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EFFICIENCY OF CONNECTED BINARY BLOCK DESIGNS WHEN A SINGLE OBSERVATION IS UNAVAILABLE

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Abstract. In this paper the problem of finding the design efficiency is considered when a single observation is unavailable in a connected binary block design. The explicit expression of efficiency is found for the resulting design when the original design is a balanced incomplete block design or a group divisible, singular or semiregular or regular with $\lambda_1 = 0$, design. The efficiency does not depend on the position of the unavailable observation. For a regular group divisible design with $\lambda_1 > 0$, the efficiency depends on the position of the unavailable observation. The bounds, both lower and upper, on the efficiency are given in this situation. The efficiencies of designs resulting from a balanced incomplete block design and a group divisible design are in fact high when a single observation is unavailable.

Key words and phrases: Balanced incomplete block design, connectedness, efficiency, group divisible design, robustness.

1. Introduction

The unavailability of data is common in scientific experiments. In statistical planning, it is never possible to anticipate beforehand which observations are going to be unavailable during the experiment. In case of unavailability of data, the experimenter can not redo the experiment with a different design because it costs money, time and effort. However, the experimenter may be interested in knowing whether all the inferences the experimenter originally planned to do can even be possible in this situation and, moreover, the efficiency of the resulting

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design relative to the original design. These facts, which are very common in real life, motivate the research in this paper. Balanced incomplete block and group divisible designs are known to have many optimal properties in the class of incomplete block designs to draw inferences on every possible comparison of treatment effects. This paper demonstrates that balanced incomplete block and group divisible designs remain quite efficient in terms of drawing inferences on treatment effects comparisons when a single observation is unavailable.

We consider a connected binary incomplete block design d with v treatments, b blocks, the common replication r for all treatments and the constant block size k $(\leq v)$. Let N $(v \times b)$ be the incidence matrix of the design d. We assume that the design d is robust against the unavailability of any single observation (see Ghosh (1982*a*)) in the sense that when any single observation is unavailable, the resulting design d_0 remains connected. The total number of observations under d is equal to bk $(= vr)$. There are bk possible cases of a single unavailable observation. We now consider a situation where a single observation is unavailable and the resulting design is d_0. We assume without loss of generality that the unavailable observation is corresponding to the treatment 1 and the block 1. It is known that if the original design d is a balanced incomplete block (BIB) or a group divisible (GD) design then d is robust against the unavailability of any single observation (see Ghosh (1982*a*), Ghosh *et al.* (1983)). For other pertinent research done in the area, see the References. In this paper we study the efficiency of the resulting design d_0 relative to the original design d. Our study goes in detail for BIB and GD designs. Let C and C_0 be the C-matrices in the adjusted normal equations under d and d_0, respectively (see Dey (1986), p. 43). Let $\psi(C)$ and $\psi(C_0)$ be the sum of the inverse of the nonzero characteristic roots of C and C_0, respectively. The efficiency of d_0 relative to that of d is defined as

$$E = \frac{\psi(C)}{\psi(C_0)} \times 100. \tag{1.1}$$

The closer the value of E to 100, the higher the efficiency of d_0. Throughout the paper we abbreviate characteristic root by CR and characteristic vector by CV.

2. Main results

Let d be a connected binary block design and d_0 be the resulting design when a single observation for treatment 1 in block 1 is unavailable. We first partition the incidence matrix of d for (treatment 1, block 1) and the remaining treatments and blocks as

$$N = \begin{bmatrix} 1 & \boldsymbol{u}' \\ \boldsymbol{g} & N^* \end{bmatrix}.$$

The incidence matrix of the design d_0 is then

$$N_0 = \begin{bmatrix} 0 & \boldsymbol{u}' \\ \boldsymbol{g} & N^* \end{bmatrix}.$$

The matrices C and C_0 are then

$$
\begin{aligned}
C &= rI_v - \frac{1}{k}\begin{bmatrix} 1+\boldsymbol{u}'\boldsymbol{u} & \boldsymbol{g}'+\boldsymbol{u}'N^{*\prime} \\ \boldsymbol{g}+N^*\boldsymbol{u} & \boldsymbol{g}\boldsymbol{g}'+N^*N^{*\prime} \end{bmatrix}, \\
C_0 &= \begin{bmatrix} r-1 & \boldsymbol{0}' \\ \boldsymbol{0} & rI_{v-1} \end{bmatrix} - \begin{bmatrix} 0 & \boldsymbol{u}' \\ \boldsymbol{g} & N^* \end{bmatrix} \begin{bmatrix} (k-1)^{-1} & \boldsymbol{0}' \\ \boldsymbol{0} & k^{-1}I_{b-1} \end{bmatrix} \begin{bmatrix} 0 & \boldsymbol{g}' \\ \boldsymbol{u} & N^{*\prime} \end{bmatrix}.
\end{aligned} \tag{2.1}
$$

We define $\boldsymbol{w}' = (k(k-1))^{-1/2}[(k-1), -\boldsymbol{g}']$. Clearly

$$C - C_0 = \boldsymbol{w}\boldsymbol{w}'. \tag{2.2}$$

We observe that $\boldsymbol{j}_v'\boldsymbol{w} = 0$ and $\boldsymbol{w}'\boldsymbol{w} = 1$, where $\boldsymbol{j}_v'$ is a $(1 \times v)$ vector with all elements unity.

THEOREM 2.1. *The design d is robust against the unavailabilty of a single observation if and only if*

$$1 - \boldsymbol{w}'C^+\boldsymbol{w} > 0, \tag{2.3}$$

where C^+ is the Moore-Penrose inverse of C. Furthermore, if (2.3) *holds then*

$$\psi(C_0) = \psi(C) + \frac{\boldsymbol{w}'C^+C^+\boldsymbol{w}}{1-\boldsymbol{w}'C^+\boldsymbol{w}}. \tag{2.4}$$

PROOF. Let $D((v-1)\times(v-1))$ be a diagonal matrix whose diagonal elements are the nonzero CR's of C. We write $C = P'DP$ where $PP' = I_{v-1}$. We have $C^+ = P'D^{-1}P$. There exists an $\boldsymbol{l}$ satisfying $\boldsymbol{w} = P'\boldsymbol{l}$. We get $C_0 = P'AP$ and $A = D - \boldsymbol{l}\boldsymbol{l}'$. Now, d_0 is connected if and only if Rank $C_0 = (v-1)$. Since C_0 is nonnegative definite, Rank $C_0 = (v-1)$ if and only if $|A| > 0$. But $|A| = |D|(1 - \boldsymbol{l}'D^{-1}\boldsymbol{l})$ and $\boldsymbol{l}'D^{-1}\boldsymbol{l} = \boldsymbol{w}'C^+\boldsymbol{w}$. Hence d_0 is connected if and only if $(1 - \boldsymbol{w}'C^+\boldsymbol{w}) > 0$. If (2.3) holds then $|A| > 0$ and $C_0^+ = P'A^{-1}P$.

We have

$$
\begin{aligned}
C_0^+ &= P'A^{-1}P \\
&= P'\left[D^{-1} + \frac{D^{-1}\boldsymbol{l}\boldsymbol{l}'D^{-1}}{1-\boldsymbol{l}'D^{-1}\boldsymbol{l}}\right]P \\
&= C^+ + \frac{C^+\boldsymbol{w}\boldsymbol{w}'C^+}{1-\boldsymbol{w}'C^+\boldsymbol{w}}.
\end{aligned}
$$

Taking trace of the above, we get the equation (2.4). This completes the proof.

Notice that $\boldsymbol{w}'C^-\boldsymbol{w} = \boldsymbol{w}'C^+\boldsymbol{w}$ for any generalized inverse C^- of C because $\boldsymbol{w}$ belongs to the column space of C.

THEOREM 2.2. *A necessary and sufficient condition for μ to be a common CR of C and C_0 with the same CV $\boldsymbol{x}$ is that $\boldsymbol{w}'\boldsymbol{x} = 0$.*

PROOF. We have

$$C_0\boldsymbol{x} = C\boldsymbol{x} - \boldsymbol{w}\boldsymbol{w}'\boldsymbol{x} = C\boldsymbol{x} - (\boldsymbol{w}'\boldsymbol{x})\boldsymbol{w}.$$

If $\boldsymbol{w}'\boldsymbol{x} = 0$ then $C_0\boldsymbol{x} = C\boldsymbol{x} = \mu\boldsymbol{x}$. If $C\boldsymbol{x} = \mu\boldsymbol{x} = C_0\boldsymbol{x}$ then $(\boldsymbol{w}'\boldsymbol{x})\boldsymbol{w} = 0$. Since $\boldsymbol{w} \neq \mathbf{0}$, we have $\boldsymbol{w}'\boldsymbol{x} = 0$. This completes the proof.

THEOREM 2.3. *The vector $\boldsymbol{w}$ is a CV of C with the CR μ if and only if $(\mu - 1)$ is a CR of C_0 with the CV $\boldsymbol{w}$.*

PROOF. If $\boldsymbol{w}$ is a CV of C with the CR μ, then $C\boldsymbol{w} = \mu\boldsymbol{w}$. Now

$$\begin{aligned} C_0\boldsymbol{w} &= C\boldsymbol{w} - \boldsymbol{w}\boldsymbol{w}'\boldsymbol{w} \\ &= \mu\boldsymbol{w} - \boldsymbol{w}, \qquad (\text{since } \boldsymbol{w}'\boldsymbol{w} = 1.) \\ &= (\mu - 1)\boldsymbol{w}. \end{aligned}$$

Thus $(\mu - 1)$ is a CR of C_0 with the CV $\boldsymbol{w}$. The rest is similar. This completes the proof.

COROLLARY 2.1. *If $\boldsymbol{w}$ is a CV of C then C_0 and C have $(v-1)$ CR's in common.*

PROOF. Notice that all CV's $\boldsymbol{x}$ of C except $\boldsymbol{w}$ can be chosen to satisfy $\boldsymbol{w}'\boldsymbol{x} = 0$. The rest is obvious from Theorems 2.2 and 2.3. This completes the proof.

Theorems 2.2, 2.3 and Corollary 2.1 imply that if $\boldsymbol{w}$ is a CV of C with the CR μ then (i) there are $(v-1)$ CR's in common for C and C_0 and (ii) the remaining CR of C_0 is $(\mu - 1)$.

Suppose that the matrix C has two nonzero CR's, namely μ_1 and μ_2. We denote $V_1 = \{\boldsymbol{x} \mid C\boldsymbol{x} = \mu_1\boldsymbol{x}, \mu_1 \neq 0\}$ and $V_2 = \{\boldsymbol{x} \mid C\boldsymbol{x} = \mu_2\boldsymbol{x}, \mu_2 \neq 0\}$, $\dim V_1 = p_1$, $\dim V_2 = p_2$, $p_1 + p_2 = (v-1)$. We consider the situation where $\boldsymbol{w}$ is not a CV of C. Then $\boldsymbol{w}$ does not belong to V_1 and V_2. We denote $V_{1w} = \{\boldsymbol{x} \mid C\boldsymbol{x} = \mu_1\boldsymbol{x}, \mu_1 \neq 0, \boldsymbol{w}'\boldsymbol{x} = 0\}$ and $V_{2w} = \{\boldsymbol{x} \mid C\boldsymbol{x} = \mu_2\boldsymbol{x}, \mu_2 \neq 0, \boldsymbol{w}'\boldsymbol{x} = 0\}$. Now $\boldsymbol{w}$ is not a CV of C and $\boldsymbol{w}'\boldsymbol{x} = 0$ imply $\dim V_{1w} = (p_1 - 1)$ and $\dim V_{2w} = (p_2 - 1)$. We know that 0 is a CR of both C and C_0. The following result now follows from Theorem 2.2.

COROLLARY 2.2. *If $\boldsymbol{w}$ is not a CV of C then the number of common CR's of C and C_0 is $(v-2)$.*

Let the columns of P_i' be an orthonormal basis for V_i, $i = 1, 2$ and $P' = [P_1', P_2']$. We have $PP' = I_{(v-1)}$. Note that $\boldsymbol{w}$ belongs to the column space of P'. We write $\boldsymbol{w} = P'\boldsymbol{l}$. Let $D((v-1) \times (v-1))$ be a diagonal matrix with the first p_1 diagonal elements are μ_1 and the remaining p_2 diagonal elements are μ_2. We have

$C = P'DP$ and $\boldsymbol{ww}' = P'\boldsymbol{ll}'P$. We denote $A = D - \boldsymbol{ll}'$, $\boldsymbol{l}' = (\boldsymbol{l}_1', \boldsymbol{l}_2')$, where $\boldsymbol{l}_1(p_1 \times 1)$, $\boldsymbol{l}_2(p_2 \times 1)$. Thus $C_0 = P'AP$ and $\boldsymbol{l}_1'\boldsymbol{l}_1 + \boldsymbol{l}_2'\boldsymbol{l}_2 = \boldsymbol{l}'\boldsymbol{l} = \boldsymbol{w}'\boldsymbol{w} = 1$. Now

$$
\begin{aligned}
0 &= |C_0 - \theta I_v| = |D - \boldsymbol{ll}' - \theta I_{(v-1)}|(-\theta) \\
&= \begin{vmatrix} D - \theta I & \boldsymbol{l} \\ \boldsymbol{l}' & 1 \end{vmatrix} (-\theta) \\
&= (\mu_1 - \theta)^{p_1 - 1}(\mu_2 - \theta)^{p_2 - 1} \\
&\quad \cdot [(\mu_1 - \theta)(\mu_2 - \theta) - (\mu_2 - \theta)\boldsymbol{l}_1'\boldsymbol{l}_1 - (\mu_1 - \theta)\boldsymbol{l}_2'\boldsymbol{l}_2](-\theta).
\end{aligned} \tag{2.5}
$$

The matrix C_0 has nonzero CR's as μ_1 with multiplicity $(p_1 - 1)$, μ_2 with multiplicity $(p_2 - 1)$ and the remaining two nonzero CR's θ_1 and θ_2 are solutions of the equation

$$
\theta^2 - \theta(\mu_1 + \mu_2 - 1) + (\mu_1\mu_2 - \mu_2\boldsymbol{l}_1'\boldsymbol{l}_1 - \mu_1\boldsymbol{l}_2'\boldsymbol{l}_2) = 0. \tag{2.6}
$$

We therefore get

$$
\begin{aligned}
(\theta_1 + \theta_2) &= (\mu_1 + \mu_2 - 1), \\
\theta_1\theta_2 &= \mu_1\mu_2 - \mu_2\boldsymbol{l}_1'\boldsymbol{l}_1 - \mu_1\boldsymbol{l}_2'\boldsymbol{l}_2 \\
&= \mu_1\mu_2 - \mu_2 - (\mu_1 - \mu_2)\boldsymbol{l}_2'\boldsymbol{l}_2.
\end{aligned} \tag{2.7}
$$

3. BIB design

In this section, we take the design d to be a BIB design (v, b, r, k, λ) (see Dey (1986), p. 32). It is well known that the matrix C has CR's as 0 with multiplicity 1 and $\lambda v k^{-1}$ with multiplicity $(v-1)$. We now have another proof of the following known result (see Whittinghill (1989)).

THEOREM 3.1. *For a BIB design (v, b, r, k, λ), $\boldsymbol{w}$ is the CV of C with the CR $\lambda v k^{-1}$. The CR's of C_0 are $\lambda v k^{-1}$ with multiplicity $(v-2)$, $(\lambda v k^{-1} - 1)$ with multiplicity* 1 *and* 0 *with multiplicity* 1.

PROOF. It is easy to see that $\boldsymbol{w}$ is a CV of C with the CR $\lambda v k^{-1}$. Now by Theorem 2.3, $(\lambda v k^{-1} - 1)$ is a CR of C_0 with the CV $\boldsymbol{w}$. Again by Corollary 2.1, C_0 and C have $(v-1)$ CR's in common. The common CR's are therefore $\lambda v k^{-1}$ with multiplicity $(v-2)$ and 0 with multiplicity 1. The rest is clear. This completes the proof.

We get $\psi(C) = (v-1)k(\lambda v)^{-1}$ and $\psi(C_0) = (v-2)k(\lambda v)^{-1} + (\lambda v k^{-1} - 1)^{-1}$. For a BIB design, it can be seen that $1 - \boldsymbol{w}'C^{+}\boldsymbol{w} = 1 - k/\lambda v \geq 1 - k/v > 0$. Hence (2.4) is true.

We define $E_0 = (v-1)(\lambda v k^{-1} - 1)$. We have the efficiency of d_0 relative to that of d as

$$
E = (1 + E_0^{-1})^{-1} \times 100. \tag{3.1}
$$

Notice that $E \geq 90$ if and only if $E_0 \geq 9$, $80 \leq E < 90$ if and only if $4 \leq E_0 < 9$ and $75 \leq E < 80$ if and only if $3 \leq E_0 < 4$. Thus the value of E_0 is the indicator for efficiency. Out of 91 BIB designs listed in Raghavarao ((1971), pp. 91–94), 88 designs (Series 4–91) have $E \geq 90$, 2 designs (Series 2 and 3) have the values of E between 80 and 90 and only the first design in the Series has 75 as the value of E.

4. GD design

In this section we take the design d to be a GD design $(v = mn, b, r, k, \lambda_1, \lambda_2)$. (See Dey (1986), p. 166.) It is well known that the matrix C for d has CR's as 0 with multiplicity 1, $\mu_1 = [r(k-1) + \lambda_1]k^{-1}$ with multiplicity $(v-m)$ and $\mu_2 = \lambda_2 v k^{-1}$ with multiplicity $(m-1)$. For a connected GD design, we have $0 \leq \boldsymbol{w}'C^+\boldsymbol{w} = \boldsymbol{l}'D^{-1}\boldsymbol{l} \leq [\text{Min}(\mu_1, \mu_2)]^{-1}\boldsymbol{l}'\boldsymbol{l} = [\text{Min}(\mu_1, \mu_2)]^{-1}$, since $\boldsymbol{l}'\boldsymbol{l} = 1$. For a connected GD design, $\lambda_2 \geq 1$ and hence $\mu_2 > 1$. It can be seen that $\mu_1 > 1$ except for the semi regular GD design with parameters $v = b = 4$, $m = n = 2$, $r = k = 2$, $\lambda_1 = 0$ and $\lambda_2 = 1$. It then follows that $1 - \boldsymbol{w}'C^+\boldsymbol{w} > 0$. Theorem 2.1 can now be used in calculating $\psi(C_0)$. Let J_n be an $(n \times n)$ matrix with all elements unity. We now denote $W^{11} = [I_m - m^{-1}J_m] \otimes [I_n - n^{-1}J_n]$, $W^{01} = [m^{-1}J_m] \otimes [I_n - n^{-1}J_n]$ and $W^{10} = [I_m - m^{-1}J_m] \otimes [n^{-1}J_n]$, where $\otimes$ denotes the Kronecker product. It is easy to check that $C = \mu_1[W^{11} + W^{01}] + \mu_2 W^{10}$. It follows that $C^+ = \mu_1^{-1}[W^{11} + W^{01}] + \mu_2^{-1}W^{10}$. Let $\rho_1 = \mu_1^{-1}$, $\rho_2 = n^{-1}[\mu_2^{-1} - \mu_1^{-1}]$ and $\rho_3 = [-v^{-1}\mu_2^{-1}]$; $\gamma_1 = \mu_1^{-2}$, $\gamma_2 = n^{-1}[\mu_2^{-2} - \mu_1^{-2}]$ and $\gamma_3 = [-v^{-1}\mu_2^{-2}]$. It follows that $C^+ = \rho_1 I_m \otimes I_n + \rho_2 I_m \otimes J_n + \rho_3 J_m \otimes J_n$ and $C^+C^+ = \gamma_1 I_m \otimes I_n + \gamma_2 I_m \otimes J_n + \gamma_3 J_m \otimes J_n$.

Suppose that the unavailable observation is for the treatment 1 of group 1 occurring in block 1. Furthermore the first block of d where the unavailability of observation occurs, contains α_i treatments from group i, $1 \leq i \leq m$. We denote $\beta_i = n - \alpha_i$ and $\Delta = (k - \alpha_1)^2 + \sum_{i=2}^m \alpha_i^2$. We write without loss of generality that $\boldsymbol{g} = [\boldsymbol{j}'_{\alpha_1 - 1}, \boldsymbol{0}'_{\beta_1}, \boldsymbol{j}'_{\alpha_2}, \boldsymbol{0}'_{\beta_2}, \ldots, \boldsymbol{j}'_{\alpha_m}, \boldsymbol{0}'_{\beta_m}]'$. It then follows from (2.4) that

$$\begin{aligned} (4.1) \qquad \psi(C_0) &= \psi(C) + [k(k-1)\gamma_1 + \gamma_2\Delta][k(k-1)(1-\rho_1) - \rho_2\Delta]^{-1} \\ &= (v - m - 1)\mu_1^{-1} + (m-2)\mu_2^{-1} \\ &\quad + [k(k-1)(\mu_1 + \mu_2 - 1)] \\ &\quad \cdot [k(k-1)\mu_2(\mu_1 - 1) + (\mu_2 - \mu_1)n^{-1}\Delta]^{-1}. \end{aligned}$$

Although we do not use Corollary 2.2 to calculate $\psi(C_0)$, it is useful in knowing that there are $(v-2)$ common CR's of C and C_0.

Table 1. The values of Δ for various types of GD designs.

type	parameters	values of α's	Δ
singular	$r = \lambda_1$, $k = nc$	$\alpha_1 = \cdots = \alpha_c = n$, $\alpha_{c+1} = \cdots = \alpha_m = 0$	$(k-n)k$
semiregular	$r > \lambda_1$, $rk = v\lambda_2$	$\alpha_1 = \cdots = \alpha_m = m^{-1}k$	$k^2(m-1)m^{-1}$
regular	$rk > v\lambda_2$, and $r > \lambda_1$, $\lambda_1 = 0$	$\alpha_1 = \cdots = \alpha_k = 1$, $\alpha_{k+1} = \cdots = \alpha_m = 0$	$k(k-1)$

Table 2. E values for singular GD designs.

No.	E	No.	E	No.	E	No.	E	No.	E	No.	E
S1	80.0	S25	98.5	S49	99.6	S73	99.2	S97	99.8	S121	99.6
S2	92.9	S26	97.6	S50	99.7	S74	99.1	S98	97.7	S122	99.7
S3	95.7	S27	95.0	S51	96.3	S75	99.4	S99	99.0	S123	99.8
S4	96.9	S28	97.5	S52	98.4	S76	99.6	S100	92.9	S124	99.9
S5	97.6	S29	98.0	S53	90.9	S77	99.5	S101	97.6		
S6	91.7	S30	98.8	S54	96.9	S78	99.6	S102	98.5		
S7	96.7	S31	98.9	S55	98.1	S79	99.7	S103	99.0		
S8	97.9	S32	95.7	S56	98.7	S80	99.7	S104	99.2		
S9	95.5	S33	98.3	S57	99.0	S81	99.7	S105	99.3		
S10	98.1	S34	98.9	S58	99.0	S82	95.5	S106	97.2		
S11	97.1	S35	97.3	S59	97.3	S83	98.2	S107	98.9		
S12	98.8	S36	98.9	S60	98.8	S84	98.8	S108	99.3		
S13	98.0	S37	97.7	S61	96.4	S85	98.5	S109	99.3		
S14	98.6	S38	98.3	S62	98.6	S86	98.4	S110	98.7		
S15	98.9	S39	99.0	S63	98.8	S87	99.3	S111	99.4		
S16	99.2	S40	99.3	S64	99.1	S88	97.3	S112	98.5		
S17	99.3	S41	99.2	S65	99.1	S89	98.9	S113	99.4		
S18	92.9	S42	98.8	S66	98.1	S90	99.3	S114	99.1		
S19	97.1	S43	99.2	S67	98.9	S91	98.6	S115	99.6		
S20	98.2	S44	99.1	S68	99.2	S92	99.4	S116	99.4		
S21	87.5	S45	99.4	S69	98.8	S93	99.5	S117	99.5		
S22	95.7	S46	99.3	S70	99.5	S94	99.4	S118	99.3		
S23	97.4	S47	99.5	S71	98.5	S95	99.6	S119	99.7		
S24	98.1	S48	99.6	S72	99.4	S96	99.8	S120	99.6		

Table 3. E values for semiregular GD designs.

No.	E	No.	E	No.	E	No.	E	No.	E
SR1	50.0	SR26	95.5	SR51	99.6	SR76	99.6	SR101	99.3
SR2	83.3	SR27	98.2	SR52	95.8	SR77	99.5	SR102	99.5
SR3	90.0	SR28	97.3	SR53	97.5	SR78	99.6	SR103	99.6
SR4	92.9	SR29	98.9	SR54	98.2	SR79	99.7	SR104	99.8
SR5	94.4	SR30	98.2	SR55	98.6	SR80	97.2	SR105	99.8
SR6	80.0	SR31	98.7	SR56	98.3	SR81	98.3	SR106	98.9
SR7	92.9	SR32	99.1	SR57	99.0	SR82	98.8	SR107	99.2
SR8	95.7	SR33	99.3	SR58	97.8	SR83	99.1	SR108	99.4
SR9	90.0	SR34	99.4	SR59	99.1	SR84	98.9	SR109	99.5
SR10	96.2	SR35	95.7	SR60	98.7	SR85	99.3	SR110	99.8
SR11	94.1	SR36	94.4	SR61	99.4	SR86	99.4		
SR12	97.6	SR37	96.7	SR62	99.4	SR87	99.6		
SR13	96.2	SR38	96.7	SR63	99.5	SR88	99.7		
SR14	97.3	SR39	97.6	SR64	99.6	SR89	99.8		
SR15	98.0	SR40	98.2	SR65	97.4	SR90	98.1		
SR16	98.5	SR41	94.1	SR66	96.7	SR91	98.6		
SR17	98.8	SR42	97.7	SR67	98.0	SR92	99.0		
SR18	75.0	SR43	98.6	SR68	98.0	SR93	99.2		
SR19	91.7	SR44	97.1	SR69	98.6	SR94	99.4		
SR20	95.0	SR45	98.8	SR70	99.0	SR95	99.5		
SR21	96.4	SR46	98.3	SR71	98.9	SR96	99.7		
SR22	97.2	SR47	99.2	SR72	98.7	SR97	99.7		
SR23	90.9	SR48	99.2	SR73	99.2	SR98	99.8		
SR24	96.6	SR49	99.4	SR74	99.3	SR99	98.8		
SR25	97.9	SR50	99.5	SR75	99.0	SR100	99.1		

Table 4. E values for regular GD designs with $\lambda_1 = 0$.

No.	E	No.	E
R18	87.5	R113	98.6
R23	95.0	R114	96.9
R29	94.4	R116	98.7
R34	95.0	R125	99.0
R36	96.9	R128	99.2
R38	97.4	R129	99.3
R39	97.7	R130	99.3
R40	98.0	R144	97.4
R41	98.4	R147	98.8
R54	90.0	R153	98.7
R55	96.2	R154	99.4
R57	97.6	R161	99.6
R70	96.7	R162	99.7
R75	98.4	R163	99.7
R77	98.6	R183	99.6
R79	97.7	R191	99.7
R81	97.9	R199	99.5
R86	98.0	R201	99.8
R88	98.8	R202	99.8
R90	99.0		
R91	99.1		
R92	99.2		
R93	99.3		
R106	98.1		
R112	96.7		

Table 5. Bounds on E values for regular GD designs with $\lambda_1 > 0$.

No.	lower	upper	No.	lower	upper	No.	lower	upper
R1	79.0	87.5	R26	92.7	97.9	R59	94.8	96.3
R2	80.0	92.3	R27	94.9	95.7	R60	95.1	97.4
R3	88.2	84.6	R28	92.7	98.3	R61	95.1	98.0
R4	80.1	94.7	R30	95.9	96.9	R62	96.8	97.3
R5	80.0	96.1	R31	96.1	97.7	R63	95.1	98.5
R6	90.7	92.9	R32	96.5	97.4	R64	95.0	98.8
R7	88.9	92.1	R33	96.2	98.2	R65	97.7	98.0
R8	79.8	97.1	R35	97.1	97.7	R66	94.9	99.0
R9	91.1	94.6	R37	97.5	98.0	R67	97.9	98.4
R10	91.9	93.2	R42	84.4	89.0	R68	97.8	98.2
R11	79.5	97.7	R43	94.2	95.5	R69	95.6	97.5
R12	91.2	95.7	R44	93.9	95.5	R71	97.1	97.6
R13	91.3	94.1	R45	94.4	96.5	R72	97.5	98.1
R14	79.3	98.1	R46	95.6	96.1	R73	97.2	98.2
R15	91.2	96.6	R47	94.4	97.2	R74	97.8	98.3
R16	94.1	95.0	R48	96.2	96.6	R76	97.7	98.9
R17	93.2	94.7	R49	94.4	97.7	R78	98.5	98.6
R19	92.0	94.4	R50	96.2	97.2	R80	98.0	98.9
R20	92.8	95.1	R51	96.6	97.1	R82	98.4	98.7
R21	92.5	96.2	R52	96.7	97.1	R83	98.5	98.8
R22	92.6	97.2	R53	94.3	98.0	R84	98.5	98.9
R24	94.3	95.2	R56	96.9	98.0	R85	98.5	99.1
R25	93.3	96.8	R58	97.5	97.8			

Table 5. (Continued).

No.	lower	upper	No.	lower	upper	No.	lower	upper
R87	98.7	98.9	R123	98.2	99.3	R157	99.2	99.5
R89	98.9	99.0	R124	99.0	99.3	R158	99.2	99.5
R94	91.5	93.3	R126	99.2	99.4	R159*	99.6	99.6
R95	96.3	97.1	R127	99.3	99.4	R160*	99.6	99.6
R96	96.8	96.9	R131	99.4	99.5	R164	98.1	98.2
R97	95.6	96.0	R132*	99.5	99.5	R165*	98.5	98.5
R98	97.4	97.7	R133	94.7	96.5	R166	96.4	97.9
R99	97.4	97.8	R134	96.0	96.3	R167	98.5	98.9
R100	97.5	98.1	R135	97.7	98.4	R168	96.7	98.7
R101	97.9	98.0	R136	98.2	98.3	R169	99.0	99.1
R102	97.6	98.4	R137	96.0	96.8	R170	99.0	99.1
R103	98.1	98.2	R138	98.2	98.6	R171	99.1	99.2
R104	92.3	95.8	R139	96.6	97.0	R172	97.8	97.9
R105	96.8	98.2	R140	97.9	98.0	R173	97.3	98.6
R107	97.8	98.7	R141	98.5	98.6	R174	98.2	98.6
R108	98.4	98.6	R142	98.5	98.7	R175*	98.4	98.4
R109	95.9	96.4	R143	95.3	97.9	R176*	98.4	98.4
R110	98.3	98.5	R145	97.2	97.5	R177*	98.6	98.6
R111	98.7	98.8	R146	98.0	99.0	R178	97.6	99.1
R115	97.7	99.0	R148	98.8	98.9	R179	99.0	99.1
R117	98.6	98.8	R149	98.8	99.2	R180*	99.0	99.0
R118	98.1	98.4	R150	99.0	99.1	R181*	99.5	99.5
R119	98.2	98.8	R151	99.1	99.3	R182	99.3	99.4
R120	98.2	99.0	R152	98.6	99.5	R184*	99.7	99.7
R121	98.2	99.2	R155	99.1	99.2	R185*	99.7	99.7
R122	98.9	99.0	R156	99.1	99.4	R186	98.6	98.7

Table 5. (Continued).

No.	lower	upper	No.	lower	upper
R187	97.9	99.0	R198	98.4	99.5
R188	98.1	99.4	R200	99.4	99.5
R189*	99.3	99.3	R203*	99.0	99.0
R190	99.6	99.7	R204*	99.1	99.1
R192*	99.8	99.8	R205*	99.1	99.1
R193*	98.8	98.8	R206	98.6	99.4
R194	99.0	99.1	R207	98.7	99.6
R195	98.3	99.2	R208	99.5	99.6
R196*	99.2	99.2	R209*	99.8	99.8
R197*	99.2	99.2			

*Indicates designs for which lower and upper bounds coincide.

For regular GD designs with $\lambda_1 > 0$, we now present bounds $\Delta_1 \leq \Delta \leq \Delta_2$ for various k. Notice that $1 \leq \alpha_1 \leq n$, $0 \leq \alpha_i \leq n$, $i = 2, 3, \ldots, m$ and $\alpha_2 + \cdots + \alpha_m = k - \alpha_1$. We denote the greatest integers in $(m-1)^{-1}(k-n)$ and $n^{-1}(k-1)$ by u and t, respectively. If $k \leq n$, we have $\Delta \geq 0$ and if $k > n$ then $\Delta \geq (k-n)^2 + (m-1)u^2 + (2u+1)[(k-n) - u(m-1)]$. Moreover, if $(k-1) \geq n(m-1)$ then $\Delta \leq n^2(m-1)m$ and if $(k-1) < n(m-1)$ then $\Delta \leq (k-1)^2 + tn^2 + (k-1-nt)^2$.

It can be checked that the expression $[k(k-1)\mu_2(\mu_1-1)+(\mu_2-\mu_1)n^{-1}\Delta]$ is in fact positive for $\Delta=\Delta_1=0$, $\Delta=\Delta_1=(k-n)^2+(m-1)u^2+(2u+1)[(k-n)-u(m-1)]$ and for $\Delta=\Delta_2=n^2(m-1)m$, $\Delta=\Delta_2=(k-1)^2+tn^2+(k-1-nt)^2$. We denote $\psi_1(C_0)$ and $\psi_2(C_0)$ as the values of $\psi(C_0)$ for $\Delta=\Delta_1$ and $\Delta=\Delta_2$, respectively in (4.1). It follows that $\psi_2(C_0)\le\psi(C_0)\le\psi_1(C_0)$ for $\lambda_2>\lambda_1$ and $\psi_1(C_0)\le\psi(C_0)\le\psi_2(C_0)$ for $\lambda_2<\lambda_1$. If $E_1=(\psi(C)/\psi_1(C_0))\times 100$ and $E_2=(\psi(C)/\psi_2(C_0))\times 100$, then the difference E_1-E_2 becomes small as λ_1 approaches λ_2.

We present in Tables 2–4, the E values for GD designs given in Clatworthy (1973), which are singular, semiregular and regular with $\lambda_1=0$. We also present in Table 5, the E_1 and E_2 values for regular GD designs with $\lambda_1>0$. We call the minimum of E_1 and E_2 as lower and the other as upper.

In Table 2, all designs except those numbered S1 and S21 have E values greater than 90. For designs numbered S1 and S21, E values are greater than or equal to 80. In Table 3, all designs except those numbered SR1, SR2, SR6 and SR18 have E values higher than 90. For designs numbered SR2 and SR6, E values are greater than or equal to 80. The design SR1 had E value 50 and the design SR18 has E value 75. In Table 4, the design numbered R18 has E value 87.5 and all other E values are higher than 90. In Table 5 the designs, numbered R1–R37, have $k=2$, $\lambda_1,\lambda_2>0$. For these designs, we have $2=k\le n$ and this implies $\Delta_1=0$ which corresponds to $\alpha_1=2$ and $\alpha_i=0$ for $i=2,\ldots,m$. This is indeed possible because $\lambda_1>0$. Also, $1=(k-1)<n(m-1)$. The greatest integer in $n^{-1}(k-1)$ is 0. Hence $\Delta_2=(k-1)^2+tn^2+(k-1-nt)^2=2$. This corresponds to the situation with $\alpha_1=1$, exactly one among $\alpha_2,\ldots,\alpha_m$ is 1 and the rest are zero. This is again possible because $\lambda_2>0$. The bounds presented in Table 5 for these designs can in fact be attained depending on the unavailable observation. We now consider the remaining 133 designs in Table 5 with $\lambda_1>0$ and $k\ge 3$. The lower and upper values in Table 5 are very close for these 133 designs. Moreover, these lower and upper values are 90 and above for all 133 designs except for the design numbered R42. The lower values for all designs in Table 5 are greater than 80 except for a few very close to 80.

5. Conclusion

There are many robustness properties of designs. BIB designs and GD designs are known to be robust against the unavailability of any observation in the sense that the resulting design is connected. In this paper we establish another robustness property of BIB designs and GD designs against the unavailability of a single observation and in terms of efficiency in the sense that the efficiency is fairly high for most resulting designs.

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