Ann. Inst. Statist. Math.
Vol. 44, No. 2, 363–378 (1992)

WAITING TIME PROBLEMS FOR A SEQUENCE OF DISCRETE RANDOM VARIABLES*

SIGEO AKI

Department of Mathematical Science, Faculty of Engineering Science, Osaka University, Machikaneyama-cho, Toyonaka, Osaka 560, Japan

(Received September 20, 1990; revised June 13, 1991)

Abstract. Let $X_1, X_2, \dots$ be a sequence of nonnegative integer valued random variables. For each nonnegative integer i, we are given a positive integer k_i. For every $i = 0, 1, 2, \dots$, E_i denotes the event that a run of i of length k_i occurs in the sequence $X_1, X_2, \dots$. For the sequence $X_1, X_2, \dots$, the generalized pgf's of the distributions of the waiting times until the r-th occurrence among the events $\{E_i\}_{i=0}^{\infty}$ are obtained. Though our situations are general, the results are very simple. For the special cases that X's are i.i.d. and $\{0,1\}$-valued, the corresponding results are consistent with previously published results.

Key words and phrases: Sooner and later problems, generalized probability generating function, discrete distributions, binary sequence of order k.

1. Introduction

Let $X_1, X_2, \dots$ be a sequence of nonnegative integer valued random variables. For the moment, we also assume that the random variables are independent and identically distributed. Let $p_i = P(X_1 = i)$, $i = 0, 1, 2, \dots$. Suppose that a sequence of positive integers $\{k_i\}_{i=0}^{\infty}$ is given. For every $i = 0, 1, 2, \dots$, we denote by E_i the event that a run of i of length k_i occurs in the sequence $X_1, X_2, \dots$. First, we study the waiting time until at least one of the events $\{E_i\}_{i=0}^{\infty}$ occurs. Next, we investigate the waiting time for the second occurrence among $\{E_i\}_{i=0}^{\infty}$. Here, "the second occurrence" means the occurrence of another event excepting the first event among the events $\{E_i\}_{i=0}^{\infty}$.

In general, the distribution of the waiting time for the r-th occurrence of the event among $\{E_i\}_{i=0}^{\infty}$ is considered. Further, we treat the corresponding problems when the condition that the random variables are independent and identically distributed is widely relaxed. The general formulas for the generalized probability generating functions of the distributions of the waiting times are obtained. As a

* This research was partially supported by the ISM Cooperative Research Program (90-ISM-CRP-11) of the Institute of Statistical Mathematics.

special case, the problem corresponding to the binary sequence of order k (cf. Aki (1985)) is treated.

Ebneshahrashoob and Sobel (1990) solved this type of problem when the sequence $X_1, X_2, \dots$ is constructed from Bernoulli trials, that is, X's are i.i.d. and $\{0,1\}$-valued random variables. They noted that the limiting distribution becomes the geometric distribution of order k_1 when k_0 tends to infinity in the situation. For the discrete distributions of order k, see, for example, Philippou *et al.* (1983), Aki *et al.* (1984) and Philippou (1986). The discrete distributions of order k, which are related to a succession event such as consecutive k successes in Bernoulli trials, have interesting applications. For example, we can mention applications of the binomial and extended binomial distributions of order k to the reliability of the consecutive-k-out-of-n: F system (cf. Aki (1985) and Aki and Hirano (1989)). Our situations in the paper can treat not only the previous examples but also finite (or infinite) succession events simultaneously. Further, more practical approach can be done, because we do not necessarily require the underlying sequence (of trials) to be independent or identically distributed. Ling (1990) studied the distribution of the waiting time for the first occurrence among E's when X's are i.i.d. and finite valued random variables and all the k's have the same value.

Our results in this paper are not only general and new but also very simple compared with the results in the previously published papers, though our situations are extensively general. For the derivation of the main part of the results, we used the method of generalized pgf (cf. Ebneshahrashoob and Sobel (1990)).

2. The first occurrence problem in the i.i.d. case

Let $X_1, X_2, \dots$ be a sequence of nonnegative integer valued i.i.d. random variables. The sequence $\{k_i\}_{i=0}^{\infty}$ is a given sequence of positive integers. Let p_i and E_i for every $i = 0, 1, 2, \dots$ be the probability and the event, respectively, described in Section 1. In this section, we consider the distribution of the waiting time for the first occurrence of an event among $\{E_i\}_{i=0}^{\infty}$. We derive a generalized probability generating function (gpgf) by adding markers x_i, $i = 0, 1, \dots$. Here, for each i, x_i represents that the first occurring event among $\{E_j\}_{j=0}^{\infty}$ is E_i. Denote by $\phi = \phi(t)$ the gpgf of the distribution of the waiting time for the occurrence of the first event among $\{E_j\}_{j=0}^{\infty}$. We set

$$G_i(t) = \frac{(p_i t)^{k_i}(1 - p_i t)}{1 - (p_i t)^{k_i}}$$

and

$$F_i(t) = \frac{p_i t - (p_i t)^{k_i}}{1 - (p_i t)^{k_i}}, \qquad i = 0, 1, 2, \dots .$$

THEOREM 2.1. *The gpgf $\phi(t)$ is given by*

$$\phi(t) = \frac{\sum_{i=0}^{\infty} G_i(t) x_i}{1 - \sum_{i=0}^{\infty} F_i(t)}.$$

PROOF. Let ϕ_{ij} be the gpgf of the conditional distribution of the waiting time given that we start with a run of i of length j. If we assume that $k_i \geq 2$, $i = 0, 1, 2, \dots$, then ϕ and ϕ_{ij}, $i = 0, 1, 2, \dots$; $j = 1, 2, \dots, k_i - 1$ satisfy the system of equations:

$$\begin{aligned}
\phi &= \sum_{j=0}^{\infty} p_j t \phi_{j1}, \\
\phi_{ij} &= p_i t \phi_{i,j+1} + \sum_{l \neq i} p_l t \phi_{l1}, \qquad i = 0, 1, 2, \dots\,; \ j = 1, 2, \dots, k_i - 2, \\
\phi_{i,k_i-1} &= p_i t x_i + \sum_{l \neq i} p_l t \phi_{l1}, \qquad i = 0, 1, 2, \dots .
\end{aligned}$$

From these equations, we have, for each $i = 0, 1, 2, \dots$,

$$\phi_{i1} = (p_i t)^{k_i - 1} x_i + (\phi - p_i t \phi_{i1}) \frac{1 - (p_i t)^{k_i - 1}}{1 - p_i t}.$$

These equations immediately imply

$$\phi_{i1} = \frac{(1 - p_i t)(p_i t)^{k_i - 1}}{1 - (p_i t)^{k_i}} x_i + \frac{1 - (p_i t)^{k_i - 1}}{1 - (p_i t)^{k_i}} \phi, \qquad i = 0, 1, 2, \dots .$$

By summing both sides of the equations after multiplying $p_i t$, we have

$$\phi = \sum_{i=0}^{\infty} \frac{(p_i t)^{k_i} (1 - p_i t)}{1 - (p_i t)^{k_i}} x_i + \phi \cdot \sum_{i=0}^{\infty} \frac{p_i t - (p_i t)^{k_i}}{1 - (p_i t)^{k_i}}.$$

This completes the proof.

Noting that $t = \sum_{i=0}^{\infty} p_i t$, we have that

$$t - \sum_{i=0}^{\infty} \frac{p_i t - (p_i t)^{k_i}}{1 - (p_i t)^{k_i}} = \sum_{i=0}^{\infty} \frac{(p_i t)^{k_i} (1 - p_i t)}{1 - (p_i t)^{k_i}},$$

and hence we can rewrite $\phi(t)$ as

$$\phi(t) = \frac{\sum_{i=0}^{\infty} G_i(t) x_i}{1 - t + \sum_{i=0}^{\infty} G_i(t)}.$$

By setting $x_i = 1$ for $i = 0, 1, 2, \dots$ in the equation in Theorem 2.1, we have the ordinary pgf of the distribution of the waiting time represented as

$$\psi(t) = \frac{\sum_{i=0}^{\infty} G_i(t)}{1 - \sum_{i=0}^{\infty} F_i(t)}.$$

Therefore, we can rewrite $\psi(t)$ as

$$\psi(t) = \frac{t - \sum_{i=0}^{\infty} F_i(t)}{1 - \sum_{i=0}^{\infty} F_i(t)}$$

or

$$\psi(t) = 1 + \frac{t-1}{1 - \sum_{i=0}^{\infty} F_i(t)}.$$

PROPOSITION 2.1. *The mean and variance of the distribution of the waiting time for the occurrence of the first event among* $\{E_i\}_{i=0}^{\infty}$ *are given, respectively, as*

$$1 \Big/ \left(1 - \sum_{i=0}^{\infty} \frac{p_i - p_i^{k_i}}{1 - p_i^{k_i}}\right)$$

and

$$\sum_{i=0}^{\infty} \frac{-p_i^{2k_i} + (2k_i - 1)p_i^{k_i+1} - (2k_i - 1)p_i^{k_i} + p_i}{(1 - p_i^{k_i})^2} \Big/ \left(1 - \sum_{i=0}^{\infty} \frac{p_i - p_i^{k_i}}{1 - p_i^{k_i}}\right)^2.$$

PROOF. By differentiating $\psi(t)$ w.r.t. t, we have

$$\psi'(t) = \frac{(1 - \sum_{i=0}^{\infty} F_i(t)) + (t-1)\sum_{i=0}^{\infty} F_i'(t)}{(1 - \sum_{i=0}^{\infty} F_i(t))^2}.$$

Letting $t = 1$ in the equation, we have the mean of the distribution given by

$$\frac{1}{1 - \sum_{i=0}^{\infty} F_i(1)}.$$

By differentiating $\psi(t)$ twice w.r.t. t and letting $t = 1$, we obtain

$$\psi''(1) = \frac{2\sum_{i=0}^{\infty} F_i'(1)}{(1 - \sum_{i=0}^{\infty} F_i(1))^2}.$$

If we calculate $\psi''(1) + \psi'(1) - (\psi'(1))^2$, we can easily derive the desired result. This completes the proof.

Remark 1. Suppose that the random variables X's are $\{0,1\}$-valued. Then, by setting $p_0 = q$, $p_1 = p$, $p_i = 0$ for $i = 2, 3, \ldots$, $k_0 = r$ and $k_1 = s$ in the formula for the mean in Proposition 2.1, we see that the corresponding mean is given by

$$\frac{1}{1 - \dfrac{p - p^s}{1 - p^s} - \dfrac{q - q^r}{1 - q^r}} = \frac{(1 - p^s)(1 - q^r)}{pq[1 - (1 - p^{s-1})(1 - q^{r-1})]}.$$

This formula agrees with the formula in p. 303 of Feller (1957).

3. The second occurrence problem in the i.i.d. case

In this section, we consider the gpgf of the distribution of the waiting time for the occurrence of the second event among $\{E_i\}_{i=0}^{\infty}$. For each i and j satisfying $i \neq j$, x_{ij} denotes the marker which means that the first occurring event is E_i and the second occurring event is E_j among $\{E_i\}_{i=0}^{\infty}$. Let $\phi = \phi(t)$ be the gpgf with markers $\{x_{ij}\}$ of the distribution of the waiting time. Let ϕ_{ij} be the gpgf of the conditional distribution of the waiting time given that we start with a run of i of length j. Let $\phi^{(l)}$ be the gpgf of the conditional distribution of the waiting time given that the first occurring event is E_l and E_l has just occurred. Further, let $\phi_{ij}^{(l)}$ be the gpgf of the conditional distribution of the waiting time given that the first occurring event is E_l and E_l has already occurred and we are currently in a run of i of length j.

THEOREM 3.1. *The gpgf ϕ of the distribution of the waiting time for the second occurring event can be written as*

$$\phi(t) = \frac{\sum_{i=0}^{\infty} G_i(t) \dfrac{\sum_{j \neq i} G_j(t) x_{ij}}{1 - t + \sum_{j \neq i} G_j(t)}}{1 - t + \sum_{j=0}^{\infty} G_j(t)}. \tag{3.1}$$

PROOF. From the definitions, ϕ, ϕ_{ij}, $\phi^{(l)}$ and $\phi_{ij}^{(l)}$, $i = 0,1,2,\ldots$; $l = 0,1,2,\ldots$; $j = 1,2,\ldots,k_i - 1$ satisfy the following system of equations;

$$\begin{aligned}
\phi &= \sum_{j=0}^{\infty} p_j t \phi_{j1}, \\
\phi_{ij} &= p_i t \phi_{i,j+1} + \sum_{l \neq i} p_l t \phi_{l1}, \qquad i = 0,1,2,\ldots\,;\ j = 1,2,\ldots,k_i - 2, \\
\phi_{i,k_i-1} &= p_i t \phi^{(i)} + \sum_{l \neq i} p_l t \phi_{l1}, \qquad i = 0,1,2,\ldots,
\end{aligned}$$

and

$$\begin{aligned}
\phi^{(i)} &= \sum_{m=0}^{\infty} p_m t \phi_{m1}^{(i)}, \qquad i = 0,1,2,\ldots, \\
\phi_{jl}^{(i)} &= p_j t \phi_{j,l+1}^{(i)} + \sum_{m \neq j} p_m t \phi_{m1}^{(i)}, \qquad i,j = 0,1,2,\ldots\,;\ l = 1,2,\ldots,k_j - 2, \\
\phi_{j,k_j-1}^{(i)} &= p_j t x_{ij} + \sum_{m \neq j} p_m t \phi_{m1}^{(i)}, \qquad i = 0,1,2,\ldots\,;\ j \neq i, \\
\phi_{i,k_i-1}^{(i)} &= p_i t \phi^{(i)} + \sum_{m \neq i} p_m t \phi_{m1}^{(i)}, \qquad i = 0,1,2,\ldots.
\end{aligned}$$

The former half of the system of equations includes only ϕ, ϕ_{ij} and $\phi^{(i)}$, $i = 0,1,2,\ldots$; $j = 1,2,\ldots,k_i - 1$ and has the same form as that of the previous section if we replace $\phi^{(i)}$ by x_i for each i. Therefore, by using Theorem 2.1, we have

$$\phi = \frac{\sum_{i=0}^{\infty} G_i(t)\phi^{(i)}}{1 - t + \sum_{i=0}^{\infty} G_i(t)}. \tag{3.2}$$

The latter half of the system of equations includes only $\phi^{(i)}$, $\phi_{j,l}^{(i)}$ and x_{ij}, $i = 0, 1, 2, \dots$; $j = 0, 1, 2, \dots$; $l = 1, 2, \dots, k_j - 1$ and has the same form as that in Section 1 if we replace $\phi^{(i)}$ and $\phi_{j,l}^{(i)}$ by ϕ and $\phi_{j,l}$, respectively. Hence, we have, from Theorem 2.1,

$$\phi^{(i)} = \frac{\sum_{j \neq i} G_j(t) x_{ij} + G_i(t)\phi^{(i)}}{1 - t + \sum_{j=0}^{\infty} G_j(t)}.$$

Thus, we obtain, for each $i = 0, 1, 2, \dots$,

$$\phi^{(i)} = \frac{\sum_{j \neq i} G_j(t) x_{ij}}{1 - t + \sum_{j \neq i} G_j(t)}. \tag{3.3}$$

From equations (3.2) and (3.3), it holds that

$$\phi = \frac{\sum_{i=0}^{\infty} G_i(t) \dfrac{\sum_{j \neq i} G_j(t) x_{ij}}{1 - t + \sum_{j \neq i} G_j(t)}}{1 - t + \sum_{j=0}^{\infty} G_j(t)},$$

which completes the proof.

Remark 2. In the equation (3.1), setting $p_0 = q$, $p_1 = p$, $p_m = 0$ for $m = 2, 3, \dots$, $x_{ij} = 1$, $k_0 = r$ and $k_1 = s$, we have the equation (11) of Ebneshahrashoob and Sobel (1990), which is the pgf of the waiting time for the later event in the case of Bernoulli trials.

Since the pgf of the distribution is fortunately very simple, we can derive the mean waiting time for the second occurring event.

PROPOSITION 3.1. *The mean waiting time for the second occurrence is represented as*

$$\left(1 + \sum_{i=0}^{\infty} \frac{1}{\dfrac{1 - p_i^{k_i}}{p_i^{k_i}(1 - p_i)} \left(\sum_{j=0}^{\infty} \dfrac{p_j^{k_j}(1 - p_j)}{1 - p_j^{k_j}} \right) - 1} \right) \Bigg/ \left(\sum_{j=0}^{\infty} \frac{p_j^{k_j}(1 - p_j)}{1 - p_j^{k_j}} \right).$$

PROOF. By setting $x_{ij} = 1$ for every $i \neq j$ in (3.1), we have the pgf of the distribution given by

$$\psi(t) = \frac{\sum_{i=0}^{\infty} G_i(t) \dfrac{\sum_{j \neq i} G_j(t)}{1 - t + \sum_{j \neq i} G_j(t)}}{1 - t + \sum_{j=0}^{\infty} G_j(t)}.$$

Noting that the derivative of the numerator can be written as

$$\sum_{i=0}^{\infty}\left\{G_i'(t)\frac{\sum_{j\neq i}G_j(t)}{1-t+\sum_{j\neq i}G_j(t)}+G_i(t)\frac{(1-t)\sum_{j\neq i}G_j'(t)+\sum_{j\neq i}G_j(t)}{(1-t+\sum_{j\neq i}G_j(t))^2}\right\},$$

we obtain

$$\begin{aligned}
&\psi'(1)\\
&=\frac{\left(\sum\limits_{i=0}^{\infty}\left(G_i'(1)+\dfrac{G_i(1)}{\sum\limits_{j\neq i}G_j(1)}\right)\right)\left(\sum\limits_{j=0}^{\infty}G_j(1)\right)-\left(\sum\limits_{i=0}^{\infty}G_i(1)\right)\left(-1+\sum\limits_{j=0}^{\infty}G_j'(1)\right)}{\left(\sum\limits_{j=0}^{\infty}G_j(1)\right)^2}\\
&=\frac{\left(\sum\limits_{i=0}^{\infty}G_i(1)\right)\left(1+\sum\limits_{i=0}^{\infty}\dfrac{G_i(1)}{\sum\limits_{j\neq i}G_j(1)}\right)}{\left(\sum\limits_{j=0}^{\infty}G_j(1)\right)^2}=\frac{1+\sum\limits_{i=0}^{\infty}\dfrac{G_i(1)}{\sum\limits_{j=0}^{\infty}G_j(1)-G_i(1)}}{\sum\limits_{j=0}^{\infty}G_j(1)}.
\end{aligned}$$

This completes the proof.

4. The r-th occurrence problem in the i.i.d. case

Let r be a positive integer greater than one. We consider the distribution of the waiting time for the occurrence of the r-th event among $\{E_i\}_{i=0}^{\infty}$. We use markers x_i, x_{ij}, $x_{ijl}, \dots$ as in the previous sections; for example, the marker x_{ijl} means that the first occurring event is E_i, the second event is E_j and the third event is E_l. In this section, we denote by $\phi_1 = \phi_1(t; x_{i_1}, i_1 = 0,1,2,\dots)$ the gpgf of the distribution of the waiting time for the first occurrence among $\{E_i\}_{i=0}^{\infty}$. In general, $\phi_r = \phi_r(t; x_{i_1,i_2,\dots,i_r}; i_1,\dots,i_r = 0,1,2,\dots)$ denotes the gpgf of the distribution of the waiting time for the r-th occurrence among the events $\{E_i\}_{i=0}^{\infty}$. From the meaning of our problem, if $j \neq l$, then $i_j \neq i_l$ holds. We set

$$\begin{aligned}
&\psi_{i_1,i_2,\dots,i_{r-1}}(t; x_{i_1,i_2,\dots,i_{r-1},l}, l \neq i_1, i_2, \dots, i_{r-1})\\
&\quad\equiv \frac{\sum_{l\neq i_1,i_2,\dots,i_{r-1}} G_l(t)x_{i_1,i_2,\dots,i_{r-1},l}}{1-t+\sum_{l\neq i_1,i_2,\dots,i_{r-1}}G_l(t)}.
\end{aligned}$$

Then, we have

THEOREM 4.1. *For each integer $r \geq 2$, it holds that*

$$\begin{aligned}
&\phi_r(t; x_{i_1,i_2,\dots,i_r}, i_1,\dots,i_r = 0,1,2,\dots)\\
&\quad= \phi_{r-1}(t; x_{i_1,i_2,\dots,i_{r-1}} = \psi_{i_1,i_2,\dots,i_{r-1}}(t; x_{i_1,i_2,\dots,i_{r-1},l})).
\end{aligned}$$

The right hand side of this equation means the formula which is obtained by replacing every marker $x_{i_1,i_2,\dots,i_{r-1}}$ *in* $\phi_{r-1}(t;x_{i_1,i_2,\dots,i_{r-1}})$ *by* $\psi_{i_1,i_2,\dots,i_{r-1}}(t;x_{i_1,i_2,\dots,i_{r-1},l})$.

PROOF. Let ϕ_{ij}, $\phi^{(l)}$ and $\phi_{ij}^{(l)}$ be the gpgf's of the conditional distributions defined in the previous section. Further, for each integer $m \le r$, let $\phi^{(i_1,i_2,\dots,i_m)}$ be the gpgf of the conditional distribution given that the n-th occurring event is E_{i_n} for $n = 1,2,\dots,m$ and the event E_m has just occurred. Let $\phi_{ij}^{(i_1,i_2,\dots,i_m)}$ be the gpgf of the conditional distribution of the waiting time given that the n-th occurring event is E_{i_n} for $n = 1,2,\dots,m$ and the event E_m has already occurred and we are currently in a run of i of length j. For the $(r-1)$-st occurrence problem, consider the system of equations for ϕ, $\phi^{(i_1,\dots,i_m)}$, $\phi_{ij}^{(i_1,\dots,i_m)}$, $m = 1,2,\dots,r-1$ and markers $x_i, x_{ij},\dots$; the system of equations for $r = 3$ was given in Section 3. If we replace every marker $x_{i_1,i_2,\dots,i_{r-1}}$ by $\phi^{(i_1,i_2,\dots,i_{r-1})}$ in the system of equations and add the following system of equations;

$$\begin{aligned}
\phi^{(i_1,i_2,\dots,i_{r-1})} &= \sum_{m=0}^{\infty} p_m t \phi_{m1}^{(i_1,i_2,\dots,i_{r-1})}, \\
\phi_{i_j,l}^{(i_1,i_2,\dots,i_{r-1})} &= p_{i_j} t \phi_{i_j,l+1}^{(i_1,i_2,\dots,i_{r-1})} + \sum_{m\ne i_j} p_m t \phi_{m1}^{(i_1,i_2,\dots,i_{r-1})}, \\
&\qquad j = 1,\dots,r-1;\ l = 1,2,\dots,k_{i_j}-2, \\
\phi_{i_j,k_{i_j}-1}^{(i_1,i_2,\dots,i_{r-1})} &= p_{i_j} t \phi^{(i_1,i_2,\dots,i_{r-1})} + \sum_{m\ne i_j} p_m t \phi_{m1}^{(i_1,i_2,\dots,i_{r-1})}, \\
&\qquad j = 1,2,\dots,r-1, \\
\phi_{n,l}^{(i_1,i_2,\dots,i_{r-1})} &= p_n t \phi_{n,l+1}^{(i_1,i_2,\dots,i_{r-1})} + \sum_{m\ne n} p_m t \phi_{m1}^{(i_1,i_2,\dots,i_{r-1})}, \\
&\qquad n \ne i_1,\dots,i_{r-1};\ l = 1,2,\dots,k_n-2, \\
\phi_{n,k_n-1}^{(i_1,i_2,\dots,i_{r-1})} &= p_n t x_{i_1,i_2,\dots,i_{r-1},n} + \sum_{m\ne n} p_m t \phi_{m1}^{(i_1,i_2,\dots,i_{r-1})}, \\
&\qquad n \ne i_1,i_2,\dots,i_{r-1},
\end{aligned}$$

then the resulting system of equations becomes the system of equations for the r-th occurrence problem. Therefore, if we calculate $\phi^{(i_1,i_2,\dots,i_{r-1})}$ by solving the above added system of equations and replace every marker $x_{i_1,\dots,i_{r-1}}$ in $\phi_{r-1}(t;x_{i_1,i_2,\dots,i_{r-1}})$ by the solution, we obtain $\phi_r(t;x_{i_1,i_2,\dots,i_r})$. We can indeed solve the above added part of the system independently and we have

$$\phi^{(i_1,i_2,\dots,i_{r-1})} = \frac{\sum_{l\ne i_1,\dots,i_{r-1}} G_l(t) x_{i_1,\dots,i_{r-1},l} + \phi^{(i_1,i_2,\dots,i_{r-1})} \sum_{j=1}^{r-1} G_{i_j}(t)}{1 - t + \sum_{j=1}^{\infty} G_j(t)},$$

and hence we obtain $\phi^{(i_1,i_2,\dots,i_{r-1})} = \psi_{i_1,i_2,\dots,i_{r-1}}(t)$. This completes the proof.

5. A non i.i.d. case

Let $X_1, X_2, \dots$ be a sequence of nonnegative integer valued random variables. As the previous sections, we are given a sequence $\{k_j\}_{j=0}^{\infty}$ of positive integers. Since we are interested in succession events such as runs of j of length k_j, $j = 0, 1, 2, \dots$, we think that the following probability law for the sequence must be most suitable. Let $\{p_i\}_{i=0}^{\infty}$ be a sequence of nonnegative real numbers satisfying $\sum_{i=0}^{\infty} p_i = 1$. Further, we assume that for every $i = 0, 1, 2, \dots$; $j = 0, 1, \dots, k_i - 1$ and $l = 0, 1, 2, \dots$, there exists a real number $p(i, j, l) \in [0, 1]$ such that for every i and l, $p(i, 0, l) = p_l$ holds and for every i and j, $\sum_{l=0}^{\infty} p(i, j, l) = 1$ holds. Suppose that the probability law of the sequence $X_1, X_2, \dots$ are given by the following system of conditional distributions:

$$
\begin{aligned}
&P(X_1 = l) = p_l, \qquad l = 0, 1, 2, \dots, \\
&P(X_x = l \mid X_{x-1} = i, X_{x-2} = i, \dots, X_{x-u} = i, X_{x-u-1} = i_{x-u-1}, \dots, X_1 = i_1) \\
&\qquad = p(i, j, l), \\
&\text{where } j = u - [u/k_i] \cdot k_i \quad \text{and} \quad i_{x-u-1} \neq i.
\end{aligned}
$$

Remark 3. If for every i, j and l, $p(i, j, l) = p_l$ holds, then X_x and $(X_1, \dots, X_{x-1})$ are independent and the corresponding problem has already treated in the previous sections.

Remark 4. If we assume that X's are $\{0, 1\}$-valued and let $k_0 = 1$ and $k_1 = k$, then we get a binary sequence of order k (cf. Aki (1985)).

Define $\{E_j\}_{j=0}^{\infty}$, ϕ, ϕ_{ij} etc. as the previous sections. We shall investigate the distribution of the waiting time for the first occurrence of an event among $\{E_j\}_{j=0}^{\infty}$. From the definition, we see that ϕ and ϕ_{ij}, $i = 0, 1, 2, \dots$; $j = 1, \dots, k_i - 1$ satisfy the system of equations:

$$
\begin{aligned}
\phi &= \sum_{j=0}^{\infty} p_j t \phi_{j1}, \\
\phi_{ij} &= p(i, j, i) t \phi_{i,j+1} + \sum_{l \neq i} p(i, j, l) t \phi_{l1}, \\
&\qquad\qquad i = 0, 1, 2, \dots; \; j = 1, 2, \dots, k_i - 2, \\
\phi_{i,k_i-1} &= p(i, k_i - 1, i) t x_i + \sum_{l \neq i} p(i, k_i - 1, l) t \phi_{l1}, \\
&\qquad\qquad i = 0, 1, 2, \dots.
\end{aligned}
$$

As it is not easy to solve the system of equations, we assume that X's are $\{0, 1, 2, \dots, m\}$-valued random variables, where m is any fixed positive integer.

Then we can obtain the following equation from the above system of equations:

$$B \cdot [\phi_{01}, \phi_{11}, \dots, \phi_{m1}]' = \begin{bmatrix} \left(\prod_{j=1}^{k_0-1} p(0,j,0)\right) t^{k_0-1} x_0 \\ \left(\prod_{j=1}^{k_1-1} p(1,j,1)\right) t^{k_1-1} x_1 \\ \vdots \\ \left(\prod_{j=1}^{k_m-1} p(m,j,m)\right) t^{k_m-1} x_m \end{bmatrix},$$

where

$$B = \begin{bmatrix} 1 & b_{01} & \cdots & b_{0m} \\ b_{10} & 1 & \cdots & b_{1m} \\ \vdots & \vdots & \vdots & \vdots \\ b_{m0} & b_{m1} & \cdots & 1 \end{bmatrix},$$

and

$$b_{ij} = \begin{cases} 1 & \text{if} \quad i = j, \\ -\sum_{l=1}^{k_i-1} \left(\prod_{n=1}^{l-1} p(i,n,i)\right) p(i,l,j) t^l & \text{if} \quad i \neq j. \end{cases}$$

Consequently, we have

THEOREM 5.1. *The gpgf $\phi(t)$ of the distribution of the waiting time for the first occurrence is given by*

$$\phi(t) = (p_0 t, \dots, p_m t) B^{-1} \begin{bmatrix} \left(\prod_{j=1}^{k_0-1} p(0,j,0)\right) t^{k_0-1} x_0 \\ \left(\prod_{j=1}^{k_1-1} p(1,j,1)\right) t^{k_1-1} x_1 \\ \vdots \\ \left(\prod_{j=1}^{k_m-1} p(m,j,m)\right) t^{k_m-1} x_m \end{bmatrix}.$$

Similarly as in the i.i.d. case, we can get the gpgf of the distribution of the waiting time for the r-th occurrence.

In the rest of the section, we investigate the sooner and later problems for the binary sequence of order k as an example of the non i.i.d. case. Here, k is a fixed positive integer. Aki (1985) defined a binary sequence of order k by extending Bernoulli trials. The sequence is suitable for considering succession events in practical situations where independence of the trials can not be assumed. The definition of the sequence is as follows:

DEFINITION 1. A sequence $\{X_i\}_{i=0}^{\infty}$ of $\{0,1\}$-valued random variables is said to be a binary sequence of order k if there exist a positive integer and k real numbers $0 < p_1, p_2, \dots, p_k < 1$ such that

(1) $X_0 = 0$ almost surely, and

(2) $P(X_n = 1 \mid X_0 = x_0, X_1 = x_1, \dots, X_{n-1} = x_{n-1}) = p_j$,

is satisfied for any positive integer n, where $j = r - [(r-1)/k] \cdot k$, and r is the smallest positive integer which satisfies $x_{n-r} = 0$.

Let $X_1, X_2, \dots$ be a binary sequence of order k. We are interested in two events E_0 and E_1, where E_0 is the event that a run of "0" of length r occurs and E_1 is the event that a run of "1" of length k occurs in the sequence $X_1, X_2, \dots$.

First, we study the sooner waiting time problem. Let y be the marker which represents the sooner (or the first) occurring event is E_0 and let x be the marker which represents the sooner occurring event is E_1 in the sequence. Denote by $\phi = \phi(t)$ the gpgf of the distribution of the waiting time for the occurrence of the sooner event. Let ϕ_i be the gpgf of the conditional distribution of the waiting time given that we start with a run of "1" of length i and let $\phi^{(j)}$ be the gpgf of the conditional distribution of the waiting time given that we start with a run of "0" of length j. From the definition, we have $\phi_0 = \phi^{(0)} = \phi$. Then, $\phi_0, \phi_1, \dots, \phi_{k-1}$, $\phi^{(1)}, \dots$, and $\phi^{(r-1)}$ satisfy the following system of equations:

$$
\begin{aligned}
\phi_0 &= p_1 t\phi_1 + q_1 t\phi^{(1)}, & \phi^{(1)} &= p_1 t\phi_1 + q_1 t\phi^{(2)},\\
\phi_1 &= p_2 t\phi_2 + q_2 t\phi^{(1)}, & \phi^{(2)} &= p_1 t\phi_1 + q_1 t\phi^{(3)},\\
&\ \vdots & &\ \vdots\\
\phi_{k-2} &= p_{k-1} t\phi_{k-1} + q_{k-1} t\phi^{(1)}, & \phi^{(r-2)} &= p_1 t\phi_1 + q_1 t\phi^{(r-1)},\\
\phi_{k-1} &= p_k tx + q_k t\phi^{(1)}, & \phi^{(r-1)} &= p_1 t\phi_1 + q_1 ty,
\end{aligned}
$$

where $q_j = 1 - p_j$, $j = 1, 2, \dots, k$. By solving the above system of equations, we have

$$
\phi = \frac{p_1 p_2 \cdots p_k q_1 t^{k+1} x \sum_{j=0}^{r-1} (q_1 t)^j + q_1^r t^r y \sum_{i=1}^{k} p_1 \cdots p_{i-1} q_i t^i}{(1 - \sum_{j=0}^{r-1} (q_1 t)^j)(\sum_{i=1}^{k} p_1 \cdots p_{i-1} q_i t^i) + \sum_{j=1}^{r} (q_1 t)^j},
$$

where we mean that $p_1 \cdots p_{i-1} = 1$ for $i = 1$. Setting $x = y = 1$ in the equation and letting $r \to \infty$, we obtain

$$
\phi \to \frac{p_1 \cdots p_k t^k}{1 - \sum_{i=1}^{k} p_1 \cdots p_{i-1} q_i t^i}.
$$

This limit agrees with the pgf of the extended geometric distribution of order k (cf. Aki (1985)).

Next, we study the later waiting time problem. We will get the gpgf of the distribution of the waiting time for the later occurring event in the sequence $X_1, X_2, \dots$. We denote by y the marker which means that the event E_0 occurs later and we denote by x the marker which means the event E_1 occurs later. Let $\phi = \phi(t)$ be the gpgf which includes the markers x and y. For $i = 0, 1, \dots, k-1$, let ϕ_i be the gpgf of the conditional distribution of the waiting time given that we start with a run of "1" of length i. For $j = 0, 1, \ldots, r-1$, let $\phi^{(j)}$ be the gpgf of the conditional distribution of the waiting time given that we start with a run of "0" of length j. For $i = 0, 1, \ldots, k-1$, let ψ_i be the gpgf of the conditional distribution of the waiting time given that the first occurring event is E_1 and E_1 has already occurred and we are currently in a run of "1" of length i. For $j = 0, 1, \ldots, r-1$, let $\psi^{(j)}$ be the gpgf of the conditional distribution given that the first occurring event is E_1 and E_1 has already occurred and we are currently in a run of "0" of length j. Further, for $i = 0, 1, \ldots, k-1$, let ξ_i be the gpgf of the conditional distribution given that the first occurring event is E_0 and E_0 has already occurred and we are currently in a run of "1" of length i and for $j = 0, 1, \ldots, r-1$, let $\xi^{(j)}$ be the gpgf of the conditional distribution given that the first occurring event is E_0 and E_0 has already occurred and we are currently in a run of "0" of length j. From the definition, we can note that $\phi = \phi_0 = \phi^{(0)}$, $\psi_0 = \psi^{(0)}$, $\xi_0 = \xi^{(0)}$ and $\xi^{(0)} = \xi^{(1)} = \cdots = \xi^{(r-1)}$. For the moment, we assume that $k \geq 2$ and $r \geq 2$. Then, $\phi_0, \ldots, \phi_{k-1}$, $\phi^{(0)}, \ldots, \phi^{(r-1)}$, $\psi_0, \ldots, \psi_{k-1}$, $\psi^{(1)}, \ldots, \psi^{(r-1)}$, $\xi^{(0)}, \xi_1, \ldots, \xi_{k-1}$ satisfy the following equations:

$$
\begin{aligned}
\phi_0 &= p_1 t\phi_1 + q_1 t\phi^{(1)}, \\
\phi_1 &= p_2 t\phi_2 + q_2 t\phi^{(1)}, \\
\phi_2 &= p_3 t\phi_3 + q_3 t\phi^{(1)}, \\
&\vdots \\
\phi_{k-2} &= p_{k-1} t\phi_{k-1} + q_{k-1} t\phi^{(1)}, \\
\phi_{k-1} &= p_k t\psi_0 + q_k t\phi^{(1)},
\end{aligned}
\tag{5.1}
$$

$$
\begin{aligned}
\psi_0 &= p_1 t\psi_1 + q_1 t\psi^{(1)}, \\
\psi_1 &= p_2 t\psi_2 + q_2 t\psi^{(1)}, \\
\psi_2 &= p_3 t\psi_3 + q_3 t\psi^{(1)}, \\
&\vdots \\
\psi_{k-2} &= p_{k-1} t\psi_{k-1} + q_{k-1} t\psi^{(1)}, \\
\psi_{k-1} &= p_k t\psi_0 + q_k t\psi^{(1)},
\end{aligned}
\tag{5.2}
$$

$$
\begin{aligned}
\psi^{(1)} &= p_1 t\psi_1 + q_1 t\psi^{(2)}, \\
\psi^{(2)} &= p_1 t\psi_1 + q_1 t\psi^{(3)}, \\
&\vdots \\
\psi^{(r-2)} &= p_1 t\psi_1 + q_1 t\psi^{(r-1)}, \\
\psi^{(r-1)} &= p_1 t\psi_1 + q_1 t y,
\end{aligned}
\tag{5.3}
$$

$$\begin{aligned}
\phi^{(0)} &= p_1 t\phi_1 + q_1 t\phi^{(1)},\\
\phi^{(1)} &= p_1 t\phi_1 + q_1 t\phi^{(2)},\\
\phi^{(2)} &= p_1 t\phi_1 + q_1 t\phi^{(3)},\\
&\vdots\\
\phi^{(r-2)} &= p_1 t\phi_1 + q_1 t\phi^{(r-1)},\\
\phi^{(r-1)} &= p_1 t\phi_1 + q_1 t\xi^{(0)},
\end{aligned} \tag{5.4}$$

$$\xi^{(0)} = p_1 t\xi_1 + q_1 \xi^{(0)}, \tag{5.5}$$

$$\begin{aligned}
\xi_1 &= p_2 t\xi_2 + q_2 t\xi^{(1)},\\
\xi_2 &= p_3 t\xi_3 + q_3 t\xi^{(1)},\\
&\vdots\\
\xi_{k-2} &= p_{k-1} t\xi_{k-1} + q_{k-1} t\xi^{(1)},\\
\xi_{k-1} &= p_k t x + q_k t\xi^{(1)}.
\end{aligned} \tag{5.6}$$

If we can solve these equations, we get the gpgf ϕ of the distribution. Here we show how to solve them. From (5.3), we have

$$\psi^{(1)} = \left(\sum_{i=0}^{r-2} q_1^i p_1 t^{i+1}\right)\psi_1 + q_1^{r-1} t^{r-1} y. \tag{5.7}$$

From (5.2) and (5.1), we get, respectively,

$$\psi_0 = p_1 \cdots p_k t^k \psi_0 + \psi^{(1)}\left(\sum_{i=1}^{k} p_1 \cdots p_{i-1} q_i t^i\right) \tag{5.8}$$

and

$$\phi_0 = p_1 \cdots p_k t^k \psi_0 + \phi^{(1)}\left(\sum_{i=1}^{k} p_1 \cdots p_{i-1} q_i t^i\right). \tag{5.9}$$

By substituting $\psi_0 = p_1 t\psi_1 + q_1 t\psi^{(1)}$ into the equations (5.8) and (5.9), we have, respectively,

$$(1 - p_1 \cdots p_k t^k)(p_1 t\psi_1 + q_1 t\psi^{(1)}) = \psi^{(1)}\left(\sum_{i=1}^{k} p_1 \cdots p_{i-1} q_i t^i\right) \tag{5.10}$$

and

$$\phi_0 = p_1 \cdots p_k t^k (p_1 t \psi_1 + q_1 t \psi^{(1)}) + \phi^{(1)} \left(\sum_{i=1}^{k} p_1 \cdots p_{i-1} q_i t^i \right). \tag{5.11}$$

By substituting the equation (5.7) into the equation (5.10), we can obtain

$$\psi_1 = \frac{q_1^{r-1} t^{r-1} y \left\{ \left(\sum_{i=2}^{k} p_1 \cdots p_{i-1} q_i t^i \right) + p_1 \cdots p_k q_1 t^{k+1} \right\}}{(1 - p_1 \cdots p_k t^k) p_1 t - \left\{ \left(\sum_{i=2}^{k} p_1 \cdots p_{i-1} q_i t^i \right) + p_1 \cdots p_k q_1 t^{k+1} \right\} \left(\sum_{i=0}^{r-2} q_1^i p_1 t^{i+1} \right)}. \tag{5.12}$$

By (5.12) and (5.7), we see that

$$\psi^{(1)} = q_1^{r-1} t^{r-1} y + \frac{q_1^{r-1} t^{r-1} y \left\{ \left(\sum_{i=2}^{k} p_1 \cdots p_{i-1} q_i t^i \right) + p_1 \cdots p_k q_1 t^{k+1} \right\} \left(\sum_{i=0}^{r-2} q_1^i p_1 t^{i+1} \right)}{(1 - p_1 \cdots p_k t^k) p_1 t - \left\{ \left(\sum_{i=2}^{k} p_1 \cdots p_{i-1} q_i t^i \right) + p_1 \cdots p_k q_1 t^{k+1} \right\} \left(\sum_{i=0}^{r-2} q_1^i p_1 t^{i+1} \right)}. \tag{5.13}$$

On the other hand, from (5.6) and (5.4), we have, respectively,

$$\xi_1 = p_2 \cdots p_k t^{k-1} x + \xi^{(1)} \left(\sum_{i=2}^{k} p_2 \cdots p_{i-1} q_i t^{i-1} \right) \tag{5.14}$$

and

$$\phi^{(0)} = \left(\sum_{i=0}^{r-1} q_1^i p_1 t^{i+1} \right) \phi_1 + q_1^r t^r \xi^{(0)}. \tag{5.15}$$

Noting that $\xi^{(0)} = \xi^{(1)}$, from (5.14) and (5.5), we see

$$\xi^{(0)} = \frac{p_1 \cdots p_k t^k x}{1 - \sum_{i=1}^{k} p_1 \cdots p_{i-1} q_i t^i}. \tag{5.16}$$

Consequently, we can calculate ϕ from equations $\phi_0 = \phi^{(0)}$, $\phi_0 = p_1 t \phi_1 + q_1 t \phi^{(1)}$, (5.11), (5.12), (5.13), (5.15) and (5.16). In fact, by substituting the equation $\phi_0 = p_1 t \phi_1 + q_1 t \phi^{(1)}$ into (5.11) and (5.15), we get

$$p_1 t \phi_1 = p_1 \cdots p_k t^k (p_1 t \psi_1 + q_1 t \psi^{(1)}) + \left(\sum_{i=2}^{k} p_1 \cdots p_{i-1} q_i t^i \right) \phi^{(1)} \tag{5.17}$$

and

$$q_1 t \phi^{(1)} = \left(\sum_{i=1}^{r-1} q_1^i p_1 t^{i+1} \right) \phi_1 + q_1^r t^r \xi^{(0)}. \tag{5.18}$$

Putting (5.17) and (5.18) together, we have

$$\phi_1 = \frac{p_1 \cdots p_k t^k (p_1 t \psi_1 + q_1 t \psi^{(1)}) + q_1^{r-1} t^{r-1} (\sum_{i=2}^k p_1 \cdots p_{i-1} q_i t^i) \xi^{(0)}}{p_1 t - (\sum_{i=2}^k p_1 \cdots p_{i-1} q_i t^i)(\sum_{i=1}^{r-1} q_1^{r-1} p_1 t^i)}.$$

Therefore, substituting into (5.15), we obtain

$$\phi(t) = q_1^r t^r \xi^{(0)} + \frac{\left(\sum_{i=0}^{r-1} q_1^i p_1 t^{i+1}\right)\left\{p_1 \cdots p_k t^k (p_1 t \psi_1 + q_1 t \psi^{(1)}) + q_1^{r-1} t^{r-1} \left(\sum_{i=2}^k p_1 \cdots p_{i-1} q_i t^i\right) \xi^{(0)}\right\}}{p_1 t - \left(\sum_{i=2}^k p_1 \cdots p_{i-1} q_i t^i\right)\left(\sum_{i=1}^{r-1} q_1^{r-1} p_1 t^i\right)},$$

where ψ_1, $\psi^{(1)}$ and $\xi^{(0)}$ were given by (5.12), (5.13) and (5.16), respectively.

Next, we consider the case that $r = 1$. The corresponding equations to (5.1)–(5.6) are as follows:

$$\begin{aligned} \phi_0 &= p_1 t \phi_1 + q_1 t \xi^{(0)}, \\ \phi_1 &= p_2 t \phi_2 + q_2 t \xi^{(0)}, \\ &\vdots \\ \phi_{k-1} &= p_k t \psi_0 + q_k t \xi^{(0)}, \end{aligned} \tag{5.19}$$

$$\begin{aligned} \psi_0 &= p_1 t \psi_1 + q_1 t y, \\ \psi_1 &= p_2 t \psi_2 + q_2 t y, \\ &\vdots \\ \psi_{k-1} &= p_k t \psi_0 + q_k t y, \end{aligned} \tag{5.20}$$

$$\begin{aligned} \xi^{(0)} &= p_1 t \xi_1 + q_1 t \xi^{(0)}, \\ \xi_1 &= p_2 t \xi_2 + q_2 t \xi^{(0)}, \\ &\vdots \\ \xi_{k-1} &= p_k t x + q_k t \xi^{(0)}. \end{aligned} \tag{5.21}$$

By (5.19), we get

$$\phi_0 = p_1 \cdots p_k t^k \psi_0 + \xi^{(0)} \left(\sum_{i=1}^k p_1 \cdots p_{i-1} q_i t^i\right). \tag{5.22}$$

The equations (5.20) and (5.21) imply, respectively,

$$\psi_0 = \frac{y(\sum_{i=1}^k p_1 \cdots p_{i-1} q_i t^i)}{1 - p_1 \cdots p_k t^k}$$

and

$$\xi^{(0)} = \frac{p_1 \cdots p_k t^k x}{1 - \sum_{i=1}^{k} p_1 \cdots p_{i-1} q_i t^i}.$$

Consequently, by substituting these equations into (5.22), we obtain

$$\phi(t) = \frac{y p_1 \cdots p_k t^k (\sum_{i=1}^{k} p_1 \cdots p_{i-1} q_i t^i)}{1 - p_1 \cdots p_k t^k} + \frac{x p_1 \cdots p_k t^k (\sum_{i=1}^{k} p_1 \cdots p_{i-1} q_i t^i)}{1 - \sum_{i=1}^{k} p_1 \cdots p_{i-1} q_i t^i}.$$

Acknowledgements

I would like to thank Professors K. Hirano, H. Kuboki and N. Kashiwagi for the stimulating discussions during the seminar held under the auspices of the Institute of Statistical Mathematics (90-ISM-CRP-11).

References

Aki, S. (1985). Discrete distribution of order k on a binary sequence, *Ann. Inst. Statist. Math.*, **37**, 205–224.

Aki, S. and Hirano, K. (1989). Estimation of parameters in the discrete distributions of order k, *Ann. Inst. Statist. Math.*, **41**, 47–61.

Aki, S., Kuboki, H. and Hirano, K. (1984). On discrete distributions of order k, *Ann. Inst. Statist. Math.*, **36**, 431–440.

Ebneshahrashoob, M. and Sobel, M. (1990). Sooner and later waiting time problems for Bernoulli trials: frequency and run quotas, *Statist. Probab. Lett.*, **9**, 5–11.

Feller, W. (1957). *An Introduction to Probability Theory and Its Applications*, Vol. 1, 2nd ed., Wiley.

Ling, K. D. (1990). On geometric distributions of order $(k_1, \ldots, k_m)$, *Statist. Probab. Lett.*, **9**, 163–171.

Philippou, A. N. (1986). Distributions and Fibonacci polynomials of order k, longest runs, and reliability of consecutive-k-out-of-n:F systems, *Fibonacci Numbers and Their Applications* (eds. A. N. Philippou, G. E. Bergum and A. F. Horadam), 203–227, Reidel, Dordrecht.

Philippou, A. N., Georghiou, C. and Philippou, G. N. (1983). A generalized geometric distribution and some of its properties, *Statist. Probab. Lett.*, **1**, 171–175.