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MSE'S OF PREDICTION IN GROWTH CURVE MODEL WITH COVARIANCE STRUCTURES*

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Abstract. We consider the growth curve model with covariance structures: positive-definite, uniform covariance structure and serial covariance structure. Two types of prediction problems are studied in this paper. One is called the conditional prediction problem and the other is called the extended prediction problem. For both types of prediction problems, the mean squared error for a serial covariance structure is obtained for the estimates based on the conditional expectation; the mean squared error for an unrestricted covariance structure is compared with the mean squared error for a uniform covariance structure or a serial covariance structure. These results are exemplified by two sets of real data.

Key words and phrases: Growth curve model, serial covariance structure, uniform covariance structure, mean squared error, conditional prediction, extended prediction.

1. Introduction

Since Potthoff and Roy (1964) first proposed a generalized multivariate analysis of variance model (GMANOVA model or a growth curve model), Rao (1965, 1987), Khatri (1966), Grizzle and Allen (1969), Lee and Geisser (1972), Reinsel (1984*a*, 1984*b*), Lee (1988) and many other authors have studied this model. The growth curve model is defined as

$$(1.1) \qquad \underset{N\times p}{Y} = \underset{N\times k}{A} \underset{k\times q}{\Xi} \underset{q\times p}{B} + \underset{N\times p}{\epsilon},$$

where Y is an observed random matrix, A and B are known design matrices of ranks k and $q \le p$, respectively, and Ξ is an unknown parameter matrix. Further, the rows of ϵ are independent and identically distributed random vectors with distribution $N_p(\mathbf{0}, \Sigma)$. In most applications of the model, p is the number of time

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points observed on each of the N subjects, $(q-1)$ is the degree of the polynomial, and k is the number of groups.

Reinsel (1984*a*, 1984*b*), Lee (1988) and others studied two types of prediction problems. One is called the conditional prediction of the unobserved portion of a partially observed vector and the other is called the extended prediction of the future values of a given vector using its past observations. Reinsel (1984*b*) considered prediction problems where the covariance matrix Σ has a linear structure represented as

$$\Sigma = \sum_{i=1}^{l} \sigma_i G_i, \tag{1.2}$$

where the G_i are known symmetric matrices which are linearly independent, and the σ_i are unknown parameters. This covariance structure contains the unrestricted covariance structure and the uniform covariance structure as special cases. However, Reinsel (1984*b*) did not treat the case when Σ has serial covariance structure. On the other hand, Lee (1988) considered prediction problems where Σ has uniform covariance structure and a serial covariance structure. We also consider two types of prediction problems with specific covariance structure.

Let $\boldsymbol{y} : p \times 1$ be $(N+1)$-st vector of future observation drawn from the model (1.1); that is, $\boldsymbol{\mu}' = E(\boldsymbol{y}') = A_1' \Xi B$, where A_1 is a known $k \times 1$ vector, $\boldsymbol{y}$ is distributed as multivariate normal with unknown covariance matrix Σ, and $\boldsymbol{\mu}'$ is the transpose of $\boldsymbol{\mu}$. Moreover, let $\boldsymbol{y}$ be partitioned as $\boldsymbol{y} = (\boldsymbol{y}_1', \boldsymbol{y}_2')'$, where $\boldsymbol{y}_i$ is $p_i \times 1$, $(i = 1, 2)$ and $p_1 + p_2 = p$. Lee and Geisser (1972), Lee (1988) and other authors considered the problem of predicting $\boldsymbol{y}_2$, given $\boldsymbol{y}_1$ and Y. This is the conditional prediction problem.

Next, let $\boldsymbol{y}_f : m \times 1$ be a future observation whose previous p-dimensional observation is a vector of Y. Rao (1987) and Lee (1988) considered the problem of predicting $\boldsymbol{y}_f$, given Y. This type of prediction is called the extended prediction problem. To study this type of prediction, the covariance structure of $\boldsymbol{y}_f$ is assumed similar to the case of the conditional prediction problem. We need some structure on Σ to predict $\boldsymbol{y}_f$. In this paper, three types of covariance structures are considered: that is, positive-definite covariance structure, uniform covariance structure and serial covariance structure. In particular, the last two covariance structures are useful for the growth curve model.

In Section 2, we give the theoretical mean squared error (MSE) of prediction for the serial covariance structure, whereas Lee (1988) gave the empirical MSE's. In Section 3, we compare the positive-definite covariance structure with the uniform covariance structure and the serial covariance structure for the MSE's. In Section 4, we apply our theoretical MSE and Reinsel's results (1984*b*) to the two sets of real data, dental measurement data and ramus height data, for both types of prediction, and make some comparisons between Lee's numerical results (1988) and our results.

2. MSE of the case for the serial covariance structure

In this section, we derive the MSE of prediction (up to order n^{-1}) for a serial covariance structure. The serial covariance structure takes the form

$$\Sigma = \sigma^2 G(\rho) = \sigma^2(\rho^{|i-j|}), \qquad i, j = 1, 2, \ldots, p, \tag{2.1}$$

where $\sigma > 0$ and $|\rho| < 1$ are unknown. Anderson (1971) has obtained the maximum likelihood estimators (MLE's) $\hat{\rho}$ and $\hat{\sigma}^2$ in a time series setting, and Azzalini (1984, 1987) has derived the MLE's for the growth curve model for an AR(1) covariance structure. Lee (1988) and Fujikoshi *et al.* (1990) have also obtained the MLE's for a growth curve model with serial covariance structure. In what follows, we use the MLE's $\hat{\Xi}$, $\hat{\sigma}^2$ and $\hat{\rho}$ of Ξ, σ^2 and ρ, respectively, based on Y which are given as the solutions of the following equations (see, e.g., Fujikoshi *et al.* (1990)):

$$\begin{aligned} &\hat{\Xi} = (A'A)^{-1}A'Y\hat{\Sigma}^{-1}B'(B\hat{\Sigma}^{-1}B')^{-1}, \\ &\hat{\sigma}^2 = \frac{n}{N}\{p(1-\hat{\rho}^2)\}^{-1}(a\hat{\rho}^2 - 2b\hat{\rho} + c), \\ &(p-1)a\hat{\rho}^3 - (p-2)b\hat{\rho}^2 - (pa+c)\hat{\rho} + pb = 0, \end{aligned} \tag{2.2}$$

where $\hat{\Sigma} = \hat{\sigma}^2 G(\hat{\rho})$, $a = \mathrm{tr}\, D_1 R$, $b = \mathrm{tr}\, D_2 R$, $c = \mathrm{tr}\, D_3 R$, $R = n^{-1}(Y - A\hat{\Xi}B)'(Y - A\hat{\Xi}B)$, $D_1 = \mathrm{diag}(0, 1, \ldots, 1, 0)$, $D_3 = I_p$ and

$$D_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 & \cdots & 0 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 1 & 0 \end{pmatrix}.$$

In order to consider the conditional prediction of $\boldsymbol{y}_2$, given $\boldsymbol{y}_1$ and Y, we partition the mean vector $\boldsymbol{\mu}$ and covariance matrix Σ as

$$\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}, \tag{2.3}$$

respectively, corresponding to the partition $(\boldsymbol{y}_1', \boldsymbol{y}_2')'$ of $\boldsymbol{y}$. We consider a conditional predictor $\hat{\boldsymbol{y}}_2$ of $\boldsymbol{y}_2$ as

$$\hat{\boldsymbol{y}}_2 = \hat{\boldsymbol{\mu}}_2 + \hat{\Sigma}_{21}\hat{\Sigma}_{11}^{-1}(\boldsymbol{y}_1 - \hat{\boldsymbol{\mu}}_1), \tag{2.4}$$

where the symbol $\hat{}$ on the right-hand-side denotes the maximum likelihood estimator of the model based on Y. To study the MSE of the prediction of $\boldsymbol{y}_2$ for the serial covariance structure, note that

$$\boldsymbol{y}_2 - \hat{\boldsymbol{y}}_2 = (-\hat{\Sigma}_{21}\hat{\Sigma}_{11}^{-1}\ I_{p_2})(\boldsymbol{y} - \boldsymbol{\mu}) + (-\hat{\Sigma}_{21}\hat{\Sigma}_{11}^{-1}\ I_{p_2})(\boldsymbol{\mu} - \hat{\boldsymbol{\mu}}),$$

the conditional expectation of $(\boldsymbol{y}_2 - \hat{\boldsymbol{y}}_2)(\boldsymbol{y}_2 - \hat{\boldsymbol{y}}_2)'$ given Y is

$$E[(\boldsymbol{y}_2 - \hat{\boldsymbol{y}}_2)(\boldsymbol{y}_2 - \hat{\boldsymbol{y}}_2)' \mid Y] = h_1(\hat{\Sigma}) + h_2(\hat{\Sigma}, \hat{\boldsymbol{\mu}}),$$

where

$$h_1(\hat{\Sigma}) = (-\hat{\Sigma}_{21}\hat{\Sigma}_{11}^{-1}\ I_{p_2})\Sigma(-\hat{\Sigma}_{21}\hat{\Sigma}_{11}^{-1}\ I_{p_2})',$$
$$h_2(\hat{\Sigma}, \hat{\boldsymbol{\mu}}) = (-\hat{\Sigma}_{21}\hat{\Sigma}_{11}^{-1}\ I_{p_2})(\boldsymbol{\mu} - \hat{\boldsymbol{\mu}})(\boldsymbol{\mu} - \hat{\boldsymbol{\mu}})'(-\hat{\Sigma}_{21}\hat{\Sigma}_{11}^{-1}\ I_{p_2})', \tag{2.5}$$

and I_p is the identity matrix of order p. Denote this conditional expectation by $h(\hat{\Sigma}, \hat{\boldsymbol{\mu}})$. Then,

$$E[h(\hat{\Sigma}, \hat{\boldsymbol{\mu}})] = E[h_1(\hat{\Sigma})] + E[h_2(\hat{\Sigma}, \hat{\boldsymbol{\mu}})]. \tag{2.6}$$

The expectation in (2.6) is the MSE of the prediction of $\boldsymbol{y}_2$ by $\hat{\boldsymbol{y}}_2$. In order to compute the expectation $E[h(\hat{\Sigma}, \hat{\boldsymbol{\mu}})]$, we define

$$U = (A'A)^{-1/2}A'(Y - A\Xi B),$$
$$V = \sqrt{n}(\Sigma^{-1/2}S\Sigma^{-1/2} - I_p),$$

where $S = n^{-1}Y'(I_N - A(A'A)^{-1}A')Y$ and $n = N - k$. Then, U and V are independent, the rows of U are distributed as $N_p(\mathbf{0}, \Sigma)$, and the limiting distribution of $V = (v_{ij})$ is normal with mean zero and $\mathrm{Var}(v_{ii}) = 2$, $\mathrm{Var}(v_{ij}) = 1$, $i \neq j$. Moreover, the $\frac{1}{2}p(p+1)$ elements are independent in the limiting distribution. It is well known that

$$E[X'WX] = (\mathrm{tr}\, W)\Sigma + (\mathbf{1}_n'W\mathbf{1}_n)\boldsymbol{\lambda}\boldsymbol{\lambda}', \tag{2.7}$$

where $X' = (X_1, \dots, X_n)$, $X_1, \dots, X_n$, are independent $N_p(\boldsymbol{\lambda}, \Sigma)$ random vectors and W is any $n \times n$ symmetric matrix and $\mathbf{1}_n = (1, \dots, 1)'$, $n \times 1$ vector. Putting $W = U'U$ and using the formula (2.7), we can easily obtain, up to order n^{-1},

$$E[h_2(\hat{\Sigma}, \hat{\boldsymbol{\mu}})] = K(\Sigma) + O(n^{-3/2}), \tag{2.8}$$

where

$$K(Z) = A_1'(A'A)^{-1}A_1(-\Sigma_{21}\Sigma_{11}^{-1}\ I_{p_2})B'(BZ^{-1}B')^{-1}B(-\Sigma_{21}\Sigma_{11}^{-1}\ I_{p_2})'$$

for $k \times 1$ vector A_1 and any $p \times p$ non-singular matrix Z. Note that $K(Z) \sim O(n^{-1})$ under the usual assumption of $A'A = O(n)$. In order to calculate the MSE of prediction up to order n^{-1}, we use the following stochastic expansion of the MLE $\hat{\rho}$ represented in term of V:

$$\hat{\rho} = \rho + n^{-1/2}\rho_1 + O_p(n^{-1}), \tag{2.9}$$

where

$$\rho_1 = -\{(p-1)r\sigma^2\}^{-1}\{(r - \rho^2)\rho a_1 - rb_1 + \rho c_1\},$$
$$a_1 = \mathrm{tr}\, \Sigma^{1/2}D_1\Sigma^{1/2}V, \qquad b_1 = \mathrm{tr}\, \Sigma^{1/2}D_2\Sigma^{1/2}V,$$
$$c_1 = \mathrm{tr}\, \Sigma V, \qquad r = p - (p-2)\rho^2.$$

Noting $\hat{\Sigma}_{21}\hat{\Sigma}_{11}^{-1} = (0, \hat{\gamma})$, $\hat{\gamma} = (\hat{\rho}, \hat{\rho}^2, \dots, \hat{\rho}^{p_2})'$, and using $E[\rho_1^2] = p(1-\rho^2)^2/\{(p-1)r\}$, we can easily obtain

$$E[h_1(\hat{\Sigma})] = \Sigma_{22\cdot 1} + \frac{1}{n} \cdot \frac{p(1-\rho^2)^2\sigma^2}{(p-1)r} \cdot D + O(n^{-3/2}), \tag{2.10}$$

where $\Sigma_{22\cdot 1} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$ and

$$D = \begin{pmatrix} 1 & 2\rho & 3\rho^2 & \cdots & p_2\rho^{p_2-1} \\ 2\rho & 4\rho^2 & 6\rho^3 & \cdots & 2p_2\rho^{p_2} \\ 3\rho^2 & 6\rho^3 & 9\rho^4 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_2\rho^{p_2-1} & 2p_2\rho^{p_2} & \cdots & \cdots & p_2^2\rho^{2(p_2-1)} \end{pmatrix}.$$

Consequently, the MSE of the prediction of $\boldsymbol{y}_2$ by $\hat{\boldsymbol{y}}_2$ is obtained as follows:

$$E[h(\hat{\Sigma}, \hat{\boldsymbol{\mu}})] = \Sigma_{22\cdot 1} + \frac{1}{n} \cdot \frac{p(1-\rho^2)^2\sigma^2}{(p-1)r} \cdot D + K(\Sigma) + O(n^{-3/2}). \tag{2.11}$$

Next, we consider the extended prediction $\boldsymbol{y}_f$, given Y. To make this type of prediction, assume that $\boldsymbol{y}_f$ has the same covariance structure as Y. Then, a predictor $\hat{\boldsymbol{y}}_f$ of $\boldsymbol{y}_f$ is defined by the same way as $\hat{\boldsymbol{y}}_2$. The MSE of the extended prediction can be obtained in a similar way of calculating the MSE of the conditional prediction for the serial covariance structure. However, the positive-definite covariance structure is not explicitly extendable to the future values of the cases observed. Thus, only the MSE of the conditional prediction is considered in Section 3.

3. Comparison of MSE's

We compare the positive-definite covariance structure with uniform covariance structure and serial covariance structure in terms of asymptotic expansion of MSE's, up to order n^{-1}. The uniform covariance structure takes the form

$$\Sigma = \sigma^2[(1-\rho)I_p + \rho\mathbf{1}_p\mathbf{1}_p'], \tag{3.1}$$

where $\sigma > 0$ and $-1/(p-1) < \rho < 1$ are unknown. Reinsel (1984*b*) has obtained an approximation including the term of order n^{-2} for prediction square error matrix in the growth curve model for more general covariance structure (1.2) that includes as special cases positive-definite covariance structure and uniform covariance structure. Using his results, up to order n^{-1}, the MSE's of prediction with positive-definite covariance structure and uniform covariance structure are

$$\Sigma_{22\cdot 1} + \frac{1}{n}p_1\Sigma_{22\cdot 1} + K(\Sigma) \tag{3.2}$$

and

$$\Sigma_{22\cdot 1} + \frac{1}{n} \cdot \frac{2p_1 s^2(1-\rho)^2\sigma^2}{p(p-1)\{1+(p_1-1)\rho\}^3}\mathbf{1}_{p_2}\mathbf{1}_{p_2}' + K(\Sigma), \tag{3.3}$$

where $s = 1 + (p-1)\rho$, respectively.

As the criterion for a comparison of two covariance structures, the quantity

$$n(\operatorname{tr}\Gamma_1 - \operatorname{tr}\Gamma_2)$$

is considered where Γ_1 is the substitution of uniform covariance structure or serial covariance structure into the MSE of prediction with positive-definite covariance structure and Γ_2 is the MSE of prediction with uniform covariance structure or serial covariance structure.

First, we make a comparison between positive-definite covariance structure and uniform covariance structure. In this case, we had only to compare $p_1\Sigma_{22\cdot1}$ (*having uniform covariance structure*) with $[2p_1s^2(1-\rho)^2\sigma^2/\{p(p-1)(1+(p_1-1)\rho)\}^3]\cdot \mathbf{1}_{p_2}\mathbf{1}'_{p_2}$ since $\Sigma_{22\cdot1}$ and $K(\Sigma)$ are common terms from (3.2) and (3.3).

Let $Q_{p,u}$ be the difference between the term $p_1\Sigma_{22\cdot1}$ in (3.2) having uniform covariance structure (3.1) and the term $[2p_1s^2(1-\rho)^2\sigma^2/\{p(p-1)(1+(p_1-1)\rho)\}^3]\cdot \mathbf{1}_{p_2}\mathbf{1}'_{p_2}$ in (3.3); that is,

$$\begin{aligned} Q_{p,u} &= p_1\Sigma_{22\cdot1}(\textit{having uniform covariance structure}) \\ &\quad - \frac{2p_1s^2(1-\rho)^2\sigma^2}{p(p-1)(1+(p_1-1)\rho)^3}\cdot \mathbf{1}_{p_2}\mathbf{1}'_{p_2} \\ &= \frac{p_1(1-\rho)\sigma^2}{p(p-1)\{1+(p_1-1)\rho\}^3}[(f_3-f_2)\cdot I_{p_2} + (f_2-f_1)\cdot \mathbf{1}_{p_2}\mathbf{1}'_{p_2}], \end{aligned}$$

where

$$\begin{aligned} f_1 &= 2s^2(1-\rho), \\ f_2 &= p(p-1)\rho\{1+(p_1-1)\rho\}^2, \\ f_3 &= p(p-1)(1+p_1\rho)\{1+(p_1-1)\rho\}^2. \end{aligned}$$

We note that $Q_{p,u}$ is positive-definite. For $p=2$, we have $Q_{p,u} = \sigma^2(1-\rho^2)\rho^2 > 0$ if $\rho \neq 0$. For $p \geq 3$, the two distinct eigenvalues of the matrix $(f_3-f_2)\cdot I_{p_2} + (f_2-f_1)\cdot \mathbf{1}_{p_2}\mathbf{1}'_{p_2}$ are f_3-f_2 and $f_3-f_1+(p_2-1)(f_2-f_1)$ which can be shown to be positive by elementary calculations. This also shows that our criterion is

$$\operatorname{tr} Q_{p,u} = \frac{p_1p_2(1-\rho)\sigma^2}{p(p-1)\{1+(p_1-1)\rho\}^3}(f_3-f_1) > 0. \tag{3.4}$$

Second, we make a comparison between positive-definite covariance structure and serial covariance structure. In order to discuss the same situation as the above case, we compare the quantity (3.2) having serial covariance structure with the quantity

$$\Sigma_{22\cdot1} + \frac{1}{n}\cdot\frac{p(1-\rho^2)^2\sigma^2}{(p-1)r}\cdot D + K(\Sigma), \tag{3.5}$$

given in (2.11). From the above, we only need compare $p_1\Sigma_{22\cdot1}$ (*having serial covariance structure*) with $[p(1-\rho^2)^2\sigma^2/\{(p-1)r\}]\cdot D$ in (2.11).

Let $Q_{p,s}$ be the difference between the term $p_1\Sigma_{22\cdot 1}$ in (3.2) having serial covariance structure (2.1) and the term $[p(1-\rho^2)^2\sigma^2/\{(p-1)r\}]\cdot D$ in (2.10); that is,

$$
\begin{aligned}
Q_{p,s} &= p_1\Sigma_{22\cdot 1}(\textit{having serial covariance structure}) - \frac{p(1-\rho^2)^2\sigma^2}{(p-1)r}\cdot D \\
&= p_1(1-\rho^2)\sigma^2 \\
&\quad\times \begin{pmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{p_2-1} \\ \rho & 1+\rho^2 & \rho(1+\rho^2) & \cdots & \rho^{p_2-2}(1+\rho^2) \\ \rho^2 & \rho(1+\rho^2) & 1+\rho^2+\rho^4 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{p_2-1} & \rho^{p_2-2}(1+\rho^2) & \cdots & \cdots & 1+\rho^2+\cdots+\rho^{2(p_2-1)} \end{pmatrix} \\
&\quad - \frac{p(1-\rho^2)^2\sigma^2}{(p-1)r} \begin{pmatrix} 1 & 2\rho & 3\rho^2 & \cdots & p_2\rho^{p_2-1} \\ 2\rho & 4\rho^2 & 6\rho^3 & \cdots & 2p_2\rho^{p_2} \\ 3\rho^2 & 6\rho^3 & 9\rho^4 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_2\rho^{p_2-1} & \cdots & \cdots & \cdots & p_2^2\rho^{2(p_2-1)} \end{pmatrix}.
\end{aligned}
$$

Then, the criterion is easily obtained as

$$
\begin{aligned}
(3.6)\qquad \operatorname{tr} Q_{p,s} = \frac{\sigma^2}{r(p-1)(1-\rho^2)}&[rp_1(p-1)\{p_2(1-\rho^2)-\rho^2+\rho^{2(p_2+1)}\} \\
&-p\{1+\rho^2-\rho^{2(p_2+1)}-\rho^{2p_2}(p_2(1-\rho^2)+1)^2\}] > 0.
\end{aligned}
$$

Furthermore, it is conjectured that $Q_{p,s} > 0$. However, its proof has not been established for general values of p and p_1.

In general, it is known that if the structure of Σ is known to be serial covariance structure or uniform covariance structure, then it is better to use it from the start. However, our results (3.4) and (3.6) show how much gain is obtained by using uniform covariance structure and serial covariance structure, respectively. Furthermore, we note that these simple covariance structures have some merit in computing MLE's and handling extended prediction of $\boldsymbol{y}_f$ (see, e.g., Lee (1988)).

4. Examples

We apply the results of Section 3 to two data sets. One is the dental measurements data on 11 girls and 16 boys, at ages 8, 10, 12 and 14 years. Each measurement is the distance, measured in *mm*, from the center of the pituitary to the pteryomaxillary fissure. The other is the ramus height data, measured in *mm*, on 20 boys at ages 8, $8\frac{1}{2}$, 9 and $9\frac{1}{2}$ years. These data sets are discussed by Rao (1987), Lee (1988) and many other authors.

Similar to Lee (1988), we consider the mean squared deviation (MSD) of the last observation of a partially observed vector; that is, $p_1 = 3$ and $p_2 = 1$ for conditional prediction. Our MSD is defined as the square root of the estimated

Table 1. The MLE's in dental and ramus data.

		Serial structure				Uniform structure			
		$p = 4$		$p = 3$		$p = 4$		$p = 3$	
		σ^2	ρ	σ^2	ρ	σ^2	ρ	σ^2	ρ
Dental measurement	1	4.8907	0.6071	5.1148	0.5706	4.9052	0.6178	4.9872	0.5881
data	2	4.8067	0.7682	4.8416	0.7315	4.6798	0.7017	4.6161	0.7143
	3G	4.6591	0.8957	4.4492	0.8663	4.4704	0.8680	4.1602	0.8530
	4B	5.1724	0.4429	5.6262	0.4105	5.2041	0.4701	5.5557	0.4517
	5B	4.9329	0.6804	5.1380	0.6450	4.8334	0.5887	4.9504	0.6288
Ramus height data	6	6.5354	0.9526	6.2846	0.9447	6.2855	0.9447	6.2375	0.9199

Note: 1 is the MLE's, with girls and boys having identical covariance structure: 2 is the same MLE's, with individual 20 excluded. 3G is the MLE's of girls. 4B is the MLE's of boys, and 5B is the MLE's of boys with individual 20 excluded.

Table 2. The MSD's: Lee's values and our values.

		Conditional prediction				Extended prediction			
		Serial structure		Uniform structure		Serial structure		Uniform structure	
		Lee's value	Our value	Lee's value	Our value	Lee's value	Our value	Lee's value	Our value
Dental measurement	MSD1	2.3643	1.8004	2.2353	1.5851	—	1.9042	—	1.6557
data	MSD2	1.3542	1.4340	2.1955	1.3765	—	1.5342	—	1.3388
	MSD3	1.9990	1.7310	2.1788	1.5709	—	1.8385	—	1.6415
	MSD4	1.2704	1.4188	2.1483	1.3700	1.8961	1.5160	2.3108	1.3347

Note: MSD1 is the MSD, with girls and boys having identical covariance structure: MSD2 is the same MSD, with individual 20 excluded. MSD3 is the MSD, with girls and boys having different covariance structures: MSD4 is the same MSD, with individual 20 excluded.

Table 3. The MSD's: Lee's values and our values.

		Conditional prediction				Extended prediction			
		Serial structure		Uniform structure		Serial structure		Uniform structure	
		Lee's value	Our value	Lee's value	Our value	Lee's value	Our value	Lee's value	Our value
Ramus height data	MSD6	0.5178	0.7833	1.4690	0.9535	0.5180	0.8284	1.3398	0.8259

MSE based on (3.3) or (3.5). In the case of the dental measurement, the design matrix B is

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -3 & -1 & 1 & 3 \end{pmatrix}, \tag{4.1}$$

and the design matrix A is a vector $\mathbf{1}_{27}$ if girls and boys are assumed to be one group. On the other hand, if both girls and boys are assumed to be two groups with common covariance matrix, the design matrix A is a 27×2 matrix composed of 11 $(1,0)$ rows, followed by 16 $(0,1)$ rows. Moreover, if girls and boys are assumed to be two groups with different covariance matrices, the design matrix A is a vector $\mathbf{1}_{11}$ for girls and a vector $\mathbf{1}_{16}$ for boys. For the ramus height data, the design matrix B is the same as (4.1) and the design matrix A is a vector $\mathbf{1}_{20}$.

For extended prediction, we consider the MSD of the last observation; that is, $p = 3$ and $m = 1$. Then the design matrix B for both data sets is the first three columns of B, as given in (4.1).

In the following, we deal with serial covariance structure and uniform covariance structure. The MLE's which we used are presented in Table 1. Our MSD's are computed up to order n^{-1}. To compare the numerical results of Lee (1988) with our results, we need to state our results for dental data. As these values are considered up to order n^{-1} and contain the term $K(\Sigma)$, they are influenced by the group to which the observation belongs. If girls and boys are assumed to be two groups with different covariance matrices, our results are defined as the weighted mean values of a girl's MSD and boy's one. Our MSD's are summarized in Tables 2 and 3 with the numerical results of Lee (1988). When $p = 4$, under the cases 1, 2, 3G, 4B, 5B and 6 in Table 1, respectively, the values of the likelihood ratio (LR) statistic for testing serial covariance structure are 19.6, 17.8, 7.7, 11.2, 12.7 and 9.5. Recently, Lee (1991) has also computed the LR ratio statistics on cases 3G and 6. The same values of the LR statistic for testing uniform covariance structure are 8.5, 19.5, 6.7, 6.0, 14.6 and 39.6. The upper 5% point of a chi-squared distribution with 8 degrees of freedom is 15.5. From these tables, it seems that the empirical MSD's (Lee's results) are more optimistic than our results for the serial covariance structure. On the other hand, it seems that our results are more sensitive than the empirical MSD's when the uniform covariance structure is accepted.

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