the matrices of the forms are rectangular are considered. Explicit expressions for the cumulants, joint cumulants and joint cumulants of bilinear and quadratic forms are given. Necessary and sufficient conditions are established for the independence of two bilinear forms as well as a bilinear and a quadratic form. Special cases are shown to agree with known results.

Key words and phrases: Bilinear forms, quadratic forms, moments, cumulants, joint cumulants, independence of bilinear and quadratic forms, nonsingular normal variables.

1. Introduction

Quadratic forms in nonsingular normal variables are widely studied in the literature. Quadratic forms, and their generalisations in the form of their matrix analogues have applications in different disciplines. Quite a large amount of work is available on the moments, cumulants, chi-squaredness, independence, distributions, approximations and asymptotic results and Chebyshev's type inequalities on one or more quadratic forms, see for example Geisser (1957), Good (1963), Hayakawa (1972), Jensen (1982), Morin-Wahhab (1985) and Provost (1989), to mention a few. Extensive results on quadratic forms are available in the singular normal case also. When it comes to bilinear forms or a collection of bilinear and quadratic forms the problem becomes complicated. Hence not many results are available in the literature in these lines. Craig (1947) considered bilinear forms of the type $X'AY$ where $A = A'$, a prime denotes a transpose, $X \sim N_p(0, I)$, $Y \sim N_p(0, I)$, $\text{cov}(X, Y) = \rho I$, X and Y are jointly normal, ρ is a scalar and I denotes an identity matrix. He established a basic result on the independence of two such bilinear forms as well as one bilinear and one quadratic form. An alternate proof is given by Aitken (1950). Ogawa (1949) gave alternate derivations of the results of Craig (1947). A simpler proof of Craig's result is given by Kawada (1950). Tan and Cheng (1981) computed the first four joint cumulants of bilinear forms and joint cumulants of bilinear and quadratic forms when the component variables are mutually independently distributed. For a detailed discussion of quadratic forms see Johnson and Kotz (1970).

In this article we consider bilinear forms of the type $X'A_iY$, A_i is $p \times q$, $i = 1, 2$ and quadratic forms of the type $Y'B_iY$, $B_i = B_i'$ where X and Y have a joint $(p+q)$-variate nonsingular normal distribution with $\mathrm{cov}(X,Y) = \Sigma_{12}$ not necessarily null. Explicit forms for the cumulants of $X'A_iY$, joint cumulants of $X'A_1Y$, $X'A_2Y$ and joint cumulants of $X'A_1Y$, $X'B_1Y$ will be given in this article. A number of results giving necessary and sufficient conditions for the independence of bilinear forms, bilinear and quadratic forms will also be given.

2. Bilinear and quadratic forms

Let the $p \times 1$ vector X and $q \times 1$ vector Y have a joint real normal distribution $N_{p+q}(0, \Sigma)$, $\Sigma > 0$. Let A_1 be a $p \times q$ real matrix and A_2 be a $q \times q$ real symmetric matrix of constants. Let $Q_1 = X'A_1Y$ and $Q_2 = Y'A_2Y$. The joint moment generating function (m.g.f.) of Q_1 and Q_2, denoted by $M_{Q_1,Q_2}(t_1,t_2)$ is given by the following expected value.

$$
\begin{aligned}
&M_{Q_1,Q_2}(t_1,t_2)\\
&\quad = E[\exp\{t_1Q_1 + t_2Q_2\}]\\
&\quad = \{|\Sigma|^{1/2}(2\pi)^{(p+q)/2}\}^{-1}\int_X\int_Y \exp\left\{-\frac{1}{2}(X',Y')\tilde{\Sigma}\binom{X}{Y}\right\}dXdY\\
&\quad = |\tilde{\Sigma}|^{-1/2}/|\Sigma|^{1/2}.
\end{aligned}
$$

where $|(\ )|$ denotes the determinant of $(\ )$ and

$$
\tilde{\Sigma} = \begin{pmatrix} \Sigma^{11} & \Sigma^{12} - t_1A_1 \\ \Sigma^{21} - t_1A_1' & \Sigma^{22} - 2t_2A_2 \end{pmatrix},
$$
$$
\Sigma^{-1} = \begin{pmatrix} \Sigma^{11} & \Sigma^{12} \\ \Sigma^{21} & \Sigma^{22} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}.
$$

Here Σ_{11} is the covariance matrix of X, that is, $\Sigma_{11} = \mathrm{cov}(X)$, $\Sigma_{22} = \mathrm{cov}(Y)$, $\Sigma_{12} = \mathrm{cov}(X,Y)$. By direct multiplication of Σ and Σ^{-1} one can get the well-known relations,

$$
\begin{aligned}
&\Sigma^{21}(\Sigma^{11})^{-1} = -\Sigma_{22}^{-1}\Sigma_{21}, && (\Sigma^{11})^{-1}\Sigma^{12} = -\Sigma_{12}\Sigma_{22}^{-1},\\
&\Sigma_{22}^{-1} = \Sigma^{22} - \Sigma^{21}(\Sigma^{11})^{-1}\Sigma^{12}, && |\Sigma| = |\Sigma_{22}||\Sigma^{11}|^{-1}
\end{aligned}
\tag{2.1}
$$

and so on. By using (2.1) one can simplify $\tilde{\Sigma}$ as follows.

$$
\begin{aligned}
|\tilde{\Sigma}| &= |\Sigma^{11}|\,|(\Sigma^{22} - 2t_2A_2) - (\Sigma^{21} - t_1A_1')(\Sigma^{11})^{-1}(\Sigma^{12} - t_1A_1)|\\
&= |\Sigma^{11}|\,|\Sigma_{22}|^{-1}|I - 2t_2\Sigma_{22}^{1/2}A_2\Sigma_{22}^{1/2}\\
&\qquad\qquad - t_1(\Sigma_{22}^{1/2}A_1'\Sigma_{12}\Sigma_{22}^{-1/2} + \Sigma_{22}^{-1/2}\Sigma_{21}A_1\Sigma_{22}^{1/2})\\
&\qquad\qquad - t_1^2\Sigma_{22}^{1/2}A_1'(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})A_1\Sigma_{22}^{1/2}|,
\end{aligned}
$$

where $\Sigma_{22}^{1/2}$ denotes the symmetric square root of the positive definite symmetric matrix Σ_{22}. One can also write $\Sigma_{22} = BB'$ and replace one of $\Sigma_{22}^{1/2}$ by B and the other by B'. For notational convenience we will use the symmetric square root. Thus the joint m.g.f. simplifies to the following.

$$M_{Q_1,Q_2}(t_1,t_2) = |I - t_1E_1 - t_2E_2 - t_1^2E_3|^{-1/2} \tag{2.2}$$

where

$$\begin{aligned}
E_1 &= E_1' = \Sigma_{22}^{-1/2}\Sigma_{21}A_1\Sigma_{22}^{1/2} + \Sigma_{22}^{1/2}A_1'\Sigma_{12}\Sigma_{22}^{-1/2},\\
E_2 &= E_2' = 2\Sigma_{22}^{1/2}A_2\Sigma_{22}^{1/2},\\
E_3 &= E_3' = \Sigma_{22}^{1/2}A_1'(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})A_1\Sigma_{22}^{1/2}.
\end{aligned}$$

The joint cumulant generating function is available by taking the logarithm of the joint m.g.f. and expanding it. That is,

$$\ln M_{Q_1,Q_2}(t_1,t_2) = \frac{1}{2}\left\{\sum_{k=1}^{\infty}\frac{1}{k}\operatorname{tr}C^k\right\} \tag{2.3}$$

where tr( ) denotes the trace of ( ) and

$$C = t_1E_1 + t_2E_2 + t_1^2E_3 \tag{2.4}$$

and without loss of generality it is assumed that $\|C\| < 1$ where $\|(\ )\|$ denotes the norm of ( ). Thus the covariance between Q_1 and Q_2 is available from (2.3) by taking the coefficient of t_1t_2 in the expansion on the right side of (2.3). This can come only from C^2.

$$\begin{aligned}
C^2 = {}& t_1^2E_1^2 + t_2^2E_2^2 + t_1^4E_3^2 + t_1t_2(E_1E_2 + E_2E_1)\\
&+ t_1^3(E_1E_3 + E_3E_1) + t_2t_1^2(E_2E_3 + E_3E_2).
\end{aligned} \tag{2.5}$$

Let the (r_1, r_2)-th joint cumulant of Q_1 and Q_2 be denoted by K_{r_1,r_2}. Then

$$\begin{aligned}
K_{1,1} &= \operatorname{cov}(Q_1,Q_2) = \frac{1}{4}\operatorname{tr}(E_1E_2 + E_2E_1) = \frac{1}{2}\operatorname{tr}(E_1E_2)\\
&= \frac{1}{2}\operatorname{tr}(\Sigma_{22}^{-1/2}\Sigma_{21}A_1\Sigma_{22}^{1/2} + \Sigma_{22}^{1/2}A_1'\Sigma_{12}\Sigma_{22}^{-1/2})(2\Sigma_{22}^{1/2}A_2\Sigma_{22}^{1/2})\\
&= 2\operatorname{tr}(\Sigma_{21}A_1\Sigma_{22}A_2).
\end{aligned} \tag{2.6}$$

In the simplification in (2.6) we have made use of the properties that for any two matrices A and B, $\operatorname{tr}AB = \operatorname{tr}BA$, $\operatorname{tr}A = \operatorname{tr}A'$ whenever the products are defined. These properties will be frequently made use of in the discussions to follow. Note that $K_{1,1} = 0$ when $\Sigma_{21} = 0$ or when $A_1\Sigma_{22}A_2' = 0$.

2.1 *Joint cumulants of bilinear and quadratic forms*

The joint cumulant generating function is given in (2.3). Now we will establish a convenient way of finding the coefficient of $t_1^{s_1} t_2^{s_2}$ from the expansion given in (2.3). Write

$$C = t_1 E_1 + t_2 E_2 + t_3 E_3, \qquad t_3 = t_1^2. \tag{2.7}$$

The terms containing $t_1^{r_1} t_2^{r_2} t_3^{r_3}$ where $r_1 + r_2 + r_3 = r$ can come only from C^r. From (2.4), (2.5) and higher powers of C note that the coefficient of $t_1^{r_1} t_2^{r_2} t_3^{r_3}$ in (2.3) is $(1/2r) \operatorname{tr} \sum_{(r_1,r_2,r_3)} (E_1 E_2 E_3)$ where the notation $\sum_{(r_1,r_2,r_3)} (E_1 E_2 E_3)$ stands for the sum of products of permutations of E_1, E_2, E_3 taking any number of them at a time so that the sum of the exponents of E_i in each term is r_i, $i = 1, 2, 3$. Thus we have the following result.

THEOREM 2.1. *Let $Q_1 = X'A_1 Y$, $Q_2 = Y'A_2 Y$, $\begin{pmatrix} X \\ Y \end{pmatrix} \sim N_{p+q}(0, \Sigma)$, $\Sigma > 0$, where $A_2 = A_2'$ and A_1 be real matrices of constants, A_1 be $p \times q$ and A_2 be $q \times q$. Then the $(r_1 + 2r_3, r_2)$-th joint cumulant of Q_1 and Q_2, denoted $K_{r_1+2r_3,r_2}$ is given by*

$$K_{r_1+2r_3,r_2} = \frac{(r_1 + 2r_3)! r_2!}{2r} \operatorname{tr}\Bigg(\sum_{(r_1,r_2,r_3)} (E_1 E_2 E_3) \Bigg) \tag{2.8}$$

where E_1, E_2, E_3 are defined in (2.2), $r = r_1 + r_2 + r_3$ and $\sum_{(r_1,r_2,r_3)}$ is explained above.

For example what is $K_{2,2}$? The possible breakdowns of (r_1, r_2, r_3) are $(0, 2, 1)$ and $(2, 2, 0)$.

$$\sum_{(0,2,1)} (E_1 E_2 E_3) = E_2^2 E_3 + E_3 E_2^2 + E_2 E_3 E_2,$$
$$r = 3, \quad r_1 + 2r_3 = 2, \quad r_2 = 2.$$
$$\sum_{(2,2,0)} (E_1 E_2 E_3) = E_1^2 E_2^2 + E_2^2 E_1^2 + E_1 E_2^2 E_1 + E_2 E_1^2 E_2$$
$$+ E_1 E_2 E_1 E_2 + E_2 E_1 E_2 E_1,$$
$$r = 4, \quad r_1 + 2r_3 = 2, \quad r_2 = 2.$$

Thus

$$\begin{aligned} K_{2,2} &= \frac{2}{3} \operatorname{tr}(E_2^2 E_3 + E_3 E_2^2 + E_2 E_3 E_2) \\ &\quad + \frac{1}{2} \operatorname{tr}(E_1^2 E_2^2 + E_2^2 E_1^2 + E_1 E_2^2 E_1 \\ &\qquad + E_2 E_1^2 E_2 + E_1 E_2 E_1 E_2 + E_2 E_1 E_2 E_1) \\ &= 2 \operatorname{tr} E_2 E_3 E_2 + 2 \operatorname{tr} E_1^2 E_2^2 + \operatorname{tr}(E_1 E_2)^2. \end{aligned} \tag{2.9}$$

Remark 2.1. When X and Y are independently distributed $\Sigma_{12}=0$ and the joint cumulants of Q_1 and Q_2 are available from (2.8) by replacing E_1 by a null matrix and E_3 by $\Sigma_{22}^{1/2}A_1'\Sigma_{11}A_1\Sigma_{22}^{1/2}$. In the sum $\sum_{(r_1,r_2,r_3)}$ replace E_1 by an identity matrix and put $r_1=0$.

Thus when X and Y are independently distributed

$$K_{2,2}=8\operatorname{tr}(\Sigma_{22}A_2\Sigma_{22}A_1'\Sigma_{11}A_1\Sigma_{22}A_2).$$

Remark 2.2. When X and Y are equicorrelated in the sense of Craig (1947) then $q=p$, $A_1=A_1'$, $A_2=A_2'$, $\Sigma_{11}=I$, $\Sigma_{22}=I$, $\Sigma_{12}=\rho I=\Sigma_{21}$ and in this case $E_1=2\rho A_1$, $E_2=2A_2$ and $E_3=(1-\rho^2)A_1^2$.

Thus when X and Y are equicorrelated in the sense of Craig (1947),

$$K_{2,2}=8\operatorname{tr}A_1^2A_2^2+24\rho^2\operatorname{tr}A_1^2A_2^2+16\rho^2\operatorname{tr}(A_1A_2)^2.$$

Remark 2.3. If the cumulants of the quadratic form Q_2 are to be obtained then in (2.8) put $r_1=0$, $r_3=0$ and replace E_1 and E_2 by identity matrices in the sum $\sum_{(r_1,r_2,r_3)}$. This will be listed as a corollary.

COROLLARY 2.1. *The r-th cumulant of the quadratic form Q_2, denoted by $K_r^{(Q_2)}$, is given by,*

$$K_r^{(Q_2)}=\frac{r!}{2r}\operatorname{tr}\sum_{(r)}(E_2)=2^{r-1}(r-1)!\operatorname{tr}(\Sigma_{22}^{1/2}A_2\Sigma_{22}^{1/2})^r. \tag{2.10}$$

Remark 2.4. If the cumulants of the bilinear form Q_1 are to be obtained then in (2.8) put $r_2=0$ and replace E_2 by I in the sum.

COROLLARY 2.2. *The (r_1+2r_3)-th cumulant of Q_1, denoted by $K_{r_1+2r_3}^{(Q_1)}$, is given by,*

$$K_{r_1+2r_3}^{(Q_1)}=\frac{(r_1+2r_3)!}{2(r_1+r_3)}\operatorname{tr}\sum_{(r_1,r_3)}(E_1E_3). \tag{2.11}$$

For approximating the density of Q_1, one may need the first four cumulants. These will be listed here explicitly.

$$K_1^{(Q_1)}=\frac{1}{2}\operatorname{tr}\sum_{(1,0)}(E_1E_3)=\frac{1}{2}\operatorname{tr}E_1=\operatorname{tr}\Sigma_{21}A_1=\textit{expected value of } Q_1,$$

$$\begin{aligned}K_2^{(Q_1)}&=\frac{2!}{2(2)}\operatorname{tr}\sum_{(2,0)}(E_1E_3)+\frac{2!}{2(1)}\operatorname{tr}\sum_{(0,1)}(E_1E_3)\\&=\frac{1}{2}\operatorname{tr}E_1^2+\operatorname{tr}E_3,\\K_3^{(Q_1)}&=\frac{3!}{2(3)}\operatorname{tr}\sum_{(3,0)}(E_1E_3)+\frac{3!}{2(2)}\operatorname{tr}\sum_{(1,1)}(E_1E_3)\\&=\operatorname{tr}E_1^3+3\operatorname{tr}E_1E_3,\\K_4^{(Q_1)}&=\frac{4!}{2(4)}\operatorname{tr}\sum_{(4,0)}(E_1E_3)+\frac{4!}{2(3)}\operatorname{tr}\sum_{(2,1)}(E_1E_3)+\frac{4!}{2(2)}\operatorname{tr}\sum_{(0,2)}(E_1E_3)\\&=3\operatorname{tr}E_1^4+12\operatorname{tr}E_3E_1^2+6\operatorname{tr}E_3^2.\end{aligned}$$

2.2 Independence of bilinear and quadratic forms

Consider the same Q_1 and Q_2 in Section 2. What is a set of necessary and sufficient conditions under which Q_1 and Q_2 are independently distributed? For the equicorrelated case, see Remark 2.2, Craig (1947) and others showed that the condition is $A_1A_2 = 0$. It is easily seen that when X and Y are independently distributed, the condition remains the same if $A_1 = A_1'$ and $\Sigma_{22} = I$. We will establish a general result with the help of the joint cumulant generating function given in (2.3). We need a result which will be stated here as a lemma.

LEMMA 2.1. *For two arbitrary $p \times p$ matrices A and B,*

$$\operatorname{tr}(AB)^2 + 2\operatorname{tr}(AB)(B'A') = 0 \Rightarrow AB = 0. \tag{2.12}$$

This result was established by Kawada (1950). The proof follows by observing that the left side of (2.12) can be written as a sum of positive definite quadratic forms of the type

$$\sum_{ij}(c_{ij}, c_{ji})\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}\begin{pmatrix} c_{ij} \\ c_{ji} \end{pmatrix}$$

which can be zero only if $c_{ij} = 0$ for all i and j, where c_{ij} is the (i,j)-th element of AB.

THEOREM 2.2. *Let $\begin{pmatrix} X \\ Y \end{pmatrix} \sim N_{p+q}(0,\Sigma)$, $\Sigma > 0$. Let $Q_1 = X'A_1Y$ and $Q_2 = Y'A_2Y$, $A_2 = A_2'$ and A_1 be $p \times q$. Let $\operatorname{cov}(X) = \Sigma_{11}$, $\operatorname{cov}(Y) = \Sigma_{22}$, $\operatorname{cov}(X,Y) = \Sigma_{12} = \Sigma_{21}'$. Then Q_1 and Q_2 are independently distributed iff $A_1\Sigma_{22}A_2 = 0$ and $A_1'\Sigma_{12}A_2 = 0$.*

PROOF. Let Q_1 and Q_2 be independently distributed. Then

$$M_{Q_1,0}(t_1,0)M_{0,Q_2}(0,t_2) = M_{Q_1,Q_2}(t_1,t_2) \tag{2.13}$$

where $M_{Q_1,Q_2}(t_1t_2)$ is given in (2.2). Take logarithms on both sides and expand as in (2.3) and then equate the coefficient of $t_1^2t_2^2$ on both sides to get the following.

$$2\operatorname{tr}(E_2E_3E_2) + 2\operatorname{tr}(E_1^2E_2^2) + \operatorname{tr}(E_1E_2)^2 = 0 \tag{2.14}$$

where E_1, E_2, E_3 are given in (2.2). Equation (2.14) is noted from (2.9). But observe that E_1, E_2, E_3 are symmetric matrices and further E_3 can be written as $B'B$ where $B = (\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{1/2}A_1\Sigma_{22}^{1/2}$. Hence (2.14) reduces to

$$[2\operatorname{tr}(E_2'B')(BE_2)] + [\operatorname{tr}(E_1E_2)^2 + 2\operatorname{tr}(E_1E_2)(E_1E_2)'] = 0. \tag{2.15}$$

But the quantities in each bracket is nonnegative and hence each is zero. That is, $E_1E_2 = 0$ and $BE_2 = 0$. But $BE_2 = 0 \Rightarrow A_1\Sigma_{22}A_2 = 0$ since $\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} > 0$, $\Sigma_{22} > 0$. Then

$$\begin{aligned} E_1E_2 &= 2(\Sigma_{22}^{-1/2}\Sigma_{21}A_1\Sigma_{22}A_2\Sigma_{22}^{1/2} + \Sigma_{22}^{1/2}A_1'\Sigma_{12}A_2\Sigma_{22}^{1/2}) \\ &= 2\Sigma_{22}^{1/2}A_1'\Sigma_{12}A_2\Sigma_{22}^{1/2}. \end{aligned}$$

Thus $E_1E_2 = 0 \Rightarrow A_1'\Sigma_{12}A_2 = 0$. This establishes the necessity. Now check $M_{Q_1,0}(t_1,0)M_{0,Q_2}(0,t_2)$ and $M_{Q_1,Q_2}(t_1,t_2)$ separately under the conditions $A_1\Sigma_{22}A_2 = 0$ and $A_1'\Sigma_{12}A_2 = 0$ to see that they are equal. This completes the proof.

COROLLARY 2.3. *The necessary and sufficient condition for the independence of Q_1 and Q_2 is* (1) $A_1\Sigma_{22}A_2 = 0$ *when X and Y are independently distributed; and* (2) $A_1A_2 = 0$ *when X and Y are equicorrelated in the sense of Craig* (1947).

Remark 2.5. In the proof of Theorem 2.2 we have made use of the explicit expressions for the cumulants. By writing bilinear forms as quadratic forms one can also make use of the results on quadratic forms, see for example Rao ((1973), p. 188), for establishing this theorem. In this case write $Q_1 = X'A_1Y = (1/2)(X',Y')\begin{pmatrix} 0 & A_1 \\ A_1' & 0 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}$ and $Q_2 = Y'A_2Y = (X',Y')\begin{pmatrix} 0 & 0 \\ 0 & A_2 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}$.

3. Bilinear forms

Let $\begin{pmatrix} X \\ Y \end{pmatrix} \sim N_{p+q}(0,\Sigma)$, $\Sigma > 0$ with $\mathrm{cov}(X) = \Sigma_{11}$, $\mathrm{cov}(Y) = \Sigma_{22}$, $\mathrm{cov}(X,Y) = \Sigma_{12} = \Sigma_{21}'$. Let $Q_1 = X'A_1Y$ and $Q_2 = X'A_2Y$ where A_1 and A_2 are $p \times q$ real matrices of constants. We will use the same notation Q_2 to denote a bilinear form here. This will not create any confusion since we will not be considering results where the Q_2's of Sections 2 and 3 both will be involved. The joint m.g.f. of Q_1 and Q_2, denoted again by $M_{Q_1,Q_2}(t_1,t_2)$ is available by following through similar steps as in (2.1) to (2.2). Then

$$M_{Q_1,Q_2}(t_1,t_2) = |B^*|^{-1/2}/|\Sigma|^{1/2} \tag{3.1}$$

where

$$\begin{aligned} |B^*| &= \begin{vmatrix} \Sigma^{11} & \Sigma^{12} - t_1A_1 - t_2A_2 \\ \Sigma^{21} - t_1A_1' - t_2A_2' & \Sigma^{22} \end{vmatrix} \\ &= |\Sigma^{11}||\Sigma_{22}^{-1}| \\ &\quad \cdot |I_q - \Sigma_{21}(t_1A_1 + t_2A_2) - \Sigma_{22}(t_1A_1' + t_2A_2')\Sigma_{12}\Sigma_{22}^{-1} \\ &\qquad - \Sigma_{22}(t_1A_1' + t_2A_2')(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})(t_1A_1 + t_2A_2)|. \end{aligned}$$

Thus

$$\begin{aligned} &M_{Q_1,Q_2}(t_1,t_2) \\ &= |I_q - \Sigma_{22}^{-1/2}\Sigma_{21}(t_1A_1 + t_2A_2)\Sigma_{22}^{1/2} - \Sigma_{22}^{1/2}(t_1A_1' + t_2A_2')\Sigma_{12}\Sigma_{22}^{-1/2} \\ &\quad - \Sigma_{22}^{-1/2}(t_1A_1 + t_2A_2)'(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})(t_1A_1 + t_2A_2)\Sigma_{22}^{1/2}|^{-1/2} \qquad (3.2) \\ &= |I_p - \Sigma_{11}^{-1/2}\Sigma_{12}(t_1A_1' + t_2A_2')\Sigma_{11}^{1/2} - \Sigma_{11}^{1/2}(t_1A_1 + t_2A_2)\Sigma_{21}\Sigma_{11}^{-1/2} \\ &\quad - \Sigma_{11}^{1/2}(t_1A_1 + t_2A_2)(\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12})(t_1A_1 + t_2A_2)'\Sigma_{11}^{1/2}|^{-1/2}. \qquad (3.3) \end{aligned}$$

From (3.2) one can write

$$M_{Q_1,Q_2}(t_1,t_2) = |I_q - t_1E_1 - t_2E_2 - t_3E_3 - t_4E_4 - t_5E_5|^{-1/2} \tag{3.4}$$

where

$$\begin{cases} t_3 = t_1^2, \quad t_4 = t_2^2, \quad t_5 = t_1t_2, \\ E_1 = E_1' = \Sigma_{22}^{-1/2}\Sigma_{21}A_1\Sigma_{22}^{1/2} + \Sigma_{22}^{1/2}A_1'\Sigma_{12}\Sigma_{22}^{-1/2}, \\ E_2 = E_2' = \Sigma_{22}^{-1/2}\Sigma_{21}A_2\Sigma_{22}^{1/2} + \Sigma_{22}^{1/2}A_2'\Sigma_{12}\Sigma_{22}^{-1/2}, \\ E_3 = E_3' = \Sigma_{22}^{1/2}A_1'DA_1\Sigma_{22}^{1/2}, \quad D = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}, \\ E_4 = E_4' = \Sigma_{22}^{1/2}A_2'DA_2\Sigma_{22}^{1/2}, \\ E_5 = E_5' = \Sigma_{22}^{1/2}A_1'DA_2\Sigma_{22}^{1/2} + \Sigma_{22}^{1/2}A_2'DA_1\Sigma_{22}^{1/2}. \end{cases} \tag{3.5}$$

For convenience we will use the same notation E_i's as in Section 2. Note that the E_i's appearing in Sections 2 and 3 are not all the same. Consider the expansion of $\ln M_{Q_1,Q_2}(t_1,t_2)$ as in (2.3) where the C here is given by

$$C = t_1E_1 + \cdots + t_5E_5.$$

The coefficient of $t_1^{r_1}\cdots t_5^{r_5}$, $r = r_1 + \cdots + r_5$ is coming from C^r only. This is given by $(1/2r)\operatorname{tr}\sum_{(r_1,\ldots,r_5)}(E_1\cdots E_5)$ where the notation is explained in Section 2. Note that $t_3 = t_1^2$, $t_4 = t_2^2$ and $t_5 = t_1t_2$ and hence the joint cumulants of the type $(r_1 + 2r_3 + r_5, r_2 + 2r_4 + r_5)$ can be obtained from this coefficient. Again denoting the (s_1, s_2)-th joint cumulant of Q_1 and Q_2 by K_{s_1,s_2} we have,

THEOREM 3.1.

$$\begin{aligned} &K_{r_1+2r_3+r_5,r_2+2r_4+r_5} \\ &\quad = \frac{(r_1+2r_3+r_5)!(r_2+2r_4+r_5)!}{2r}\operatorname{tr}\Bigg(\sum_{(r_1,\ldots,r_5)}(E_1\cdots E_5)\Bigg) \end{aligned} \tag{3.6}$$

where $r = r_1 + \cdots + r_5$.

Remark 3.1. Various cumulants of Q_1 are available from (3.6) by putting $r_2 = 0$, $r_4 = 0$, $r_5 = 0$ and replacing E_2, E_4 and E_5 by identities in the sum $\sum(E_1\cdots E_5)$.

Remark 3.2. When X and Y are independently distributed various joint cumulants of Q_1 and Q_2 are available from (3.6) by putting $E_1 = 0$, $E_2 = 0$ and replacing D by Σ_{11}. In the sum $\sum(E_1\cdots E_5)$ put $r_1 = 0$, $r_2 = 0$ and replace E_1 and E_2 by identity matrices.

Remark 3.3. When X and Y are equicorrelated in the sense of Craig (1947) the joint cumulants of Q_1 and Q_2 are available from (3.6) by putting $E_1 = 2\rho A_1$,

$E_2 = 2\rho A_2$, $E_3 = (1-\rho^2)A_1^2$, $E_4 = (1-\rho^2)A_2^2$, $E_5 = 2(1-\rho^2)A_1A_2$. Note that in this case $q = p$, $A_1 = A_1'$, $A_2 = A_2'$.

The first few cumulants will be needed if someone is trying to approximate the distributions of Q_1 and Q_2 or the joint distribution of Q_1 and Q_2. Hence a few of these will be listed here explicitly.

Take $K_{1,0}$, $K_{0,1}$, $K_{2,0}$, $K_{0,2}$ from (2.11) or from the explicit forms given there.

$$
(3.7)\quad \begin{cases}
K_{1,1} = \dfrac{1}{2(2)}\operatorname{tr}(E_1E_2 + E_2E_1) + \dfrac{1}{2(1)}\operatorname{tr} E_5 = \dfrac{1}{2}\operatorname{tr} E_1E_2 + \dfrac{1}{2}\operatorname{tr} E_5 \\
\qquad = \operatorname{tr}(\Sigma_{21}A_1\Sigma_{21}A_2) + \operatorname{tr}(\Sigma_{22}A_1'\Sigma_{11}A_2) = \operatorname{cov}(Q_1, Q_2); \\
K_{2,1} = \dfrac{2!}{2(3)}\operatorname{tr}(E_1^2E_2 + E_2E_1^2 + E_1E_2E_1) \\
\qquad + \dfrac{2!}{2(2)}\operatorname{tr}(E_1E_5 + E_5E_1) + \dfrac{2!}{2(2)}\operatorname{tr}(E_2E_3 + E_3E_2) \\
\qquad = \operatorname{tr}(E_1^2E_2) + \operatorname{tr}(E_1E_5) + \operatorname{tr}(E_2E_3); \\
K_{1,2} = \operatorname{tr}(E_2^2E_1) + \operatorname{tr}(E_2E_5) + \operatorname{tr}(E_1E_4); \\
K_{2,2} = 2\operatorname{tr} E_1^2E_2^2 + 2\operatorname{tr} E_1^2E_4 + \operatorname{tr} E_5^2 + 2\operatorname{tr} E_3E_4 \\
\qquad + 2\operatorname{tr} E_2^2E_3 + 4\operatorname{tr} E_1E_2E_5 + \operatorname{tr}(E_1E_2)^2.
\end{cases}
$$

Remark 3.4. When X and Y are independently distributed

$$
(3.8)\qquad K_{2,2} = \operatorname{tr} E_5^2 + 2\operatorname{tr} E_3E_4.
$$

THEOREM 3.2. *Let* $\begin{pmatrix} X \\ Y \end{pmatrix} \sim N_{p+q}(0, \Sigma)$, $\Sigma > 0$, $\operatorname{cov}(X) = \Sigma_{11}$, $\operatorname{cov}(Y) = \Sigma_{22}$, $\operatorname{cov}(X, Y) = \Sigma_{12} = \Sigma_{21}'$. *Consider* $Q_i = X'A_iY$, *where* A_i *is a* $p \times q$ *real matrix of constants,* $i = 1, 2$. *Then the necessary and sufficient conditions for the independence of* Q_1 *and* Q_2 *are*

$$
A_1\Sigma_{22}A_2' = 0, \qquad A_1'\Sigma_{11}A_2 = 0, \qquad A_2\Sigma_{21}A_1 = 0, \qquad A_1\Sigma_{21}A_2 = 0.
$$

PROOF. Necessity. Let Q_1 and Q_2 be independently distributed. Then

$$
(3.9)\qquad M_{Q_1,0}(t_1, 0)M_{0,Q_2}(0, t_2) = M_{Q_1,Q_2}(t_1, t_2)
$$

where $M_{Q_1,Q_2}(t_1, t_2)$ is given in (3.2). Take logarithms on both sides of (3.9), expand and compare the coefficient of $t_1^2t_2^2$ on both sides to get $K_{2,2} = 0$ where $K_{2,2}$ is given in (3.7). Since $K_{2,2} = 0$ for all Σ_{12} it should hold for $\Sigma_{12} = 0$ also. Then from (3.8)

$$
(3.10)\qquad \operatorname{tr} E_5^2 + 2\operatorname{tr} E_3E_4 = 0.
$$

But note that

$$
\begin{aligned}
&\operatorname{tr} E_5^2 = \operatorname{tr}(E_5)(E_5)' \geq 0, \\
&\operatorname{tr}(E_3E_4) = \operatorname{tr}(D^{1/2}A_1\Sigma_{22}A_2'D^{1/2})(D^{1/2}A_1\Sigma_{22}A_2'D^{1/2})' \geq 0.
\end{aligned}
$$

Hence

$$D^{1/2}A_1\Sigma_{22}A_2'D^{1/2} = 0 \Rightarrow A_1\Sigma_{22}A_2' = 0.$$

Then from (3.3) one has $A_1'\Sigma_{11}A_2 = 0$. Now impose these conditions on (3.7) and simplify. After a lot of algebra $K_{2,2}$ of (3.7) can be written as follows.

$$\begin{aligned} K_{2,2} &= 2\,\mathrm{tr}\, U_1'(\Sigma_{22}^{1/2}A_2\Sigma_{11}A_2\Sigma_{22}^{1/2})U_1 \\ &\quad + 2\,\mathrm{tr}\, U_2'(\Sigma_{22}^{1/2}A_1'\Sigma_{11}A_1\Sigma_{22}^{1/2})U_2 \\ &\quad + 2[\mathrm{tr}(U_1U_2)^2 + 2\,\mathrm{tr}(U_1U_2)(U_2U_1)] \end{aligned} \tag{3.11}$$

where

$$U_1 = \Sigma_{22}^{-1/2}\Sigma_{21}A_1\Sigma_{22}^{1/2}, \qquad U_2 = \Sigma_{22}^{-1/2}\Sigma_{21}A_2\Sigma_{22}^{1/2}.$$

Note that $U_1U_2' = 0$. Also independence of Q_1 and Q_2 implies that $K_{2,2} = 0$ for all A_1, A_2, Σ_{11}, Σ_{22}. Put $U_2 = 0$, that is, select an A_2 such that $\Sigma_{21}A_2 = 0$. Then $K_{2,2} = 0 \Rightarrow \Sigma_{11}^{1/2}A_2\Sigma_{22}^{1/2}U_1 = 0 \Rightarrow A_2\Sigma_{21}A_1 = 0$. Similarly $A_1\Sigma_{21}A_2 = 0$. Under the conditions $A_2\Sigma_{21}A_1 = 0$ and $A_1\Sigma_{21}A_2 = 0$ note that $U_1U_2 = 0$, $U_2U_1 = 0$ and $K_{2,2} = 0$. Hence, the conditions $A_2\Sigma_{21}A_1 = 0$ and $A_1\Sigma_{21}A_2 = 0$ are necessary.

To see the sufficiency consider $M_{Q_1,0}(t_1,0)M_{0,Q_2}(0,t_2)$ and $M_{Q_1,Q_2}(t_1,t_2)$ separately and impose the conditions $A_1\Sigma_{22}A_2' = 0$, $A_1'\Sigma_{11}A_2 = 0$, $A_2\Sigma_{21}A_1 = 0$ and $A_1\Sigma_{21}A_2 = 0$. Under these conditions the E_5 of $M_{Q_1,Q_2}(t_1,t_2)$ reduces to

$$E_5 = -\Sigma_{22}^{1/2}A_1'\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}A_2\Sigma_{22}^{1/2} - \Sigma_{22}^{1/2}A_2'\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}A_1\Sigma_{22}^{1/2}. \tag{3.12}$$

$$\begin{aligned} &M_{Q_1,0}(t_1,0)M_{0,Q_2}(0,t_2) \\ &\quad = |I - t_1E_1 - t_2E_2 - t_1^2E_3 - t_2^2E_4 \\ &\qquad + t_1^2t_2E_3E_2 + t_1t_2E_1E_2 + t_1t_2^2E_1E_4 + t_1^2t_2^2E_3E_4|^{-1/2}. \end{aligned}$$

But note that under the necessary conditions $E_1E_4 = 0$, $E_3E_2 = 0$, $E_3E_4 = 0$ and $E_1E_2 = -E_5$ of (3.12) which establishes sufficiency.

COROLLARY 3.1. *When X and Y are independently distributed then Q_1 and Q_2 are independently distributed iff $A_1\Sigma_{22}A_2' = 0$ and $A_1'\Sigma_{11}A_2 = 0$.*

COROLLARY 3.2. *When X and Y are equicorrelated in the sense $q = p$, $\Sigma_{11} = \sigma_1^2I$, $\Sigma_{22} = \sigma_2^2I$, $\Sigma_{21} = \rho I = \Sigma_{12}$ then Q_1 and Q_2 are independently distributed iff $A_1A_2' = 0$, $A_1'A_2 = 0$, $A_1A_2 = 0$, $A_2A_1 = 0$.*

COROLLARY 3.3. *When X and Y are equicorrelated in the sense of Craig* (1947) *then Q_1 and Q_2 are independently distributed iff $A_1A_2 = 0$.*

Remark 3.5. Similar comments as in Remark 2.5 are also applicable here.

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