Ann. Inst. Statist. Math.
Vol. 44, No. 4, 613–622 (1992)

TRUNCATED SELECTION PROCEDURES FOR THE MOST PROBABLE EVENT AND THE LEAST PROBABLE EVENT

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(Received November 6, 1989; revised September 30, 1991)

Abstract. This paper proposes two sequential procedures for selecting respectively the multinomial cell with the largest cell probability and the multinomial cell with the smallest cell probability. The stopping rule for both procedures uses truncation of the procedure studied by Ramey and Alam (1979, *Biometrika*, **66**, 171–173). A property of the least favorable configuration of the proposed procedures is proved, which partially solves a conjecture given in Ramey and Alam (1979). The proposed procedures are compared with other procedures which have been considered in the literature and are found to be better in certain respects.

Key words and phrases: Multinomial distribution, ranking, selection.

1. Motivations

Ramey and Alam (1979) studied a sequential procedure for selecting the cell with the largest probability of occurrence from a multinomial distribution with k cells. Their procedure was the combination of stopping rules considered by Cacoullos and Sobel (1966) and Alam (1971). Cacoullos and Sobel's procedure stops sampling when the frequency of any cell reaches a given positive integer M. Alam's procedure stops sampling when the difference between the largest frequency and the second largest frequency is equal to a given positive integer r. The combination of these two stopping rules, which was studied by Ramey and Alam (1979), stops sampling when either of these two stopping criteria is satisfied with the sample size bounded by $kM - k + 1$. When sampling is terminated, the procedure selects the cell with the unique largest cell frequency as corresponding to the most probable event. There is no possibility of a tie for the first place. Bechhofer and Kulkarni (1984) studied the performance of the curtailment of the fixed sample size procedure considered in Bechhofer *et al.* (1959). The curtailed procedure stops sampling once the frequency of any cell is large enough to guarantee the selection of a particular cell. The probability of a correct selection for the curtailed procedure is the same for the fixed sample size procedure uniformly in the cell probabilities. However, curtailed procedure always stops sampling for an equal or

smaller sample size. In two more recent papers, Bechhofer and Goldsman (1985*b*, 1986) used truncation to improve Bechhofer-Kiefer-Sobel's sequential procedure based on likelihood ratios. Our procedures R_1 and R_2 were partially motivated by their results and we also inherit some of their notations and terminology.

In this paper, we propose a sequential procedure R_1, whose stopping rule is the combination of Ramey and Alam's stopping rule, and Bechhofer and Kulkarni's stopping rule. The probability of a correct selection for procedure R_1 may not always be as large as that of the fixed sample size procedure due to the earlier stopping time caused by Ramey and Alam's stopping rule. However, by choosing appropriate stopping parameters, it always achieves the same probability requirement ((2.1) below) as the fixed sample size procedure with a small expected sample size. The concept of combining stopping rules can also be employed in selecting the least probable multinomial cell. We propose a sequential procedure R_2 which uses a combined stopping rule modified from procedure R_1 to select the cell with the smallest cell probability. A similar result on the least favorable configuration can also be obtained for procedure R_2 as for procedure R_1.

2. Notation, terminology and procedures

A multinomial distribution with k cells $\pi_1, \pi_2, \ldots, \pi_k$ is given; let the ordered values of the unknown cell probabilities $p_i \geq 0$ $(1 \leq i \leq k)$ with $\sum_{i=1}^{k} p_i = 1$ be denoted by $p_{[1]} \leq p_{[2]} \leq \cdots \leq p_{[k]}$, and the corresponding cells be denoted by $\pi_{(1)}, \pi_{(2)}, \ldots, \pi_{(k)}$. It is assumed that the values of the p_i and $p_{[j]}$ $(1 \leq i, j \leq k)$ are unknown, and the pairings of the p_i and $p_{[j]}$ $(1 \leq i, j \leq k)$ are completely unknown. Let P^* with $1/k < P^* < 1$, δ_1^*, δ_2^* denote three specified constants. For selecting the cell $\pi_{(k)}$, we require procedure R_1 which guarantees that

$$(2.1) \qquad P(\text{correct selection} \mid R_1) \geq P^* \quad \text{whenever} \quad p_{[k]}/p_{[k-1]} \geq \delta_1^*.$$

For selecting the cell $\pi_{(1)}$, we require procedure R_2 which guarantees that

$$(2.2) \qquad P(\text{correct selection} \mid R_2) \geq P^* \quad \text{whenever} \quad p_{[2]} - p_{[1]} \geq \delta_2^*.$$

We should notice that traditionally, two different measures for the distance between the preferred cell ($\pi_{(k)}$ or $\pi_{(1)}$) and the remaining cells are used (ratio for R_1 and difference for R_2). These two measures defined the so-called preference zones (PZ) as follows:

$$(2.3.1) \qquad \text{PZ for } R_1 = \{\boldsymbol{P} : p_{[k]}/p_{[k-1]} \geq \delta_1^*\},$$

$$(2.3.2) \qquad \text{PZ for } R_2 = \{\boldsymbol{P} : p_{[2]} - p_{[1]} \geq \delta_2^*\}.$$

Both procedures R_1 and R_2 take (multinomial) vector-observations one-at-a-time until certain stopping requirements are satisfied. Let n denote the maximum number of vector-observations that the experimenter is allowed to take. The value of n may be based on economical conditions. By stage m $(m \leq n)$, we shall mean that a total of m vector-observations have already been taken. Let $X_{i,m}$ $(1 \leq i \leq k,\ 1 \leq m \leq n)$ denote the frequency in cell π_i through stage m and

$X_{[1],m} \leq X_{[2],m} \leq \cdots \leq X_{[k],m}$ denote their ordered values. Let M_1, M_2, r_1 and r_2 denote positive integers. Procedures R_1 and R_2 are defined as follows:

PROCEDURE R_1.

Sampling rule. Take vector-observations one-at-a-time.

Stopping rule. Stop sampling at the first stage m at which any one of the following conditions is satisfied:

$$X_{[k],m} = M_1; \tag{2.4}$$

$$X_{[k],m} - X_{[k-1],m} = r_1; \tag{2.5}$$

$$X_{[k],m} \geq X_{[k-1],m} + n - m; \tag{2.6}$$

$$m = n. \tag{2.7}$$

Selection rule. Having stopped sampling, select the cell π_i if and only if $X_{i,m} = X_{[k],m}$. If there is a tie for the first place, use randomization to break it.

Remark 2.1. It is clear from the above definition that a tie for the first place can occur only when (2.7) is satisfied.

PROCEDURE R_2.

Sampling rule. Take vector-observations one-at-a-time.

Stopping rule. Stop sampling at the first stage m at which any one of the following conditions is satisfied:

$$X_{[2],m} = M_2; \tag{2.8}$$

$$X_{[2],m} - X_{[1],m} = r_2; \tag{2.9}$$

$$X_{[1],m} \leq X_{[2],m} - (n - m); \tag{2.10}$$

$$m = n. \tag{2.11}$$

Selection rule. Having stopped sampling, select the cell π_i if and only if $X_{i,m} = X_{[1],m}$. If there is a tie for the last place, use randomization to break it.

Remark 2.2. It is clear from the above definition that a tie for the last place can occur only when (2.11) is satisfied.

Remark 2.3. For procedure R_1, the probability of a correct selection is the same for all the M_1 such as that $M_1 \geq n/2$ with fixed r_1 and n. The probability of a correct selection is the same for all the r_1 such as that $r_1 \geq M_1$ with fixed M_1 and n. Similarly, for procedure R_2, the probability of a correct selection is the same for all the M_2 such as that $M_2 \geq n/k$, with fixed r_2 and n. The probability of a correct selection stays the same for all the r_2 such as that $r_2 \geq M_2$ with fixed M_2 and n. In particular when $r_1 \geq M_1 \geq n/2$, R_1 always makes the same decision as fixed sample size procedure. When $r_2 \geq M_2 \geq n/k$, R_2 always makes the same decision as fixed sample size procedure. Thus the computation of the procedure parameters (r_1, M_1, n) for R_1 and (r_2, M_2, n) for R_2 to satisfy requirements (2.1) and (2.2) can be restricted to those cases with $r_1 \leq M_1 \leq n/2$ and $r_2 \leq M_2 \leq n/k$ respectively which saves the computing time significantly.

3. The least favorable configurations

The parameter vector $\boldsymbol{p} = (p_1, \ldots, p_k)$ which minimizes the probability of a correct selection under the preference zone (2.3) is called the least favorable configuration (LFC). It was shown by Kesten and Morse (1959) that the LFC for the fixed sample size procedure of Bechhofer *et al.* (1959) for selecting the most probable cell is given by the so-called slippage configuration under (2.3.1):

$$p_{[k]} = \frac{\delta_1^*}{\delta_1^* + k - 1}, \qquad p_{[1]} = \cdots = p_{[k-1]} = \frac{1}{\delta_1^* + k - 1}. \tag{3.1}$$

Alam (1971) showed that for the fixed sample size procedure for selecting the least probable cell, the LFC is given by the slippage configuration under (2.3.2):

$$p_{[1]} = \frac{1 - (k-1)\delta_2^*}{k}, \qquad p_{[2]} = \cdots = p_{[k]} = \frac{1 + \delta_2^*}{k}. \tag{3.2}$$

Bechhofer and Kulkarni (1984) proved that since early curtailment employed in the stopping rule of selecting the most probable cell does not change the final decision under any configuration, the LFC should stay the same as if curtailment is not used. (The result can also be applied to curtailing the fixed sample size procedure for selecting the least probable cell.) Ramey and Alam (1979) conjectured that (3.1) is the LFC for their procedure. Since our procedure R_1 has stopping rule which is a combination of curtailed procedure and Ramey and Alam's procedure, we infer that the LFC for R_1 is also (3.1). A partial result is given in Theorem 3.1 to solve the conjecture. Numerical calculation based on the theorem is given in Section 4 to support our conjecture. Since procedure R_1 is a generalization of Ramey and Alam's procedure (cf. Remark 2.3), Theorem 3.1 also solved Ramey and Alam's conjecture partially. With a straightforward modification to Theorem 3.1, we can easily obtain Theorem 3.2 which deals with the LFC for procedure R_2.

THEOREM 3.1. *The LFC for procedure R_1 is of the form*

$$\boldsymbol{P} = (0, 0, \ldots, 0, s, p, p, \ldots, p, p\delta_1^*) \quad \text{where} \quad 0 \le s \le p.$$

PROOF. We begin with some notations developed in Bechhofer and Goldsman (1985*a*). Let $X_{1t}, X_{2t}, \ldots, X_{kt}$ denote the frequencies of the cell probabilities $p_1 \le p_2 \le \cdots \le p_{k-1} < p_k$ at stage t of sampling, $t = 1, 2, \ldots$. Here we assume that $p_1 \le p_2 \le \cdots \le p_{k-1} \le p_k$ without loss of generality. Procedure R_1 terminates sampling when any of the condition in (2.4)–(2.7) is satisfied. Define $\#(l_1, l_2, \ldots, l_k)$ to be the number of distinct paths of the sampling process $\{\boldsymbol{X}_t, t = 1, 2, \ldots\}$ which lead to procedure termination exactly when $\boldsymbol{X}_t = (X_{1t}, X_{2t}, \ldots, X_{kt}) = (l_1, \ldots, l_k)$. Then it is clear that

$$\#(l_1, l_2, \ldots, l_k) \tag{3.3}$$

$$= \binom{\sum_{i=1}^{k} l_i}{l_1, l_2, \ldots, l_k} - \sum_{j_1=0}^{l_1} \sum_{j_2=0}^{l_2} \cdots \sum_{j_k=0}^{l_k} \binom{\sum_{i=1}^{k} (l_i - j_i)}{l_1 - j_1, l_2 - j_2, \ldots, l_k - j_k} \times \#(j_1, j_2, \ldots, j_k)$$

where $\binom{a}{a_1, a_2, \ldots, a_k} = a! / \prod_{i=1}^{k} a_i!$ is the multinomial coefficient.

Thus the probability of a correct selection $P(CS \mid R_1)$ can be written as

$$P(CS \mid R_1) = \sum_{1} \#(l_1, l_2, \ldots, l_k) \frac{1}{R(l_1, \ldots, l_k)} \prod_{i=1}^{k} p_i^{l_i} \tag{3.4}$$

where $\sum_1$ is taken over all the possible stopping points $(l_1, l_2, \ldots, l_k)$ with $l_k = \max_{1 \le i \le k} l_i$ and $R(l_1, \ldots, l_k) = \#$ of l_i's whose values are equal to l_k.

Since $\max_{(l_1, l_2, \ldots, l_k)} \sum_{i=1}^{k} l_i = n$, we can make the terms in $P(CS \mid R_1)$ in (3.4) all have order n by multiplying each term in $\sum_1$ a factor $(p_1 + p_2 + \cdots + p_k)^{n - \sum_{i=1}^{k} l_i} = 1$. Thus $P(CS \mid R_1)$ can be rewritten as

$$P(CS \mid R_1) = \sum_{1} \#(l_1, l_2, \ldots, l_k) \times \frac{1}{R(l_1, l_2, \ldots, l_k)} \sum_{2} \binom{n - \sum_{i=1}^{k} l_i}{n_1, n_2, \ldots, n_k} \prod_{i=1}^{k} p_i^{l_i + n_i} \tag{3.5}$$

where $\sum_2$ is over all the possible partitions $(n_1, n_2, \ldots, n_k)$ of $n - \sum_{i=1}^{k} l_i$ in k cells.

In $P(CS \mid R_1)$, we fix $p_3, p_4, \ldots, p_k$ and let $p_1 = x$ and $p_2 = 1 - x - \sum_{i=3}^{k} p_i$. Thus $P(CS \mid R_1)$ becomes a function of x only. We can prove that $P(CS \mid R_1)$ is a non-decreasing function in x by showing that the derivative of $P(CS \mid R_1)$ with respect to x is non-negative. The detailed proof is tedious and it is omitted here. But the complete proof is available from the author upon request.

Thus, we can push p_1 toward 0 and p_2 toward p_k / δ^* and the $P(CS \mid R_1)$ value will not increase by doing this process. Since the proof holds for any $x = p_i < p_j = 1 - x - \sum_{\substack{m \ne i \\ m \ne j}} p_m$, we can apply the same pushing process repeatedly until the LFC in the statement of the theorem is reached. □

Remark 3.1. The results of Theorems 3.1 and 3.2 are the same as the results for fixed-sample-size procedures that were given by Propositions 13.C.1.b and 13.C.2.b of Marshall and Olkin (1979) where the authors have used Schur-Convexity of the decision function to reprove the Theorem 1 of Kesten and Morse (1959) and Theorem 1 of Alam and Thompson (1972) more elegantly. Here our

proofs have followed the approaches in Kesten and Morse (1959) and it is not known to the author at this point that the Schur-Convexity approach will work for our *sequential* decision rule.

Since the proof of Theorem 3.1 does not involve the measure we used to define the PZ in (2.3.1), the same argument can be applied to the PZ in (2.3.2) for procedure R_2. We will only state the result in Theorem 3.2 and omit the lengthy proof.

THEOREM 3.2. *The LFC for procedure R_2 is of the form* $\boldsymbol{P} = (p-\delta_2^*, p, p, \ldots, p, s)$ *where* $0 \leq p - \delta_2^* < p \leq s$.

Remark 3.2. Theorems 3.1 and 3.2 give exactly the same results as given in Lemma 1 of Kesten and Morse (1959) and Lemma 2.1 of Alam and Thompson (1972). Their results were for fixed-sample-size procedures for selecting the most probable and least probable cells respectively. To prove the LFC is the slippage configurations for both selection goals, they went on to prove a second lemma in the respective papers. For our sequential procedures R_1 and R_2 we could neither prove the conjecture that the slippage configurations are LFC's, nor could we find counterexamples to deny the conjecture for general k. However, numerical evidence for $k = 3$ and 4 justifies our conjecture. We present some computational results in Section 4.

4. Tables and concluding remarks

Table 1 presents the combinations (r_1, M_1, n) whose $P(CS \mid R_1, LFC)$'s achieve $P^* = .75, .90, .95$ with $\delta_1^* = 2.0, 2.4, 3.0$ and $k = 3, 4$. The LFC for each individual case is identified in the following manner. By Theorem 3.1, the LFC is for the form $(s, p, p\delta_1^*)$ for $k = 3$ and $(s, p, p, p\delta_1^*)$ for $k = 4$. We first wrote a Fortran program to calculate the exact $P(CS \mid R_1)$ under a general configuration $(p_1, p_2, \ldots, p_k)$. Then for each k and each δ_1^*, we divided the interval where s may take values (i.e., $[0, 1/(k - 1 + \delta_1^*)]$) into 100 equal length subintervals and calculated $P(CS \mid R_1)$ for all configurations whose s's are taken to be the end points of these 100 subintervals. The LFC should be the configuration which minimizes $P(CS \mid R_1)$ under each (k, δ_1^*). For all the cases considered in our table, the LFC's are slippage. For each P^* and δ_1^*, we also tabulated $E(N \mid R_1, LFC)$, the expected sample size under the LFC, and $E(N \mid EPC)$, the expected sample size under the equal parameter configuration $(p, p, \ldots, p)$. The last column of the table presents N_0, the sample size required to guarantee P^* under the fixed-sample-size procedure.

Remark 4.1. The result in Theorem 3.1 saves significant computing time in locating LFC even though the LFC has not been completely identified.

Remark 4.2. By truncating the sample size at n, our procedure R_1 improves upon Ramey-Alam's procedure (see Table IIA and Table IIIA in Bechhofer and Goldsman (1985*a*)) in both criteria $E(N \mid LFC)$ and $E(N \mid EPC)$ in every case

Table 1.

	P^*	δ_1^*	(r_1, M_1, n)	$P(CS \mid R_1, LFC)$	$E(N \mid R_1, LFC)$	$\text{eff}(R_1/RA \mid LFC)$	$E(N \mid R_1, EPC)$	$\text{eff}(R_1/RA \mid EPC)$	N
$k = 3$	.75	3.0	(2, 3, 5)	.75744	3.24160	1.136	3.48148	1.220	5
		2.4	(2, 5, 10)	.75073	4.53410	1.039	5.18702	1.069	7
		2.0	(4, 5, 12)	.75008	8.77598	1.004	9.56423	1.007	12
	.90	3.0	(3, 5, 13)	.90037	6.76178	1.000	8.74049	1.000	11
		2.4	(4, 8, 19)	.90384	11.63743	1.013	14.85723	1.044	18
		2.0	(4, 15, 40)	.90035	16.40898	1.112	23.39602	1.009	29
	.95	3.0	(4, 7, 19)	.95051	9.76760	1.000	13.73120	1.000	17
		2.4	(4, 14, 36)	.95001	13.81188	1.002	22.37740	1.020	26
		2.0	(5, 22, 60)	.95015	22.38875	1.001	36.74166	1.008	42
$k = 4$	.75	3.0	(2, 4, 9)	.75129	4.89512	1.052	5.73975	1.113	8
		2.4	(3, 5, 13)	.75548	8.96387	1.029	10.27102	1.061	12
		2.0	(3, 8, 26)	.75056	13.34458	1.002	16.07583	1.006	20
	.90	3.0	(3, 7, 19)	.90160	9.73890	1.013	13.76206	1.069	16
		2.4	(4, 10, 29)	.90247	16.75011	1.014	22.90395	1.063	26
		2.0	(5, 15, 47)	.90004	27.67384	1.009	37.59982	1.043	41
	.95	3.0	(4, 9, 25)	.95186	13.60210	1.011	20.45754	1.085	23
		2.4	(5, 13, 4)	.95062	22.22853	1.003	33.55981	1.033	37
		2.0	(6, 21, 69)	.95023	34.17621	1.061	49.27084	1.168	53

under consideration. The procedure R_1 also requires smaller maximum sample size than Ramey-Alam's procedure.

Remark 4.3. The n values for procedure R_1 are always taken to be no less than N_0, the size for the fixed sample size procedure of Bechhofer, Elmaghraby and Morse in order to reduce $E(N)$ to its smallest value. However, we do not have to restrict ourselves to choose those n's which are larger than N_0. For example, in the case of $k = 3$, $P^* = .75$, $\delta_1^* = 2.4$,

$$P\{CS \mid R_1, LFC, (r_1, M_1, n) = (4, 4, 7)\} = .75016,$$
$$E\{N \mid R_1, LFC, (r_2, M_1, n) = (4, 4, 7)\} = 5.55940,$$
$$E\{N \mid R_1, EPC, (r_1, M_1, n) = (4, 4, 7)\} = 5.78601.$$

By reducing n to 7 which is the size for fixed-sample-size procedure, we will get larger $E(N)$ values. In our table, we always choose the combination (r_1, M_1, n) that minimizes $E(N \mid R_1, LFC)$.

Table 2 presents the combinations (r_2, M_2, n) whose $P(CS \mid R_2, LFC)$'s achieve $P^* = .75, .90, .95$, with $\delta_2^* = .2, .3$ and $k = 3, 4$. The LFC is identified in a similar manner as in Table 1 described above. Again, LFC's are all slippage. Analogous remarks for Table 2 can also be made as Remarks 4.1 and 4.3 for Table 1.

Table 2.

	P^*	δ_2^*	(r_2, M_2, n)	$P(CS \mid R_2, LFC)$	$E(N \mid R_2, LFC)$	$E(N \mid R_2, EPC)$	N
$k = 3$	.75	.3	(2, 3, 7)	.76636	5.54179	5.68313	7
		.2	(4, 6, 17)	.75586	13.51177	14.05518	16
	.90	.3	(3, 5, 15)	.90230	10.65978	12.13003	15
		.2	(5, 13, 38)	.90164	26.23318	31.30759	37
	.95	.3	(5, 7, 21)	.95002	16.44052	18.79208	21
		.2	(7, 18, 56)	.95021	34.57044	45.73080	54
$k = 4$	.75	.3	(1, 1, 6)	.77591	4.60219	4.14844	6
		.2	(2, 4, 16)	.76072	11.57677	12.07290	15
	.90	.3	(2, 2, 10)	.90809	8.36650	7.83524	10
		.2	(4, 7, 31)	.90026	20.26012	23.94635	29
	.95	.3	(2, 3, 13)	.95132	9.59149	10.48805	13
		.2	(5, 10, 42)	.95078	26.65183	31.26086	40

Remark 4.4. In compliance with a referee's request, we have looked up the $E(N)$ values for Ramey-Alam's procedure (RA) from Bechhofer and Goldsman

(1985*a*) and calculated the relative efficiency of R_1 to RA. We regard this as a measurement to compare these two procedures. The relative efficiency $\text{eff}(R_1/RA)$ is defined as follows:

$$\text{eff}(R_1/RA \mid \boldsymbol{p}) = \frac{E(N \mid RA, \boldsymbol{p})}{E(N \mid R_1, \boldsymbol{p})} \quad \text{where} \quad \boldsymbol{p} \text{ is either LFC or EPC.}$$

The relative efficiency was not computed for R_2 since the fixed-sample-size procedure of Alam and Thompson (1972) was the only existing procedure for selecting $P_{[1]}$ and the fixed-sample-sizes required are shown in the last column of Table 2. We do not regard the fully sequential procedure based on likelihood ratios of Bechhofer *et al.* (1968) as an appropriate procedure to compare with because R_1 and R_2 are not fully sequential and the stopping rules (2.4)–(2.7) and (2.8)–(2.11) are easier to implement than traditional sequential procedures.

Remark 4.5. Asymptotic results based on the largest order statistics and the second order statistics for R_1 and the smallest order statistics and the second smallest order statistics for R_2 require the application of Gnedenko type limit laws and it is very unlikely that they will give close approximation. However, we believe that in this paper, we have fulfilled the purposes of showing that (1) it is possible to improve upon Ramey-Alam's procedure by truncating and curtailing, (2) that the LFC of the proposed procedures are slippage and thus reduce the computing time for the procedure parameters and (3) that the idea of using the difference of order statistics can also be applied to select the smallest cell probability.

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