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CONSECUTIVE k-OUT-OF-n SYSTEMS WITH MAINTENANCE

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Abstract. A consecutive k-out-of-n system consists of n linearly or cyclically ordered components such that the system fails if and only if at least k consecutive components fail. In this paper we consider a maintained system where each component is repaired independently of the others according to an exponential distribution. Assuming general lifetime distributions for system's components we prove a limit theorem for the time to first failure of both linear and circular systems.

Key words and phrases: Consecutive k-out-of-n systems, reliability bounds, maintenance, time to failure, Weibull limit theorem.

1. Introduction

Consecutive k-out-of-n systems have been used to model telecommunications, oil pipelines, vacuum systems in accelerators, computer ring networks and space relay stations. A consecutive k-out-of-n system consists of n components arranged on a line or on a circle. Each component has two states: up (working) and down (failed). The system is considered to be down if and only if at least k consecutive components are down.

During the last decade, a lot of research has been done to compute various reliability characteristics of consecutive k-out-of-n systems for the case of the non-maintained systems, Derman *et al.* (1982), Fu (1985, 1986), Hwang (1986), Papastavridis (1987). In the present paper we examine the situation of a maintained consecutive k-out-of-n system where, each component is separately maintained and undergoes a perfect repair every time it goes down. Repair starts immediately after a component's breakdown, and the repair time is independent of the other components.

In Section 2 we derive lower and upper bounds for the distribution of the time T_n of first failure of a consecutive k-out-of-n linear system with maintenance. Both bounds are expressed via the distribution of time of first failure of parallel subsystems. We mention here that a lot of research work is available for the first failure time of parallel systems, Gaver (1964), Branson and Shah (1971), Bhat (1973), Ross and Schechtman (1979), which can be efficiently used for the evaluation of our bounds. In Section 3 we prove that in the case of a consecutive

k-out-of-n linear system with general lifetimes and exponential repair times, the distribution of T_n approaches the Weibull distribution, as n becomes large. The proof is based on the inequalities given in Section 2 and Brown's analysis of parallel systems (1975). Finally, in Section 4 we show that the same limit theorem applies also in the circular system. The corresponding results for the non-maintained case have been proved by Papastavridis (1987). Recently Chao and Fu (1989) proved a limiting theorem of similar nature for a very large class of systems which includes consecutive k-out-of-n systems as a special case, using a different set of assumptions.

2. Bounds for the linear case

Consider a system consisting of n separately maintained independent components arranged on a line and numbered $1, 2, \ldots, n$. The i-th component is assumed to have a lifetime distribution F_i and repair distribution G_i, $i = 1, 2, \ldots, n$. At time $t = 0$ all components are up (working), and thereon, each component alternates between intervals in which it is up and in which it is down. Suppose that the system works according to the "consecutive k-out-of-n" principle (i.e. it is considered down at time t, if and only if there are at least k consecutive failed components at time t) and denote by T_n the time of the first system failure.

For a fixed $t \geq 0$ we introduce the next events related to the components' performance in the time interval $[0, t]$.

(a) C_i, $i = 1, 2, \ldots, n$ is the event that the i-th component went down at least once in the time interval $[0, t]$. Conventionally we set $C_0 = \emptyset$.

(b) S_i, $i = 1, 2, \ldots, n - k + 1$ is the event that there is an instance in the time interval $[0, t]$, when all components $i, i+1, \ldots, i+k-1$ are down.

It is obvious that the failure of the maintained consecutive k-out-of-n system in the time interval $[0, t]$, means the occurrence of at least one of the events S_i, $i = 1, 2, \ldots, n - k + 1$.

Throughout this paper $\Pr(E)$ denotes the probability of the event E. If E_1, E_2 are any two events then $E_1 - E_2$ will be used for the set theoretic difference among sets, $E_1 E_2$ for the intersection, E' for the complement etc.

Before going to the proof of the main result of this section, we will take care of a small detail, that will be needed in the sequel.

LEMMA 2.1. *If A_i, $i = k, k+1, \ldots, n$ is the event that the consecutive k-out-of-i system consisting of components $1, 2, \ldots, i$ stays continuously up in time interval $[0, t]$, then $\Pr(C_{i-k+1} \mid A_i) \leq \Pr(C_{i-k+1})$.*

PROOF. The assertion is equivalent to $\Pr(A_i' \mid C_{i-k+1}') \leq \Pr(A_i')$. If $k \leq i < 2k$ we have $C_{i-k+1}' \subseteq A_i$, and the above inequality is trivial. For $i \geq 2k$ the proof follows immediately from inequality

$$\Pr(A_i' C_{i-k+1}') = \Pr(A_{i-k}') \Pr(C_{i-k+1}') \leq \Pr(A_i') \Pr(C_{i-k+1}'). \qquad \square$$

We are now ready to prove the following theorem providing an *upper bound* for the reliability of a linear consecutive k-out-of-n system with maintenance.

THEOREM 2.1. *For the time T_n of the first system failure it holds that*

$$\Pr(T_n \geq t) \leq \prod_{i=1}^{n-k+1} \{1 - \Pr(S_i) + \Pr(S_i)\Pr(C_{i-1})\}. \tag{2.1}$$

PROOF. Since $\{A_i\}_{i=k,\dots,n}$ is a monotone decreasing sequence of events (sets) we have that $\Pr(A_i A'_{i+1}) = \Pr(A_i) - \Pr(A_{i+1})$ and

$$\begin{aligned} \Pr(T_n \geq t) &= \Pr(A_n) = \Pr(A_k) \prod_{i=k}^{n-1} \frac{\Pr(A_{i+1})}{\Pr(A_i)} \\ &= \Pr(A_k) \prod_{i=k}^{n-1} \{1 - \Pr(A'_{i+1} \mid A_i)\}. \end{aligned} \tag{2.2}$$

Conditioning on the event C'_{i-k+1} that the $i-k+1$ component is continuously up in the time interval $[0,t]$, and taking Lemma 2.1 into account, we obtain

$$\begin{aligned} \Pr(A'_{i+1} \mid A_i) &\geq \Pr(A'_{i+1} \mid A_i C'_{i-k+1}) \Pr(C'_{i-k+1} \mid A_i) \\ &\geq \Pr(A'_{i+1} \mid A_i C'_{i-k+1}) \{1 - \Pr(C_{i-k+1})\}. \end{aligned}$$

But, it is not difficult to observe that

$$\Pr(A'_{i+1} \mid A_i C'_{i-k+1}) = \Pr(S_{i-k+2}) \tag{2.3}$$

and the proof of the theorem follows immediately. □

The theorem above is an improvement of a result given by Fu ((1986), p. 317, Theorem 1), which is stated there for the non-maintained case. Our proof consists of a sharpening of Fu's ideas. As numerical computations performed by Fu indicate, the previous theorem provides a good approximation of system's reliability for the non-maintained case, and we believe that it is of independent interest.

In the next theorem we give two *lower bounds* for the reliability of a linear consecutive k-out-of-n system with maintenance. The first of them is valid for any life and repair time distributions of the components. The second provides a significantly better approximation to system's reliability on the cost of certain distributional restrictions on components' life and repair times.

THEOREM 2.2. (a) *For the time T_n of the first system failure it holds that*

$$\Pr(T_n \geq t) \geq \prod_{i=1}^{n-k+1} \{1 - \Pr(S_i) - \Pr(C_{i-1})\}.$$

(b) *If each component has exponential failure distribution and the repair time distributions have decreasing repair rate (i.e. repair distributions are DFR) then*

$$\Pr(T_n \geq t) \geq \prod_{i=1}^{n-k+1} \{1 - \Pr(S_i)\}.$$

PROOF. (a) From the total probability theorem we obtain

$$\begin{aligned}\Pr(A'_{i+1} \mid A_i) &= \Pr(A'_{i+1} \mid A_i C_{i-k+1}) \Pr(C_{i-k+1} \mid A_i) \\ &\quad + \Pr(A'_{i+1} \mid A_i C'_{i-k+1}) \Pr(C'_{i-k+1} \mid A_i) \\ &\leq \Pr(C_{i-k+1} \mid A_i) + \Pr(A'_{i+1} \mid A_i C'_{i-k+1})\end{aligned}$$

and making use of Lemma 2.1 and (2.3) we conclude that

$$\Pr(A'_{i+1} \mid A_i) \leq \Pr(C_{i-k+1}) + \Pr(S_{i-k+2}).$$

The assertion is now an immediate consequence of (2.2).

(b) It is a special case of a general result given by Barlow and Proschan ((1976), p. 40, Lemma 4.3). □

It is worth mentioning that the probabilities $\Pr(S_i)$ which appear in Theorems 2.1 and 2.2 represent the distribution of the time to first failure of a parallel subsystem. So, our bounds can be computed for the cases where the corresponding problem for the parallel system has been solved.

3. The limit distribution for the linear system

Before moving to the study of the limit distribution of a consecutive k-out-of-n system with maintenance, we are going to develop some mathematical results useful for the achievement of our final goal. Let us consider first a simple stochastic system consisting of one component which can be either "up" (working) or "down" (failed). When the component goes up (or down) then, independent of the past, it remains up (or down) for a random length of time, having distribution F (or G) and then goes down (or up).

Denote by $p(t)$ ($q(t)$) the probability that the component is up (down) at time t, given it was working at time $t = 0$, by $\lambda(t)$ the instantaneous failure rate of the component, i.e.

$$\lambda(t)dt = \Pr[\text{failure in } (t, t + dt) \mid \text{component is good at time } t]$$

and by h the density function of the time at which the repair of the first breakdown of the component *is completed.* Obviously h is the first derivative of the convolution $F * G'$ with respect to t (here G' denotes the derivative of G).

LEMMA 3.1. (a) *The functions p and λ satisfy the next integral equations*

$$p(t) = [1 - F(t)] + \int_0^t p(t - \tau)h(\tau)d\tau. \tag{3.1}$$

$$\lambda(t)p(t) = F'(t) + \int_0^t \lambda(t - \tau)p(t - \tau)h(\tau)d\tau. \tag{3.2}$$

(b) *If F is of the form $F(t) = (\lambda t)^a + o(t^a)$ then*

$$q(t) = (\lambda t)^a + o(t^a), \qquad \lambda(t)p(t) = a\lambda(\lambda t)^{a-1} + o(t^{a-1}). \tag{3.3}$$

PROOF. (a) For (3.1), condition first on the events of zero or greater than zero component crashes in the time interval $[0, t]$, and in the sequel compute the second term by conditioning on the time of the first repair. For (3.2), observe that $\lambda(t)p(t)dt$ expresses the probability that component is good at time t and fails in the interval $(t, t + dt)$, and proceed as before.

(b) It is not difficult to verify that (3.1) and (3.2) are equivalent to

$$F - F * G' = q - q * h, \qquad \lambda \cdot p = f + (\lambda \cdot p) * h.$$

Equations (3.3) are now a direct consequence of the relation $F'(t) = a\lambda(\lambda t)^{a-1} + o(t^{a-1})$ and the fact that the first term of the convolution's McLaurin expansion is of higher degree than the first term of both convoluted functions. □

Henceforth we are going to restrict ourselves to the case of *exponential repair* times with mean $1/\mu$, i.e. $G(t) = 1 - \exp(-\mu t)$, $t \geq 0$. Furthermore, let F_D denote the cumulative distribution function of the time to first failure of a parallel system with k components, which are separately maintained. The proof of the main result of this section requires the knowledge of the McLaurin's expansion first term of F_D. Adjusting Brown's arguments (1975) to our model, we may introduce the following function

$$P(t) = k\lambda(t)p(t)q^{k-1}(t)$$

which expresses the probability that the system is working at time t and goes down in the time interval $[t, t + dt]$, divided by dt. In other words, $P(t)$ is the derivative of the expected number of failures of the system in $[0, t]$.

The next two lemmas will be crucial in the sequel.

LEMMA 3.2. *There is a function M such that*

$$\int_0^t P(s)ds = F_D(t) + (F_D * M)(t). \tag{3.4}$$

PROOF. See Brown (1975) p. 368 and p. 392, Appendix 8. □

LEMMA 3.3. *If F is of the form $F(t) = (\lambda t)^a + o(t^a)$ and the repair times are exponential with mean $1/\mu$, then*

$$F_D(t) = (\lambda t)^{ak} + o(t^{ak}). \tag{3.5}$$

PROOF. The results of Lemma 3.1 allow us to write

$$P(t) = ka\lambda(\lambda t)^{a-1} \cdot 1 \cdot (\lambda t^a)^{k-1} + o(t^{ak-1}) = ka\lambda^{ak}t^{ak-1} + o(t^{ak-1}).$$

Hence

$$\int_0^t P(s)ds = (\lambda t)^{ak} + o(t^{ak})$$

and (3.5) is now easily deduced from (3.4). □

We conjecture that (3.5) is true for general repair distribution too, but we were unable to prove it. This is the only obstacle in stating the limit theorems of this paper for general repair distribution.

We are now ready to prove a limit theorem concerning the time T_n to first failure of a linear consecutive k-out-of-n system with iid components having arbitrary lifetime distributions and exponential repair times.

THEOREM 3.1. *If F is of the form $F(t) = (\lambda t)^a + o(t^a)$ and the repair times are exponential with mean $1/\mu$, then the random variable $n^{1/ak}T_n$ is asymptotically Weibull i.e.*

$$\lim_{n\to\infty} \Pr(n^{1/ak}T_n \le t) = 1 - \exp[-(\lambda t)^{ak}].$$

PROOF. Let $t_n = tn^{-1/ak}$. Theorem 2.1 gives

$$\Pr(T_n \ge t) \le [1 - F_D(t_n) + F_D(t_n)F(t_n)]^{n-k+1}.$$

On the other hand, from Barlow and Proschan ((1981), p. 34, Theorem 3.4), we know that the reliability of the *non-maintained* linear consecutive k-out-of-n system is bounded below by the quantity $[1 - F^k]^{n-k+1}$ and therefore $[1 - F^k(t_n)]^{n-k+1} \le \Pr(T_n \ge t_n)$. Making use of the form of F and Lemma 3.3, we may verify that

$$\lim_{n\to\infty} (n-k+1)F^k(t_n) = (\lambda t)^{ak}, \qquad \lim_{n\to\infty} F(t_n) = 0,$$
$$\lim_{n\to\infty} (n-k+1)F_D(t_n) = (\lambda t)^{ak}, \qquad F_D(t_n) = \frac{1}{n}(\lambda t)^{ak} + \frac{1}{n}o(n^{-1/ak})$$

and the convergence (as n tends to $+\infty$) of the probability $\Pr(T_n \ge t_n)$ to $\exp[-(\lambda t)^{ak}]$ is now obvious. This completes the proof. □

4. The limit distribution for the circular system

We are now going to state a limit theorem similar to Theorem 3.1, for the case of a *circular* consecutive k-out-of-n system with maintenance. The only difference from the model that we have already studied is that the components $1, 2, \ldots, n$ are arranged on a circle instead of a line.

In order to simplify subsequent calculations, we modify the arguments used in Section 2 as follows: Right after the n-th component of the system consider $k-1$ linearly arranged components, labeled as $n+1, n+2, \ldots, n+k-1$, which are treated as duplications of components $1, 2, \ldots, k-1$ respectively. Therefore the event S_i that there is an instance at the time interval $[0, t]$, when all components $i, i+1, \ldots, i+k-1$ are down, becomes meaningful for all $i = 1, 2, \ldots, n$ and the failure of the circular consecutive k-out-of-n system in the time interval $[0, t]$ is expressed as the union $S_1 \cup S_2 \cup \cdots \cup S_n$.

Denoting by T_n^c the time to first failure of the circular consecutive k-out-of-n system (with maintenance) we may state the next lemma which provides upper and lower bounds for the probability $\Pr(T_n^c \leq t)$.

LEMMA 4.1. *The times T_n^c and T_n satisfy the next double inequality*

$$\Pr(T_n \leq t) \leq \Pr(T_n^c \leq t) \leq \Pr(T_n \leq t) + \sum_{i=n-k+2}^{n} \Pr(S_i). \tag{4.1}$$

PROOF. The inequality follows immediately from the obvious relation

$$\bigcup_{i=1}^{n-k+1} S_i \subseteq \bigcup_{i=1}^{n} S_i \subseteq \left[\bigcup_{i=1}^{n-k+1} S_i\right] \cup \left[\bigcup_{i=n-k+2}^{n} S_i\right]$$

and the subadditivity property of the probability measure. □

The next theorem describes the limiting distribution of T_n^c.

THEOREM 4.1. *If F is of the form $F(t) = (\lambda t)^a + o(t^a)$ and the repair times are exponential with mean $1/\mu$, then the random variable $n^{1/ak}T_n^c$ is asymptotically Weibull i.e.*

$$\lim_{n\to\infty} \Pr(n^{1/ak}T_n^c < t) = 1 - \exp[-(\lambda t)^{ak}].$$

PROOF. Let $t_n = tn^{-1/ak}$. Ineqality (4.1) yields

$$\Pr(n^{1/ak}T_n < t) \leq \Pr(n^{1/ak}T_n^c \leq t) \leq \Pr(n^{1/ak}T_n \leq t) + (k-1)F^k(t_n)$$

and therefore

$$\lim_{n\to\infty} \Pr(n^{1/ak}T_n^c \leq t) = \lim_{n\to\infty} \Pr(n^{1/ak}T_n \leq t) = 1 - \exp[-(\lambda t)^{ak}]. \qquad \square$$

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