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# SUFFICIENCY AND FUZZINESS IN RANDOM EXPERIMENTS*

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**Abstract.** In previous papers, the consequences of the "presence of fuzziness" in the experimental information on which statistical inferences are based were discussed. Thus, the intuitive assertion ⟨⟨fuzziness entails a loss of information⟩⟩ was formalized, by comparing the information in the "exact case" with that in the "fuzzy case". This comparison was carried out through different criteria to compare experiments (in particular, that based on the "pattern" one, Blackwell's sufficiency criterion). Our purpose now is slightly different, in the sense that we try to compare two "fuzzy cases". More precisely, the question we are interested in is the following: how will different "degrees of fuzziness" in the experimental information affect the sufficiency? In this paper, a study of this question is carried out by constructing an alternative criterion (equivalent to sufficiency under comparability conditions), but whose interpretation is more intuitive in the fuzzy case. The study is first developed for Bernoulli experiments, and the coherence with the axiomatic requirements for measures of fuzziness is also analyzed in such a situation. Then it is generalized to other random experiments and a simple example is examined.

*Key words and phrases*: Blackwell's sufficiency, fuzziness, fuzzy information, random experiment, probability of a fuzzy event.

## 1. Preliminary concepts

The essential element in statistical problems is the *random experiment*, that is a process by which an observation is made, resulting in an outcome that cannot be previously predicted. In addition, it is often assumed that the experiment can be repeated under more or less identical conditions and there is statistical regularity. In such a situation, the components of a model for a random experiment are: (i) the

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identification of all experimental outcomes; (ii) the identification of all observable events; (iii) the assignment of probabilities to these events.

According to the ability to observe the experimental outcomes, the traditional approach often admits that the observer is able to perceive the outcome after each experimental performance with exactness, and the observable events are statements regarding the experimental outcome, so that after the experiment has been conducted one can answer YES or NOT to the occurrence of each of those statements. The model associated with this "traditional experiment" is then given by a probability space $(X, \beta_X, P_\theta)$, $\theta \in \Theta$, where $X$ is the sample space (or set of all possible exact outcomes), $\beta_X$ is the $\sigma$-field of all events of interest (so that, each observable event may be mathematically identified with a measurable subset of the sample space $X$), and $\theta$ is the state or parameter value governing the experimental distribution $P_\theta$. Furthermore, it is usually supposed that $X$ is a set of real numbers and $\beta_X$ is the smallest Borel $\sigma$-field on $X$ (in other words, the elementary observable events are all the singletons of exact outcomes).

### 1.1 *Comparing traditional experiments through sufficiency*

Given two experiments, $\boldsymbol{E} = (X, \beta_X, P_\theta)$, $\theta \in \Theta$, $\boldsymbol{F} = (Y, \beta_Y, Q_\theta)$, $\theta \in \Theta$, whose distributions are governed by the same state of nature or parameter value $\theta$, the idea of comparing such experiments was introduced by Bohnenblust, Shapley and Sherman, in a private communication whose basic results are collected by Blackwell (1951), and developed into a theory by Blackwell (1951, 1953). Many preference relations to compare experiments have been then examined and connected with the previous ones (see, for instance, papers referenced in Lehmann (1988)).

Among all the relations, the one based on Blackwell's sufficiency has become the "pattern criterion". Thus, it has a desirable intuitive meaning and it has been considered plausible, in all the studies concerning the topic, that any other preference relation has to agree with Blackwell's one (when applicable). Consequently, to ensure the suitability of new comparison criteria, it is very usual to analyze the implications of the sufficiency in terms of them.

Blackwell's (1951, 1953) method for comparing experiments states that

DEFINITION 1.1.1. The experiment $\boldsymbol{E}$ is *sufficient for* the experiment $\boldsymbol{F}$ if there exists a nonnegative function $h$ on $X \times Y$, so that the density function associated with $Q_\theta$ with respect to a $\sigma$-finite measure $\nu$ on $\beta_X \times \beta_Y$ is given by

$$g_\theta(y) = \int_X h(x,y) f_\theta(x) d\nu(x), \qquad \text{for all} \quad \theta \in \Theta, \quad y \in Y$$

where

$$\int_Y h(x,y) d\nu(y) = 1, \qquad \text{for all} \quad x \in X$$

and $h$ is integrable with respect to $x$ ($f_\theta(x)$ being the density function associated with $P_\theta$ with respect to the $\sigma$-finite measure $\nu$).

This preference relation has a very intuitive interpretation. Thus, since the function $h$ (called stochastic transformation) does not depend on $\theta$, the above sufficiency condition indicates that an outcome from $\boldsymbol{F}$ could be generated from an observation on $\boldsymbol{E}$ and an auxiliary randomization according to $h$ (in other words, to observe $\boldsymbol{F}$ does not add any probabilistic information about $\theta$ to what is contained in $\boldsymbol{E}$).

### 1.2 *Modeling random experiments involving fuzziness*

In previous papers (Gil (1987, 1988*a*, 1988*b*), Gil *et al.* (1988)), we have analyzed an approach in which fuzziness is incorporated to the random experiment. The necessity for incorporating fuzziness to random experiments usually derives in practice from one of the two following sources of imprecision:

(i) either the lack of precision in the observer report of experimental data does not allow us to answer YES or NOT to the occurrence of events assimilable with measurable subsets of the sample space (and, consequently, we have to introduce a new type of events for which we can decide if each of them is true of false),

(ii) or the events themselves, so that after the exact observation of the experimental outcome is known, the observer cannot answer YES or NOT to their occurrence, but rather he can specify the degree with which each of them is true (or false).

In both situations, to describe events associated with the random experiment, we can often use fuzzy subsets of the original sample space.

A model for random experiments involving fuzziness starts with the mathematical identification of the available information or event (Okuda *et al.* (1978), Tanaka *et al.* (1979), Zadeh (1978)). Let $\boldsymbol{E} = (X, \beta_X, P_\theta)$, $\theta \in \Theta$.

DEFINITION 1.2.1. A fuzzy event $\tilde{e}$ on $X$, characterized by a Borel-measurable membership function $\mu_{\tilde{e}}$ from $X$ to $[0,1]$, where $\mu_{\tilde{e}}(x)$ represents the "degree of compatibility" of $x$ with $\tilde{e}$ (or degree to which $\tilde{e}$ is satisfied when $x$ is the outcome in the performance of $\boldsymbol{E}$), is called *fuzzy information associated with the experiment* $\boldsymbol{E}$.

Another relevant element to model the new situation is the assignment of "probabilities" to the observable (fuzzy) events. Zadeh (1968) suggested to quantify the "induced probability" of a fuzzy event as follows:

DEFINITION 1.2.2. The *probability of* $\tilde{e}$ *induced by* $P_\theta$ is given by

$$\mathcal{P}_\theta(\tilde{e}) = \int_X \mu_{\tilde{e}}(x) f_\theta(x) d\nu(x).$$

According to Zadeh (1978), the value $\mathcal{P}_\theta(\tilde{e})$ could be interpreted as the "degree of consistency" of the probability distribution $P_\theta$ with the possibility distribution (Zadeh (1978)) associated with the membership function $\mu_{\tilde{e}}$.

*Remark.* The use of the preceding definition could be justified because of the two following reasons: (i) it is the most immediate extension from the non-fuzzy case, in which we replace the indicator function of a measurable exact observation or event by the membership function of a fuzzy observation or event; (ii) Zadeh's definition is coherent with Le Cam's (1964, 1986) definition of the "probability" of bounded numerical functions in a *single stage experiment*. (Thus, Le Cam extended the structure of a probability space to a weaker structure in which the class of indicator functions of the classical events associated with the experiment is enlarged to the class of bounded numerical functions from the space $X$. Whenever this last class contains the membership function of a given fuzzy event, the extension suggested by Le Cam would coincide with the one in Zadeh's probabilistic definition.)

In previous papers (Gil (1987, 1988*a*)), we discussed the consequences of the presence of fuzziness in the experimental information, by formalizing the idea that the "exact case" (in which fuzziness is completely absent) is "more informative" than the "fuzzy case". This formalization was carried out through different criteria to compare experiments, such as the one based on sufficiency (Blackwell (1951, 1953)) and, consequently, those based on Shannon's information measure (Lindley (1956)), expected value of sample information (Raiffa and Schlaifer (1961)), Fisher's amount of information (Stone (1961)), and others and represented an extension of studies from Ferentinos and Papaioannou (1979), Kale (1964) and Kullback (1968) for grouped data.

The aim of this paper is to develop a similar study by comparing through sufficiency two situations, associated with the same population, in both of which different "degrees of fuzziness" can be present. Broadly speaking, we want to formalize the idea that the "sharper" an observation the "more informative" (according to sufficiency) it is. The comparison through sufficiency in the fuzzy case will not be as immediate to interpret as in the non-fuzzy case (since the framework of the first one not only contains probabilities, but membership degrees). Nevertheless, we are next going to establish an equivalent criterion much easier to interpret when fuzziness is involved in random experiments. Such an equivalence allows us to connect sufficiency and fuzziness (as intended, for instance, by Klir and Folger (1988)), under some conditions.

## 2. Equivalent comparison of experiments involving fuzziness

To connect fuzziness in the experimental observations or events with the probabilisitic notion of sufficiency, it should be first noted that the degree of fuzziness of a fuzzy set is usually expressed (Klir and Folger (1988)), in the most natural way, in terms of the lack of distinction between the set and its complement, since the less a set differs from its complement, the fuzzier it is. (Although the definition of the complement of a fuzzy set is not unique, we herein will employ that most commonly used, the fuzzy set $\tilde{e}^c$ described by the membership function $\mu_{\tilde{e}^c}(x) = 1 - \mu_{\tilde{e}}(x)$, for all $x \in X$.) Then, the degree of fuzziness of the fuzzy information $\tilde{e}$ could be interpreted as the lack of distinction in the fuzzy 2-partition (Bezdek (1987)), $\chi = \{\tilde{e}, \tilde{e}^c\}$.

On the other hand, on the basis of this fuzzy 2-partition and Zadeh's probabilitistic definition, (Definition 1.2.2) it is possible to induce a new probability space, $\mathcal{E} = (\chi, \mathcal{F}(\chi), \mathcal{P}_\theta)$, $\theta \in \Theta$ (where $\mathcal{F}(\chi)$ = parts of $\chi$), which may be regarded as a "probability space induced by $\chi$".

Let $\boldsymbol{E} = (X, \beta_X, P_\theta)$, $\theta \in \Theta$, be a random experiment and let $\tilde{e}$ and $\tilde{e}'$ denote two fuzzy observations associated with $\boldsymbol{E}$. Let $\mathcal{E} = (\chi, \mathcal{F}(\chi), \mathcal{P}_\theta)$, $\mathcal{E}' = (\chi, \mathcal{F}(\chi'), \mathcal{P}_\theta)$, $\theta \in \Theta$, where $\chi = \{\tilde{e}, \tilde{e}^c\}$, $\chi' = \{\tilde{e}', \tilde{e}'^c\}$. Then, the notion of sufficiency may be immediately applied as follows:

DEFINITION 2.1. We will say that $\mathcal{E}$ is *sufficient for* $\mathcal{E}'$ if there exists a non-negative function $h$ on $\chi \times \chi'$ such that

$$\mathcal{P}_\theta(\tilde{e}') = h(\tilde{e}, \tilde{e}')\mathcal{P}_\theta(\tilde{e}) + h(\tilde{e}^c, \tilde{e}')\mathcal{P}_\theta(\tilde{e}^c),$$
$$\mathcal{P}_\theta(\tilde{e}'^c) = h(\tilde{e}, \tilde{e}'^c)\mathcal{P}_\theta(\tilde{e}) + h(\tilde{e}^c, \tilde{e}'^c)\mathcal{P}_\theta(\tilde{e}^c)$$

where $h(\tilde{e}, \tilde{e}') + h(\tilde{e}, \tilde{e}'^c) = 1$, $h(\tilde{e}^c, \tilde{e}') + h(\tilde{e}^c, \tilde{e}'^c) = 1$.

Obviously, the conditions concerning $\mathcal{P}_\theta(\tilde{e}')$ and $\mathcal{P}_\theta(\tilde{e}'^c)$ are equivalent (so, we could remove one of them in the above definition).

As we have previously commented, the interpretation of this definition is not intuitive.

On the other hand, the question of how to measure the fuzziness of a particular fuzzy observation (or, in general, of a fuzzy subset) has been exhaustively studied in the literature of Fuzzy Sets Theory (see, for instance, Klir and Folger (1988)). Formally,

DEFINITION 2.2. A *measure of fuzziness* is a real function $f$ defined on $\tilde{\mathcal{F}}(X)$ (set of all fuzzy subsets of $X$) satisfying the following requirements:

*Axiom* 1. $f(\tilde{e}) = 0$ if and only if $\tilde{e}$ is a crisp set.

*Axiom* 2. If $\tilde{e}, \tilde{e}' \in \tilde{\mathcal{F}}(X)$ and $\tilde{e}$ is "sharper" than $\tilde{e}'$, then $f(\tilde{e}) \leq f(\tilde{e}')$.

*Axiom* 3. $f(\tilde{e})$ assumes the maximum value if and only if $\tilde{e}$ is "maximally fuzzy".

The notions "sharper" and "maximally fuzzy" above are usually interpreted as follows:

(1) $\tilde{e}$ is intended as "sharper" than $\tilde{e}'$ if $\mu_{\tilde{e}}(x) \leq \mu_{\tilde{e}'}(x)$ for $\mu_{\tilde{e}'}(x) \leq 1/2$, and $\mu_{\tilde{e}}(x) \geq \mu_{\tilde{e}'}(x)$ for $\mu_{\tilde{e}'}(x) \geq 1/2$, for all $x \in X$.

(2) $\tilde{e}$ is intended as "maximally fuzzy" if and only if $\mu_{\tilde{e}}(x) = 1/2$, for all $x \in X$.

As the two fuzzy observations we have just compared in Definition 2.1 are associated with the same experiment (and, consequently, with the same probabilistic information), the comparison *via* sufficiency must be mainly dependent on the membership functions of those observations. We are now going to formalize this assertion. Thus, we first consider the simplest case in which the referential experiment is Bernoulli.

## 2.1 *Sufficiency and fuzziness in Bernoulli experiments*

In particular, when $\boldsymbol{E} = (X, \beta_X, P_\theta)$, $\theta \in \Theta$, is a Bernoulli experiment, it involves only two outcomes (often coded by the real values 0 and 1). The probability measure is then defined by $P_\theta(0) = 1 - \theta$, $P_\theta(1) = \theta$, $\Theta \subset [0, 1]$. Many experiments are of this type: a vaccine is effective or it is not; a patient has a symptom or does not have it; a pathological condition is present or absent.

A fuzzy observation or event $\tilde{e}$ associated with the Bernoulli experiment $\boldsymbol{E}$ may be described by means of a pair $(\mu_0, \mu_1)$, where $\mu_0 = \mu_{\tilde{e}}(0)$ and $\mu_1 = \mu_{\tilde{e}}(1)$. The induced "probability" in this case would be given by $\mathcal{P}_\theta(\tilde{e}) = \mu_0 + \theta(\mu_1 - \mu_0)$. Examples of fuzzy observations or events associated with a Bernoulli experiment are, for instance, the following ones: a given patient sometimes cannot be diagnosed as having a particular malady or not, but having it with a specified degree; some organisms cannot be classified as belonging to a certain species or not, but rather belonging to it with a specified degree.

The following theorem establishes an equivalent criterion to Blackwell's sufficiency, under comparability conditions and for Bernoulli experiments.

THEOREM 2.1. *Let $\boldsymbol{E} = (X, \beta_X, P_\theta)$, $\theta \in \Theta$, be a Bernoulli experiment and let $\tilde{e}$ and $\tilde{e}'$ denote two fuzzy observations associated with $\boldsymbol{E}$. Let $\mathcal{E} = (\chi, \mathcal{F}(\chi), \mathcal{P}_\theta)$, $\mathcal{E}' = (\chi', \mathcal{F}(\chi'), \mathcal{P}_\theta)$, $\theta \in \Theta$, where $\chi = \{\tilde{e}, \tilde{e}^c\}$, $\chi' = \{\tilde{e}', \tilde{e}'^c\}$. Then, if $\mathcal{E}$ and $\mathcal{E}'$ are comparable, we have*

*$\mathcal{E}$ is sufficient for $\mathcal{E}'$ if and only if*

$$|\mu_1' - \mu_0'| \leq |\mu_1 - \mu_0| \tag{2.1}$$

*(where $\mu_0 = \mu_{\tilde{e}}(0)$, $\mu_1 = \mu_{\tilde{e}}(1)$ and $\mu_0' = \mu_{\tilde{e}'}(0)$ and $\mu_1' = \mu_{\tilde{e}'}(1)$).*

The interpretation of this alternative comparison is obvious, since it means that $\mu_{\tilde{e}}$ "discriminates" more between 0 and 1 than $\mu_{\tilde{e}'}$. We are now going to connect it with the measurement of fuzziness.

To examine the relationships between sufficiency and fuzziness (in the sense of Definition 2.2) we have previously to assume some particular constraints. Although theoretically we have $0 \leq \mu_0 \leq 1$, $0 \leq \mu_1 \leq 1$, we can often constraint our study to cases in which $0 \leq \mu_0 \leq 1/2$ and $1/2 \leq \mu_1 \leq 1$ (or $1/2 \leq \mu_0 \leq 1$ and $0 \leq \mu_1 \leq 1/2$). Thus, in the observation from a Bernoulli trial, one could: (a) obtain quite fuzzy information so that the outcomes 0 and 1 are equally compatible with the information (that could be often represented by $\mu_0 = \mu_1 = 1/2$); (b) obtain fuzzy information so that 0 is less compatible with the information than 1 (that could be often represented by $0 \leq \mu_0 < 1/2$ and $1/2 < \mu_1 \leq 1$); (c) obtain fuzzy information so that 0 is more compatible with the information than 1 (that could be often represented by $1/2 < \mu_0 \leq 1$ and $0 \leq \mu_1 < 1/2$).

On the basis of Theorem 2.1, and under the preceding assumptions, the following results state that the comparison of fuzzy observations by means of the notion of sufficiency is coherent with the axiomatic requirements that every measure of fuzziness must satisfy.

THEOREM 2.2. *Let $\boldsymbol{E} = (X, \beta_X, P_\theta)$, $\theta \in \Theta$, be a random experiment and let $\tilde{e}$ and $\tilde{e}'$ denote two fuzzy observations associated with $\boldsymbol{E}$. Let $\mathcal{E} = (\chi, \mathcal{F}(\chi), \mathcal{P}_\theta)$, $\mathcal{E}' = (\chi', \mathcal{F}(\chi'), \mathcal{P}_\theta)$, $\theta \in \Theta$, where $\chi = \{\tilde{e}, \tilde{e}^c\}$, $\chi' = \{\tilde{e}', \tilde{e}'^c\}$. Then, if* $\operatorname{sgn}\{\mu_{\tilde{e}}(0) - 1/2\} \neq \operatorname{sgn}\{\mu_{\tilde{e}}(1) - 1/2\}$, *and* $\operatorname{sgn}\{\mu_{\tilde{e}'}(0) - 1/2\} \neq \operatorname{sgn}\{\mu_{\tilde{e}'}(1) - 1/2\}$, *we have*

(i) *if $\tilde{e}$ is a crisp set, then $\mathcal{E}$ is sufficient for $\mathcal{E}'$, whatever the fuzzy set $\tilde{e}'$ may be (that is, exact experimental information is always sufficient for fuzzy experimental information);*

(ii) *if $\tilde{e}$ is sharper than $\tilde{e}'$, then $\mathcal{E}$ is sufficient for $\mathcal{E}'$;*

(iii) *if $\tilde{e}'$ is maximally fuzzy, then $\mathcal{E}$ is sufficient for $\mathcal{E}'$ (that is, fuzzy experimental information is always sufficient for uniformly fuzzy information).*

The result (ii) in Theorem 2.2 is now illustrated by means of an example:

*Example.* Consider a population of mice, a fraction $\theta$ of which has a character $C$.

Assume that the character $C$ may be recognized through two different symptoms $A$ and $B$, each one of which determines the presence of character $C$.

However, suppose that after examining each mouse for presence of $C$, the accessible mechanisms of detection of $A$ and $B$ do not allow us to state them exactly, but it is only possible to conclude $\tilde{a} =$ "the mouse has $A$ quite sharply" or $\tilde{b} =$ "the mouse seems more or less to have $B$". If these imprecise propositions are assimilated with the fuzzy events characterized by the membership functions $\mu_{\tilde{a}}(1) = 0.9$, $\mu_{\tilde{a}}(0) = 0.2$, $\mu_{\tilde{b}}(1) = 0.6$, $\mu_{\tilde{b}}(0) = 0.3$ (quantifying the degree to which the available propositions agree with having or not each symptom, where $0 = C$ is absent, $1 = C$ is present), and we are interested in drawing conclusions about $\theta$, it is preferred to try to detect $A$ than $B$. Thus, if we define $h(\tilde{a}, \tilde{b}) = 45/7$, $h(\tilde{a}^c, \tilde{b}) = 15/7$, then $\mathcal{P}_\theta(\tilde{b}) = h(\tilde{a}, \tilde{b})\mathcal{P}_\theta(\tilde{a}) + h(\tilde{a}^c, \tilde{b})\mathcal{P}_\theta(\tilde{a}^c)$, whence $\mathcal{E} = (\mathcal{A}, \mathcal{F}(\mathcal{A}), \mathcal{P}_\theta)$, $\theta \in \Theta = [0,1]$ $(\mathcal{A} = \{\tilde{a}, \tilde{a}^c\})$, is sufficient for $\mathcal{E}' = (\beta, \mathcal{F}(\beta), \mathcal{P}_\theta)$, $\theta \in \Theta = [0,1]$ $(\beta = \{\tilde{b}, \tilde{b}^c\})$.

### 2.2 *Sufficiency and fuzziness in other experiments*

We are now going to extend previous results to the case of other random experiments. Difficulties in the extension of this study for other experiments arise because of the non-comparability of the fuzzy data, unless some restricted conditions are satisfied. Thus, the generalization of the present study to experiments like binomial, Pascal, Poisson, exponential or normal ones indicates that two fuzzy data associated with the same experiment become comparable only in a special situation (according to which the membership function has to be uniquely determined up to a particular linear transformation). We now illustrate such an assertion by examining the case of binomial experiments.

Let $\boldsymbol{E} = (X, \beta_X, P_\theta)$, $\theta \in \Theta$ be a binomial experiment, involving $n+1$ outcomes (assimilated with the real values $0, 1, 2, \ldots, n$). The probability measure is then defined by $P_\theta(k) = \binom{n}{k} \theta^k (1-\theta)^{n-k}$, $k = 0, 1, 2, \ldots, n$ and $\Theta \subset [0,1]$. Many experiments are of this type: to observe number of defective pieces in a sample of eighteen; to count the number of people having a certain contagious disease in a sample of twenty five people exposed to that disease, etc.

A fuzzy observation or event $\tilde{e}$ associated with the binomial experiment $\boldsymbol{E}$ may be described by means of an $(n+1)$-tuple $(\mu_0, \mu_1, \mu_2, \ldots, \mu_n)$, where $\mu_k = \mu_{\tilde{e}}(k)$. The induced "probability" in this case would be given by $\mathcal{P}_\theta(\tilde{e}) = \sum_k \mu_k P_\theta(k)$.

Examples of fuzzy observations or events associated with a binomial experiment are, for instance, the following ones: the number of defective pieces in the sample is *moderate*; there are *many* people in the sample having the contagious disease, etc.

By using the inversion formula theorem of the Fourier integral (see Papoulis (1962)), we can obtain the following result, establishing conditions of applicability for the sufficiency criterion (comparability conditions) and extending the equivalent criterion in Theorem 2.1.

THEOREM 2.3. *Let $\boldsymbol{E} = (X, \beta_X, P_\theta)$, $\theta \in \Theta$, be a binomial experiment and let $\tilde{e}$ and $\tilde{e}'$ denote two fuzzy observations associated with $\boldsymbol{E}$. Let $\mathcal{E} = (\chi, \mathcal{F}(\chi), \mathcal{P}_\theta)$, $\mathcal{E}' = (\chi', \mathcal{F}(\chi'), \mathcal{P}_\theta)$, $\theta \in \Theta$, where $\chi = \{\tilde{e}, \tilde{e}^c\}$, $\chi' = \{\tilde{e}', \tilde{e}'^c\}$. Then,*

(i) *$\mathcal{E}$ and $\mathcal{E}'$ are comparable if and only if there exist $\alpha, \beta \in [0,1]$ (independent of $\theta$) such that*

$$\mu_{\tilde{e}'}(x) = \beta + (\alpha - \beta)\mu_{\tilde{e}}(x), \qquad \text{for all} \qquad x \in X \text{ and,}$$

(ii) *if $\mathcal{E}$ and $\mathcal{E}'$ are comparable, then we have that $\mathcal{E}$ is sufficient for $\mathcal{E}'$ if and only if*

$$|\mu'_r - \mu'_s| \le |\mu_r - \mu_s|, \qquad \text{for all} \quad r, s \in \{0, 1, 2, \ldots, n\} \tag{2.2}$$

*(where $\mu_k = \mu_{\tilde{e}}(k)$ and $\mu'_k = \mu_{\tilde{e}'}(k)$).*

The preceding result could analogously be established for other random experiments. The connections with fuzziness stated in Theorem 2.2 are not so immediate to extend because of the difficulties in generalizing the conditions assumed before it. Nevertheless, due to the shape of membership functions that are employed in practice, when both, sufficiency and fuzziness comparisons are applicable, they lead usually to the same preference relation.

## 3. Illustrative example

To illustrate the results in Section 2, we are now going to consider the following situation:

*Example.* Suppose that the time of attention (in minutes) to a concrete game in a population of ten-year-old children has an exponential distribution with unknown parameter $\theta$ ($\theta$ = inverse of the population mean time). A psychologist wants to draw conclusions about $\theta$, but as the loss of interest in a game does not usually happen in an instantaneous way, he cannot measure the time of attention exactly. Assume that he expresses the perception of the outcome after a measurement by means of propositions such as "too much time", or "around 20 minutes", or "a moderate time".

The imprecision associated with these propositions is non-probabilistic in nature (since it means uncertainty regarding concepts or definition of events, not regarding occurrence of exact events), but it could be easily characterized by means

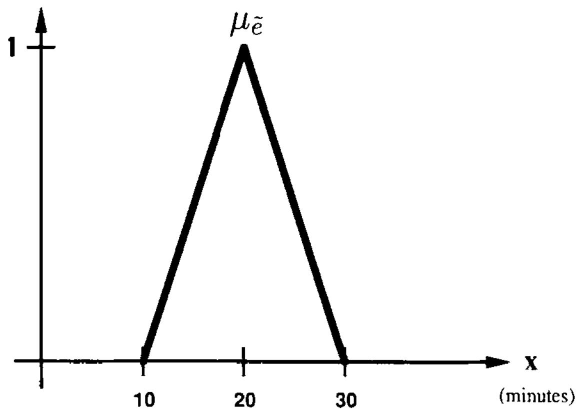


Fig. 1. Membership function of the fuzzy information $\tilde{e}$ = "around 20 minutes".

of fuzzy events. So, the proposition $\tilde{e}$ = "around 20 minutes" could be assimilated with the fuzzy information $\tilde{e}$ characterized, for instance, by the membership function $\mu_{\tilde{e}}(x) = (x - 10)/10$ if $x \in (10, 20]$, $= (30 - x)/10$ if $x \in (20, 30)$, $= 0$ otherwise (see Fig. 1).

Figure 2 shows some of the fuzzy observations or events that can be compared (and consequently, whose complements can be also compared) with the fuzzy information $\tilde{e}$ = "around 20 minutes" through Blackwell's sufficiency. For all of them $\tilde{e}$ is sufficient.

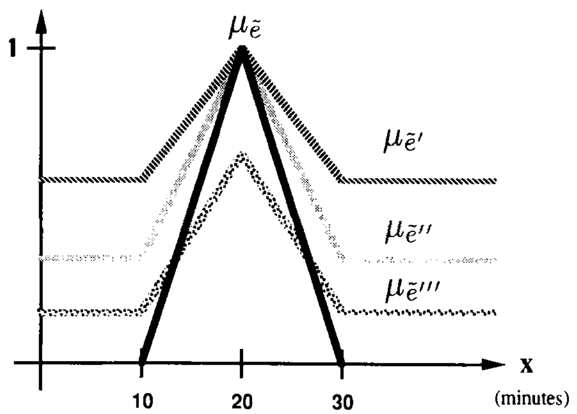


Fig. 2. Some fuzzy observations ($\tilde{e}'$, $\tilde{e}''$ and $\tilde{e}'''$) for which the fuzzy information $\tilde{e}$ = "around 20 minutes" is sufficient.

Figure 3 shows some of the fuzzy observations or events that can be compared with the fuzzy information $\tilde{e}$ = "around 20 minutes" through fuzziness (in the sense of Definition 2.2). $\tilde{e}$ is sharper than each of them.

Finally, Fig. 4 shows a situation in which both, sufficiency and fuzziness, are

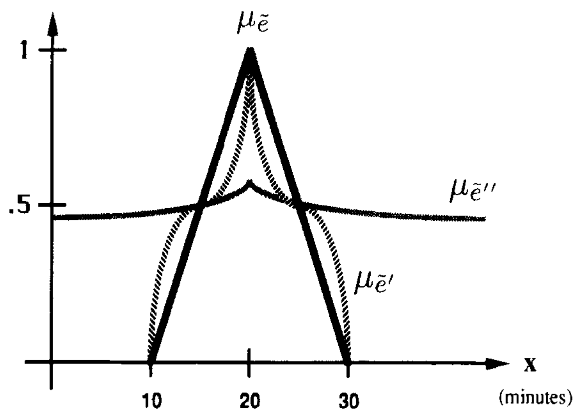


Fig. 3. Some fuzzy observations ($\tilde{e}'$, and $\tilde{e}''$) for which the fuzzy information $\tilde{e}$ = "around 20 minutes" is sharper than each of them.

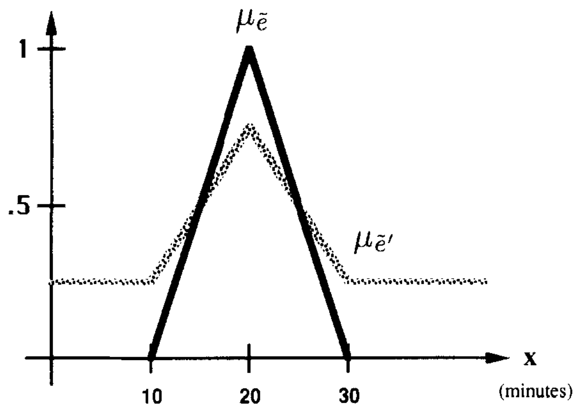


Fig. 4. A fuzzy observation ($\tilde{e}'$) for which the fuzzy information $\tilde{e}$ = "around 20 minutes" is sufficient and sharper than it.

applicable and lead to coherent conclusions ($\tilde{e}$ sufficient for $\tilde{e}'$, and $\tilde{e}$ sharper than $\tilde{e}'$).

## 4. Concluding remarks

As in the non-fuzzy case, the inconveniences in connecting directly sufficiency and fuzziness (following the ideas in Theorem 2.2) for general experiments are due, in part, because of non-comparability problems.

It should be hence interesting to analyze in the near future questions similar to those discussed in this paper, but based on comparisons avoiding non-comparability inconveniences (that is, establishing complete preorderings) such as those based on probabilistic information measures (for instance, the expected Shannon's amount of information, for a particular prior distribution) instead of sufficiency, and the ones based on measures of fuzziness (such as the De Luca and Termini non-probabilistic entropy (1972)).

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