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ASYMPTOTIC EXPANSIONS FOR TWO-STAGE RANK TESTS

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Abstract. Stein's two-stage procedure produces a t-test which can realize a prescribed power against a given alternative, regardless of the unknown variance of the underlying normal distribution. This is achieved by determining the size of a second sample on the basis of a variance estimate derived from the first sample. In the paper we introduce a nonparametric competitor of this classical procedure by replacing the t-test by a rank test. For rank tests, the most precise information available are asymptotic expansions for their power to order n^{-1}, where n is the sample size. Using results on combinations of rank tests for sub-samples, we obtain the same level of precision for the two-stage case. In this way we can determine the size of the additional sample to the natural order and moreover compare the nonparametric and the classical procedure in terms of expected additional numbers of observations required.

Key words and phrases: One-sample problem, Stein's two-stage procedure.

1. Introduction

Consider the following one-sample problem: let $X_1, X_2, \ldots$ be independent identically distributed (iid) random variables (rv's) from a continuous distribution function (df) $F(x-\theta)$. Suppose that the distribution determined by F is symmetric about zero, i.e. $F(-x) = 1 - F(x)$ for all x. Then we are interested in testing $H_0 : \theta = 0$ based on a sample from the sequence $X_1, X_2, \ldots$. For the special case where $F(x) = \Phi(x/\sigma)$, in which Φ is the standard normal df, Stein has proposed the following two-stage procedure (see e.g. Lehmann (1986), pp. 258–260). Take an initial sample of size m and evaluate its sample variance S_m^2. Then take a second sample of size $N - m$, where

$$N = \max(m, [S_m^2/c] + 1), \tag{1.1}$$

in which $c > 0$ is any given constant and $[y]$ denotes the largest integer $\leq y$. Stein has shown that $N^{1/2}(\bar{X}_N - \theta)/S_m$, where $\bar{X}_N = N^{-1}\sum_{i=1}^{N} X_i$, has a t_{m-1} distribution. Consequently, for any given alternative, the constant c from (1.1) can be chosen such that the corresponding test has at least a given power against that alternative, independent of the scale parameter σ.

The purpose of the present paper is to investigate to what extent this nice feature of having a power independent of certain aspects of the unknown underlying df can be generalized to the case of rank tests. This would provide a more balanced situation in the following sense: fixed sample size rank tests, being distributionfree, have levels which are completely independent of F, whereas the powers are directly dependent of F. Hence it would be nice to have independence of e.g. scale parameter here too.

Clearly, the results obtained cannot be expected to be more precise than those with the fixed sample cases, and hence we will have to be contented with asymptotic rather than exact results. However, the asymptotic results available for fixed sample sizes n are not the mere first order normality results, but also asymptotic expansions to $o(n^{-1})$ from Albers, Bickel and van Zwet (1976) (to be denoted by ABZ in the sequel). Hence we shall derive such expansions for two-stage rank tests as well. Perhaps it is useful to remark already at this point that we shall concentrate on the case where N exceeds m with large probability, i.e. where the second sample is not-empty. For if this is not true, we will, with positive probability, use more observations than are necessary to attain the prescribed power, and hence the realized power will exceed it by a non-negligible amount, even to $o(1)$. But if there is already a discrepancy to first order, there is no need to pursue second order results.

Another introductory remark is the following. Even taking into account that the results from ABZ provide an excellent starting point, the derivation of expansions for two-stage rank tests poses tedious technical problems. Therefore, we shall use a device which simplifies matters considerably. Instead of evaluating the rank statistic in question for the total sample of size N, we let the two-stage character persist, then we evaluate separate rank statistics for the initial and the second sample. These two are then combined to a total statistic in an optimal manner.

The likely objection to this approach is that, simple as it may be, it will lead to an inferior procedure. However, this is definitely not the case. A similar device was applied by Albers and Akritas (1987) in the context of censored rank tests and was seen to work well. Moreover, Albers (1992) demonstrated that for rank tests, the loss due to splitting the sample can typically be compensated by as little as one additional observation. Incidentally, note that an explanation for the negligibility of such losses is also suggested by observing that Stein's statistic $N^{1/2}(\bar{X}_N - \theta)/S_m$ itself can be viewed as a linear combination of the separate statistics $m^{1/2}(\bar{X}_m - \theta)/S_m$ and $(N-m)^{1/2}(\bar{X}^*_{N-m} - \theta)/S_m$, with $\bar{X}_m$ and $\bar{X}^*_{N-m}$ the averages of the first and second sample, respectively.

In Section 2 we shall derive the desired expansion along the following lines. Using a suitable conditioning argument enables us to apply results from ABZ to each of the two separate rank statistics. The conditional expansions thus obtained in their turn lead through application of the results of Albers (1992) to a conditional expansion for the combined statistic. The result intended then follows by taking expectations. A specific choice of N as a function of the first sample, closely related to (1.1), is considered in Section 3. Some examples and a comparison to the performance of Stein's procedure are the subject of Section 4. Moreover, by way of illustration, a small simulation study is presented there for the case of

Wilcoxon's one-sample test. It turns out that the proposed procedure behaves quite satisfactory. In particular, it drastically improves the rather poor procedure based on mere first order approximations. Finally, the proofs are collected in the Appendix.

2. The expansion for the two-stage test

First we introduce some notation. As in the introduction, let $X_1, X_2, \ldots$ be iid rv's with continuous df $F(x-\theta)$, where F satisfies $F(-x) = 1 - F(x)$ for all x. For any given sample size n, let $0 < Z_1 < \cdots < Z_n$ denote the order statistics of $|X_1|, \ldots, |X_n|$. Moreover, let

$$(2.1) \qquad V_j = \begin{cases} 1 & \text{if the } X_i \text{ corresponding to } Z_j \text{ is positive,} \\ 0 & \text{otherwise,} \end{cases}$$

for $j = 1, \ldots, n$. Finally, let J be a continuous function on $(0,1)$ and let $U_{1:n} < \cdots < U_{n:n}$ be the order statistics of a sample of size n from the uniform distribution on $(0,1)$. Then we have the exact scores

$$(2.2) \qquad a_j = EJ(U_{j:n}),$$

$j = 1, \ldots, n$. The one-sample linear rank statistic for testing $H_0 : \theta = 0$ is now given by

$$(2.3) \qquad T = \sum_{j=1}^{n} a_j V_j.$$

Next we move on to the two-stage situation, where we have an initial sample of size m and a second sample of size $N - m$, in which $N = N(X_1, \ldots, X_m)$ in general. The first and rather obvious restriction we impose is that $N = N(Z_{(m)})$ where $Z_{(m)} = (Z_1, \ldots, Z_m)$ is the vector of absolute order statistics of the first sample. In this way it remains possible to preserve the distributionfree character of the test, as the Vj's and $Z_{(m)}$ are independent under H_0. The second restriction is the one announced in the introduction: to keep things tractable, we shall not consider a single rank statistic of the form (2.3) for the total sample, but instead work with separate rank statistics T_1 and T_2 for the first and second sample, respectively. As T from (2.3) under standard regularity conditions is asymptotically normal with mean and variance of the form $\theta n \mu$ and $n\sigma^2$, respectively, it is straightforward to verify (also see Albers (1992)) that the optimal combination T^* of T_1 and T_2 simply equals

$$(2.4) \qquad T^* = T_1 + T_2.$$

In passing we remark that our approach does allow the selection of different score functions for T_1 and T_2. This option for example, can be interesting in connection with adaptive rank tests (see Albers (1980)): the information contained in the first sample can also be used to select a hopefully better second score function.

However interesting it may be, it constitutes a digression involving considerable additional technicalities and therefore we shall not pursue it here.

The purpose of the present section now is to obtain an asymptotic expansion for the df of the statistic in (2.4), being the sum of two ordinary rank statistics which are linked through the relation $N = N(Z_{(m)})$. In the next section we shall investigate special choices of $N(Z_{(m)})$ which are of particular interest for testing applications, but here we shall only impose certain regularity conditions. In addition to the trivially needed constraint $N \geq m$, the main assumption entails that for some $\epsilon > 0$

$$P((1+\epsilon) \leq N/m \leq \epsilon^{-1}) = 1 - o(m^{-1}). \tag{2.5}$$

This condition ensures that if the first sample size $m \to \infty$, the size of the second sample will tend to infinity as the same rate, except on a set of negligible probability in an analysis to order m^{-1}.

The first step towards the desired expansion is to condition on $Z_{(m)}$. As T_2 depends on the first sample only through $N = N(Z_{(m)})$, it is immediate that conditional on $Z_{(m)} = z_{(m)}$, this statistic is independent of T_1 and moreover has the same distribution as the usual rank statistic from (2.3) for sample size $N(z_{(m)}) - m$. Consequently the asymptotic expansion for the df of T_2 given $Z_{(m)} = z_{(m)}$ is readily available from Theorem 4.1 of ABZ, which we shall now quote.

First we give the conditions on the df F and the score function J. Let Q be the class of twice continuously differentiable functions Q on $(0,1)$ that satisfy

$$\limsup_{t \to 0,1} t(1-t) \left| \frac{Q''(t)}{Q'(t)} \right| < \frac{3}{2}. \tag{2.6}$$

Let $\mathcal{F}$ be the class of df's on $\mathbb{R}^1$ with positive densities that are symmetric about zero, four times differentiable, and such that, for $\psi_i = f^{(i)}/f$, $\Psi_i(t) = \psi_i(F^{-1}((1+t)/2))$, $m_1 = 6$, $m_2 = 3$, $m_3 = 4/3$, $m_4 = 1$, we have $\Psi_1 \in Q$ (see (2.6)) and

$$\limsup_{y \to 0} \int_{-\infty}^{\infty} |\psi_i(x+y)|^{m_i} f(x)dx < \infty, \qquad i = 1, \ldots, 4. \tag{2.7}$$

Let $\mathcal{J}$ be the class of nonconstant functions J on $(0,1)$ that satisfy $J \in Q$ and $\int_0^1 J^4(t)dt < \infty$.

Next we introduce the expansion. Let $\phi^{(k)}$ be the $(k+1)$-th derivative of Φ, $k = 0, 1, 2, \ldots$. Simply denote $\phi^{(0)}$ by ϕ. The Hermite polynomial of degree k is defined through

$$H_k(x) = (-1)^k \phi^{(k)}(x)/\phi(x), \qquad k = 0, 1, 2, \ldots. \tag{2.8}$$

Hence $H_0(x) = 1$, $H_1(x) = x$, $H_2(x) = x^2 - 1$, $H_3(x) = x^3 - 3x$, etc. Moreover, we define using the convention that integration will be over $(0,1)$, unless stated otherwise,

$$
(2.9)\qquad
\begin{aligned}
\eta_n &= -\frac{n^{1/2}\theta \int J\Psi_1}{\left(\int J^2\right)^{1/2}},\\
b_0 &= -\frac{\int J(3\Psi_1^3 - 6\Psi_1\Psi_2 + \Psi_3)\int J^2}{6\left(\int J\Psi_1\right)^3},\\
b_1 &= \frac{\int J^2\Psi_1^2 - \iint J(s)\Psi_1'(s)J(t)\Psi_1'(t)(s\wedge t - st)dsdt}{2\left(\int J\Psi_1\right)^2},\\
b_2 &= \frac{\int J^3\Psi_1}{3\left(\int J^2\right)\left(\int J\Psi_1\right)},\\
b_3 &= \frac{\int J^4}{12\left(\int J^2\right)^2},\\
\tilde{b}_{0,n} &= \frac{2\sum_{j=1}^n \mathrm{Cov}_j}{\int J\Psi_1} - \frac{\sum_{j=1}^n \sigma_j^2}{\int J^2},
\end{aligned}
$$

where $\mathrm{Cov}_j = \mathrm{Cov}(J(U_{j:n}),\ \Psi_1(U_{j:n}))$ and $\sigma_j^2 = \sigma^2(J(U_{j:n}))$ (cf. (2.2)). Using (2.8) and (2.9) we arrive at the expansion $\tilde{G}(x-\eta_n)$, where

$$
(2.10)\qquad \tilde{G}(x) = \Phi(x) + n^{-1}\phi(x)\left\{\frac{1}{2}\eta_n\tilde{b}_{0,n} + \sum_{k=0}^{3}\eta_n^{(3-k)}b_kH_k(x)\right\}.
$$

Then we have:

LEMMA 2.1. *Let $F \in \mathcal{F}$, $J \in \mathcal{J}$ and $0 \le \theta \le Cn^{-1/2}$ for some $C > 0$. Then T from* (2.3) *satisfies*

$$
(2.11)\qquad \sup_x \left| P\left(\frac{2T - \sum_{j=1}^n a_j}{\left(\sum_{j=1}^n a_j^2\right)^{1/2}} \le x\right) - \tilde{G}(x-\eta_n)\right| = o(n^{-1}).
$$

PROOF. This result is contained in Theorem 4.1 of ABZ. □

Application of Lemma 2.1 for sample size $N(z_{(m)}) - m$ gives in view of (2.5) the conditional expansion for the df of T_2 except on a set of probability $o(m^{-1})$. For T_1, the matter is more complicated, as it obviously depends on $Z_{(m)} = (Z_1, \ldots, Z_m)$ in a more essential way than T_2. Fortunately, however, the program carried out in ABZ to obtain the expansion for the df of the rank statistic in (2.3) precisely begins with conditioning on the absolute order statistics and establishing a conditional expansion (cf. p. 111). Hence, although Theorem 4.1 from ABZ cannot be applied to T_1, earlier results, like Theorems 2.1 and 2.3 from ABZ, are relevant now. Of course, considerable modification is required for the present purpose. In particular, the result for T_1 has to be formulated in such a way that it lends itself to combination through the methods of Albers (1992) with the result for T_2, thus producing a conditional expansion for the df of their sum T^* (cf. (2.4)).

Before we can formulate the result, we need some additional notations. For the statistic T in (2.3), let $Z_{(n)} = (Z_1, \dots, Z_n)$ and define, for $j = 1, \dots, n$,

$$P_j = P(V_j = 1 \mid Z_{(n)}), \qquad \bar{\pi}_j = EP_j, \tag{2.12}$$

with V_j as in (2.1). Moreover, let

$$U = \sum_{j=1}^{n} a_j (P_j - \bar{\pi}_j) \Bigg/ \left(\sum_{j=1}^{n} a_j^2 \right)^{1/2}. \tag{2.13}$$

Then, using (2.10), (2.12) and (2.13) we introduce the conditional expansion

$$\begin{aligned} \tilde{K}(x) = \tilde{G}(x) + \phi(x) \Bigg\{ & -2U + \Bigg[-2(U^2 - EU^2) \\ & + \frac{1}{2} \frac{\sum_{j=1}^{n} a_j^2 \{(2P_j - 1)^2 - E(2P_j - 1)^2\}}{\sum_{j=1}^{n} a_j^2} \Bigg] H_1(x) \\ & + \frac{2}{3} \frac{\sum_{j=1}^{n} a_j^3 (P_j - \bar{\pi}_j)}{\left(\sum_{j=1}^{n} a_j^2 \right)^{3/2}} H_2(x) \Bigg\}. \end{aligned} \tag{2.14}$$

We then have, with η_n as in (2.9),

LEMMA 2.2. *Under the assumptions of Lemma* 2.1 *the rank statistic* T *from* (2.3) *satisfies*

$$\begin{aligned} \sup_x & \left| P\left(\frac{2T - \sum_{j=1}^{n} a_j}{\left(\sum_{j=1}^{n} a_j^2 \right)^{1/2}} \le x \;\Bigg|\; Z_{(n)} \right) - \tilde{K}(x - \eta_n) \right| \\ & = o(n^{-1}) + O\left(\sum_{j=1}^{n} |2P_j - 1|^5 + |U|^3 \right), \end{aligned} \tag{2.15}$$

except on a set of $z_{(n)}$*-values with probability* $o(n^{-1})$.

PROOF. See the Appendix. □

The implication of Lemma 2.2 is that the conditional expansion $\tilde{K}(x - \eta_n)$ largely agrees with the unconditional expansion $\tilde{G}(x - \eta_n)$. The difference terms in (2.14) not only have vanishing expectations, but, apart from the first one, will moreover turn out to be too small to cause any effect in the sequel. The term with

U, however, is in probability of order $n^{-1/2}$ and therefore has to be handled with care.

Summarizing the progress up to this point, we have that conditional on $Z_{(m)} = z_{(m)}$, the rank statistics T_1 and T_2 are independent, that Lemma 2.1 for sample size $N(Z_{(m)}) - m$ provides an expansion for the df of T_2, while Lemma 2.2 for sample size m does that for the df of T_1. The next step is to use the results of Albers (1992) to obtain a conditional expansion for the df of T^* from (2.4). To distinguish between the two parts involved, we denote (cf. (2.2))

$$
\begin{aligned}
a_{1j} &= EJ(U_{j:m}), && j = 1, \ldots, m, \\
a_{2j} &= EJ(U_{j:N-m}), && j = 1, \ldots, N-m.
\end{aligned} \tag{2.16}
$$

Likewise, we define $(V_{11}, \ldots, V_{1m})$ in analogy to (2.1) and modify (2.12) into

$$
P_{1j} = P(V_{1j} = 1 \mid Z_{(m)}), \qquad \pi_{1j} = EP_{1j}. \tag{2.17}
$$

This in its turn leads to (cf. (2.13))

$$
\tilde{U} = \sum a_{1j}(P_{1j} - \pi_{1j}) \Big/ \left(\sum a_{1j}^2 + \sum a_{2j}^2\right)^{1/2}, \tag{2.18}
$$

where we adapt the convention that summation involving a_{1j} or a_{2j} from (2.16) runs from 1 to m or $N - m$, respectively. Now let

$$
\begin{aligned}
\tilde{H}(x) = \Phi(x) + \phi(x)\Bigg\{ & \frac{1}{2}N^{-1}\eta_N(\tilde{b}_{0,m} + \tilde{b}_{0,N-m}) \\
& + N^{-1}\sum_{k=0}^{3} \eta_N^{(3-k)} b_k H_k(x) - 2\tilde{U} \\
& + \Bigg(\Bigg[-2\left\{\sum a_{1j}(P_{1j} - \pi_{1j})\right\}^2 \\
& + 2E\left\{\sum a_{1j}(P_{1j} - \pi_{1j})\right\}^2 \\
& + \frac{1}{2}\sum a_{1j}^2\{(2P_{1j} - 1)^2 - E(2P_{1j} - 1)^2\}\Bigg] \Big/ \\
& \left(\sum a_{1j}^2 + \sum a_{2j}^2\right)\Bigg) H_1(x) \\
& + \frac{2}{3}\Bigg(\sum a_{1j}^3(P_{1j} - \pi_{1j}) \Big/ \\
& \left(\sum a_{1j}^2 + \sum a_{2j}^2\right)^{3/2}\Bigg) H_2(x)\Bigg\},
\end{aligned} \tag{2.19}
$$

with η_N as in (2.9), with n replaced by N. Then we have

LEMMA 2.3. *Let $F \in \mathcal{F}$ and $J \in \mathcal{J}$. Suppose that $0 \le \theta \le Cm^{-1/2}$ for some $C > 0$ and N statisfies (2.5) for some $\epsilon > 0$. Then T^* from (2.4) satisfies*

$$
\begin{aligned}
\sup_x & \left| P\left(\frac{2T^* - \sum a_{1j} - \sum a_{2j}}{(\sum a_{1j}^2 + \sum a_{2j}^2)^{1/2}} \le x \;\middle|\; Z_{(m)} \right) - \tilde{H}(x - \eta_n) \right| \\
& = o(N^{-1}) + O\left(\sum |2P_{1j} - 1|^5 + |\tilde{U}|^3\right),
\end{aligned} \tag{2.20}
$$

except on a set of $z_{(m)}$-values with probability $o(m^{-1})$.

PROOF. See the Appendix. □

Note that the complexity of the expansion has increased remarkably little in going from (2.14) to (2.19), thus demonstrating that using separate statistics indeed has a very limited effect. In fact, if we replace $\tilde{b}_{0,m} + \tilde{b}_{0,N-m}$ in (2.19) by $\tilde{b}_{0,N}$ the first part of $\tilde{H}$ coincides with $\tilde{G}$ from (2.10) applied with N rather than n. More in particular, under the hypothesis we obtain that $\eta_N = 0$ in (2.20), $P_{1j} = \pi_{1j} = 1/2$ in (2.17) and $\tilde{U} = 0$ in (2.18). Hence under H_0 the expansion $\tilde{H}(x - \eta_N)$ in (2.20) boils down to $\Phi(x) + N^{-1}b_3H_3(x)\phi(x)$. Consequently, the test which rejects $H_0 : \theta = 0$ in favor of $H_1 : \theta > 0$ for large values of $(2T^* - \sum a_{1j} - \sum a_{2j})/(\sum a_{1j}^2 + \sum a_{2j}^2)^{1/2}$ has critical value $\xi_\alpha = \tilde{\xi}_\alpha + o(N^{-1})$, where

$$\tilde{\xi}_\alpha = u_\alpha - N^{-1}b_3H_3(u_\alpha), \tag{2.21}$$

with $u_\alpha = \Phi^{-1}(1-\alpha)$. But this is precisely the same result as for the fixed sample case!

Moreover note that (2.21) also shows that the standardized critical value depends in a very limited way on the conditioning. In fact, as we will check in more detail later on, replacement of N^{-1} in $\tilde{\xi}_\alpha$ by a fixed value like (something sufficiently close to) $(EN)^{-1}$, typically will result in changes of $o(m^{-1})$. Hence the requirement that the test, just like a permutation test, is performed conditionally, is apparently met not merely to $o(1)$, but even to $o(m^{-1})$, by the standardization of T^* through the conditional mean $(\sum a_{1j} + \sum a_{2j})/2$ and the conditional variance $(\sum a_{1j}^2 + \sum a_{2j}^2)/4$. Hence a test based on the unconditional distribution of $(2T^* - \sum a_{1j} - \sum a_{2j})/(\sum a_{1j}^2 + \sum a_{2j}^2)^{1/2}$, will agree to $o(m^{-1})$ with the exact conditional test. In particular, it will be distributionfree to this order.

In view of these last remarks it makes sense to set as the next goal the replacement of the conditional expansion in (2.20) by an unconditional one, which clearly can be achieved by taking the expectation with respect to $Z_{(m)}$.

This can be done under various conditions on N, resulting also in a variety of expansions. Here we shall concentrate, however, on the case of main interest, in which $N - EN = O_P(m^{1/2})$. The resulting expansion then is the natural one, in the sense that the second order terms typically are of order m^{-1}, while the remainder is $o(m^{-1})$. To be more precise, we shall in addition to (2.5) suppose that for certain $\beta > 1$

$$E|N - EN|^{2\beta} = O(m^\beta). \tag{2.22}$$

Let $r \geq (1 + 2\epsilon)m$ and let η_r be as in (2.9), with n replaced by r. Define

$$\bar{U} = (N/r)^{1/2} - 1, \tag{2.23}$$

$$\bar{H}(x) = \Phi(x) + \phi(x)\left\{\frac{1}{2}r^{-1}\eta_r(\tilde{b}_{0,m} + \tilde{b}_{0,[r]-m})\right. \tag{2.24}$$

$$
\begin{aligned}
&+r^{-1}\sum_{k=0}^{3}\eta_r^{(3-k)}b_kH_k(x)-\eta_rE\bar{U}\\
&-\frac{1}{2}\eta_r^2E\bar{U}^2H_1(x)\\
&+\left.\frac{r^{-1}\eta_rE\left(\bar{U}\sum a_{1j}(\psi_1(Z_{1j})-E\psi_1(Z_{1j}))\right)}{\int J\Psi_1}\right\}.
\end{aligned}
$$

Then we finally arrive at

THEOREM 2.1. *Let $F \in \mathcal{F}$ and $J \in \mathcal{J}$. Suppose that $0 \le \theta \le Cm^{-1/2}$ for some $C > 0$ and N satisfies (2.5) for some $\epsilon > 0$ and (2.22) for some $\beta > 1$. If r in (2.23) and (2.24) is chosen such that $r = EN + o(m^{1/2})$, we have for T^* from (2.4) that*

$$
\sup_x\left|P\left(\frac{2T^*-\sum a_{1j}-\sum a_{2j}}{(\sum a_{1j}^2+\sum a_{2j}^2)^{1/2}}\le x\right)-\bar{H}(x-\eta_r)\right|=o(m^{-1}), \tag{2.25}
$$

while all terms in $\bar{H}(x-\eta_r)$ beyond $\Phi(x-\eta_r)$ are $o(m^{-1/2})$.

PROOF. See the Appendix. □

Making a final comparison to the expansion from (2.10) for the fixed sample case, we conclude that the present result is obtained from the former by

i) using the (to order $m^{1/2}$) expected sample size r,

ii) replacing $\tilde{b}_{0,r}$ by $(\tilde{b}_{0,m}+\tilde{b}_{0,[r]-m})$ to account for the splitting of the statistic,

iii) adding the three terms involving $\bar{U}$ from (2.23) to account for the sample size being stochastic rather than fixed.

As concerns the terms involving $\bar{U}$, also observe that the first two of these terms simply result from expanding the leading term $E\Phi(x-\eta_N)$. The complicated last one, however, reflects the interaction between the two stages of the procedure, and is less easy to predict. In fact, it is precisely the derivation of this term which requires the delicate conditional analysis given before.

To conclude this section we briefly return to (2.21) and show that N^{-1} indeed can be replaced by a suitable fixed value. Let r again satisfy $r = EN + o(m^{1/2})$ and define

$$
\bar{\xi}_\alpha = u_\alpha - r^{-1}b_3H_3(u_\alpha). \tag{2.26}
$$

From (2.22) it follows that $P(|N-r| > m^{1-\delta}) = O(m^{-2(1-\delta)\beta}m^{\beta}) = O(m^{-(1-2\delta)\beta})$. Hence, by choosing $0 < \delta < (\beta-1)/(2\beta)$, we obtain that $|\bar{\xi}_\alpha - \tilde{\xi}_\alpha| = O(|N^{-1}-r^{-1}|) = O(m^{-2}|N-r|)$ is uniformly $o(m^{-1})$, except on a set of probability $o(m^{-1})$.

3. Tests with guaranteed power

The expansion in Theorem 2.1 for the df of T^* from (2.4) enables us to derive how $N = N(Z_{(m)})$ should be chosen to ensure to $o(m^{-1})$ a prescribed power against a given alternative. An outline of the program involved is the following. First determine an r such that the power requirement is met to first order, using the leading term $1 - \Phi(u_\alpha - \eta_r)$ in the power expansion. The r thus obtained involves $\int J\Psi_1$, which, just as Ψ_1 itself, is unknown. Replacement of this quantity by a suitable estimator provides the first candidate N_1 for N. (Note that the case where $\int J\Psi_1$ is not completely arbitrary but restricted to some parametric family, will typically lead to the natural situation mentioned in Section 2, where $N_1 - EN_1 = O_p(m^{1/2})$.) A correction term $\hat{f}_r$ is then added to this first choice, selected through (2.24) in such a way that it precisely cancels the lower order terms. Hence choosing $N_1 + \hat{f}_r$ will produce the required power to $o(m^{-1})$. (Note that $\hat{f}_r$ also involves through the b_k from (2.9) the estimation of integrals involving Ψ_i, $i = 1, 2, 3$.) The final touch then consists of setting $N = \max(m, [N_1 + \hat{f}_r + 1/2])$ (cf. (1.1)).

The execution of this program is straightforward, as no new technical obstacles need to be tackled. Hence it does not seem sufficiently interesting to do so, especially as writing it down in some detail would require quite a bit of space. Consequently, we shall restrict ourselves to the outline above as far as the general case is concerned and now specialize right away to the situation corresponding to Stein's procedure, which is our motivating example. This means that we assume the underlying df F to be a member of a scale family $\{\tilde{F}(\cdot/\sigma), \sigma > 0\}$, for some standard df $\tilde{F}$ with $\int_{-\infty}^{\infty} x^2 d\tilde{F}(x) = 1$. It follows that $\Psi_i = \sigma^{-i}\tilde{\Psi}_i$, $i = 1, 2, 3$ and thus $\int J\Psi_1 = \sigma^{-1}\int J\tilde{\Psi}_1$. Now Theorem 2.1 implies that the power of the level α-test based on T^* satisfies $\pi^*(\theta) = 1 - \Phi(u_\alpha - \eta_r) + o(m^{-1/2})$ for all r such that $r = EN + o(m^{1/2})$. Hence to obtain $\pi^*(\kappa m^{-1/2}) = \pi_1$ for given κ and π_1, we need that to $o(m^{1/2})$ the following holds:

$$r = \frac{m(u_\alpha - u_\pi)^2 \int J^2}{\left(\kappa \int J\tilde{\Psi}_1\right)^2}\sigma^2, \tag{3.1}$$

where $u_\pi = \Phi^{-1}(1 - \pi_1)$. To ensure that there exists an $\epsilon > 0$ such that $r \geq (1 + 2\epsilon)m$, it suffices in view of (3.1) if κ in $\theta_1 = \kappa m^{-1/2}$ is chosen such that

$$\kappa < (u_\alpha - u_\pi)\left(\int J^2\right)^{1/2}\sigma \Big/ \left(-\int J\tilde{\Psi}_1\right), \tag{3.2}$$

which upper bound reflects the fact that the alternative should be sufficiently close to the hypothesis to indeed require a second sample of size proportional to m.

As $r - EN = o(m^{1/2})$, we can without loss of generality, use in the sequel the convenient choice in (3.1) for r. Of course, many estimators of σ^2 in (3.1) are possible, but again we specialize immediately to a single case for briefness' sake. To stay as close as possible to Stein's procedure, we essentially select the sample variance S_m^2 (cf. (1.1)). However, as N should depend on $X_1, \ldots, X_m$ through $Z_{(m)} = (Z_{11}, \ldots, Z_{1m})$ only, we shall use the modified version

$$\bar{S}_m^2 = \frac{1}{m}\sum_{j=1}^{m} Z_{1j}^2 = \frac{1}{m}\sum_{i=1}^{m} X_i^2. \tag{3.3}$$

This immediately gives as our initial estimator

$$N_1 = r\frac{\bar{S}_m^2}{\sigma^2} = \frac{m(u_\alpha - u_\pi)^2 \int J^2}{(\kappa \int J\tilde{\Psi}_1)^2}\bar{S}_m^2. \tag{3.4}$$

To obtain the correction term $\hat{f}_r$ to N_1 we need some further notation. Let κ_4 be the fourth cumulant,

$$\kappa_4 = \int_{-\infty}^{\infty} x^4 dF(x) \Big/ \left(\int_{-\infty}^{\infty} x^2 dF(x)\right)^2 - 3. \tag{3.5}$$

As κ_4 is scale invariant, it does not have to be estimated, but can be evaluated using $\tilde{F}$ in (3.5). Moreover, define

$$L(t) = (\tilde{F}^{-1}((1+t)/2))^2, \qquad M(t) = \int_0^t J(u)d\tilde{\Psi}_1(u). \tag{3.6}$$

Then define through (3.2)–(3.6)

$$\begin{aligned} f_r = {} & (\tilde{b}_{0,m} + \tilde{b}_{0,[r]-m}) + 2\sum_{k=0}^{2}(u_\alpha - u_\pi)^{(2-k)} b_k H_k(u_\pi) \\ & - 2(u_\alpha^2 + u_\alpha u_\pi + u_\pi^2 - 3)b_3 - (u_\alpha - u_\pi)^2 \int J^2 \Big/ \left(\int J\tilde{\Psi}_1\right)^2 \\ & + \frac{1}{4} r m^{-1}(2+\kappa_4)(1 - u_\alpha u_\pi + u_\pi^2) \\ & + \left(\int LM - \int L \int M\right) \Big/ \int J\tilde{\Psi}_1. \end{aligned} \tag{3.7}$$

Let $\hat{f}_r$ be obtained from f_r by substituting $[N_1]$ for $[r]$ and N_1 for r, respectively. Note that no further replacements are necessary to compute $\hat{f}_r$, as $\tilde{b}_{0,n}$ and b_k, $k = 1,2,3$, from (2.9) are scale invariant which allow replacement of Ψ_i, $i = 1,2,3$ and Ψ_1' by $\tilde{\Psi}_i$ and $\tilde{\Psi}_1'$.

Now we can formulate the main result of this section

THEOREM 3.1. *Let $J \in \mathcal{J}$ and $F \in \{\tilde{F}(\cdot/\sigma), \sigma > 0\}$, where $\tilde{F} \in \mathcal{F}$, $\int_{-\infty}^{\infty} x^2 d\tilde{F}(x) = 1$ and $\int_{-\infty}^{\infty} x^6 d\tilde{F}(x) < \infty$. If κ satisfies (3.2), then the level α-test based on T^* from (2.4) has power $\pi_1 + o(m^{-1})$ against $\theta_1 = \kappa m^{-1/2}$ if we select*

$$N = \max\left(m, \left[\frac{m(u_\alpha - u_\pi)^2 \int J^2}{\left(\kappa \int J\tilde{\Psi}_1\right)^2}\bar{S}_m^2 + \hat{f}_r + \frac{1}{2}\right]\right). \tag{3.8}$$

PROOF. See the Appendix. □

Note that, not surprisingly, the correction term $\hat{f}_r$ is strongly related to a deficiency in terms of Hodges and Lehmann (1970). Moreover, in analogy to the remark following Theorem 2.1, we observe that in (3.7) the terms involving $\tilde{b}_{0,\cdot}$ and b_k, $k = 1, 2, 3$ are the type of corrections required already for the ordinary rank test to improve the precision in the power determination from $o(1)$ to $o(m^{-1})$. The splitting of the statistics implies the penalty of $\tilde{b}_{0,m} + \tilde{b}_{0,N-m} - \tilde{b}_{0,N}$ additional observations, which number typically is one at most (cf. Albers (1992)). The remaining three terms again reflect the effect of estimating the sample size. The first represents the bias and the second the variance contribution of $\bar{S}_m^2$, which is made transparent by noting that they can be omitted from $\hat{f}_r$ if we replace $\bar{S}_m^2$ in (3.8) by

$$(3.9) \qquad (\bar{S}_m^2 - \kappa^2 m^{-1})\left\{1 + \frac{1}{4}m^{-1}(2+\kappa_4)(1 - u_\alpha u_\pi + u_\pi^2)\right\}.$$

The third and last of these terms again is the interaction term (cf. (3.6)).

It is also interesting to observe that the interpretation above of the various components of $\hat{f}_r$ opens the way to an heuristic assessment of the direction the correction will point to. To be precise, we'll compare N from (3.8) to the simple first order choice

$$\bar{N} = \max\left(m, \left[m(u_\alpha - u_\pi)^2 \int J^2 \Big/ \left(\kappa \int J\tilde{\Psi}_1\right)^2 (\bar{S}_m^2 - \kappa^2/m)\right]\right),$$

which only takes the obvious bias correction term κ^2/m into account. To begin with, we note that the deficiency-related terms involving $\tilde{b}_0$ and b_k, $k = 1, 2, 3$, will typically result in a positive contribution to $\hat{f}_r$, as these terms express the loss (be it of second order) we incur by using a rank test instead of a parametric test. As is immediate from the above, the splitting causes yet another positive contribution to $\hat{f}_r$.

It remains to consider the effect of estimating the sample size. To this end, we first observe that it is readily verified that for x near 1 the function

$$q(x) = 1 - \Phi(u_\pi - \eta_r(x^{1/2} - 1))$$

is concave, as $u_\pi < 0 < \eta_r$. Hence if the rv X is close to 1 with large probability and $EX = 1$, we have $Eq(X) \le q(EX) = q(1) = 1 - \Phi(u_\pi) = \pi_1$. For $X = (\bar{S}_m^2 - \kappa^2/m)/\sigma^2$, however, $Eq(X)$ is nothing but $E(1 - \Phi(u_\alpha - \eta_{\bar{N}}))$, which shows that also in this respect a positive correction of the first order choice $\bar{N}$ is called for. (If one feels this reason to be enlightening but rather sloppy, then note that (3.9) provides a proof: as $-u_\pi$, u_α and $(2 + \kappa_4)$ are non-negative, the same will hold for the correction $(2 + \kappa_4)(1 - u_\alpha u_\pi + u_\pi^2)$.)

The final term to deal with is the interaction term. It originates as $-2U$ in (2.14), with U as in (2.13). After convolution it becomes $-2\tilde{U}$ in (2.19), with $\tilde{U}$

as in (2.18) and thus it contributes $2E\tilde{U}$ to the power. As the numerator of $\tilde{U}$ has mean zero, while its denominator can be replaced by a multiple of $N^{1/2} = r^{1/2}(1+\bar{U})$, we next observe that this leads to a power term which is proportional to $-E\bar{U}\sum a_{1j}(P_{1j}-\pi_{1j})$. It is intuitively clear that $\bar{S}_m^2$ and $\sum a_{1j}P_{1j}$, and thus $\bar{U} = (N/r)^{1/2}-1$ and $\sum a_{1j}P_{1j}$ are typically positively correlated (see the Appendix for the actual computations again) and hence this power contribution will be negative too, thus requiring a positive term in $\hat{f}_r$. Summarizing, we observe that $\hat{f}_r$ will characteristically be positive and that $\bar{N}$ will lead to a power which systematically falls short of the prescribed π_1.

For the special case of the locally most powerful rank test against location alternatives of type F, considerable simplification of the results is possible. Let $J = -\Psi_1$ (or without loss of generality, $J = -\tilde{\Psi}_1$, if desired) and introduce

$$\zeta_1 = \int \tilde{\Psi}_1^4 \Big/ \left(\int \tilde{\Psi}_1^2\right)^2, \quad \zeta_2 = \int \tilde{\Psi}_2^2 \Big/ \left(\int \tilde{\Psi}_1^2\right)^2. \tag{3.10}$$

Then let

$$\begin{aligned}
\hat{f}_r^* = & \left\{ \sum_{j=1}^{m} \sigma^2(\tilde{\Psi}_1^2(U_{j:m})) + \sum_{j=1}^{[N_1]-m} \sigma^2(\tilde{\Psi}_1^2(U_{j:[N_1]-m})) \right\} \Big/ \int \tilde{\Psi}_1^2 \\
& + \frac{1}{36}\zeta_1(-2u_\alpha^2 + 13u_\alpha u_\pi - 5u_\pi^2 - 6) + \frac{1}{3}\zeta_2(u_\alpha - u_\pi)^2 \\
& + \frac{1}{4}(u_\alpha - u_\pi)u_\pi - (u_\alpha - u_\pi)^2 \Big/ \int \tilde{\Psi}_1^2 \\
& + \frac{1}{4}N_1 m^{-1}(2+\kappa_4)(1 - u_\alpha u_\pi + u_\pi^2) \\
& + \frac{1}{2}\left(\int L\tilde{\Psi}_1^2 \Big/ \int \tilde{\Psi}_1^2 - 1\right),
\end{aligned} \tag{3.11}$$

and obtain

COROLLARY 3.1. *If in Theorem* 3.1 *we let* $J = -\Psi_1$, *we can in* (3.8) *replace* $\int J^2/(\int J\tilde{\Psi}_1)^2$ *by* $1/(\int \tilde{\Psi}_1^2)$ *and* $\hat{f}_r$ *by* $\hat{f}_r^*$ *from* (3.11).

PROOF. See the Appendix. □

Finally, we remark that for $J = -\Psi_1$ the results not only hold for exact scores, but also for approximate scores $a_j = J(j/(n+1))$ (cf. ABZ, Theorem 4.2).

4. Examples and a numerical illustration

Next we present some explicit examples. First consider the normal case $\tilde{F} = \Phi$. If we apply the corollary above, T^* from (2.4) will be a combination of two normal scores statistics. Here $\int \tilde{\Psi}_1^2 = 1$, $\kappa_4 = 0$, $\zeta_1 = 3$, $\zeta_2 = 2$ and $(\int L\tilde{\Psi}_1^2/\int \tilde{\Psi}_1^2 - 1)/2 = 1$, while

$$\sum_{j=1}^{n} \sigma^2(\tilde{\Psi}_1^2(U_{j:n})) = \frac{1}{2}\log\log n + \frac{1}{2}\gamma + o(1),$$

where γ is Euler's constant $\lim_{k\to\infty}(\sum_{i=1}^{k} i^{-1} - \log k) = 0.577216\ldots$ (cf. ABZ and Bickel and van Zwet (1978), p. 974). Insertion of these results into (3.8) and (3.11) leads to

$$(4.1)\qquad N = \max\left(m, \left[\frac{m(u_\alpha - u_\pi)^2}{\kappa^2}\left(\bar{S}_m^2 - \frac{\kappa^2}{m}\right)\left(1 + \frac{1 - u_\alpha u_\pi + u_\pi^2}{2m}\right)\right.\right.$$
$$\left.\left. + \frac{1}{2}\log\log m + \frac{1}{2}\log\log([N_1] - m) + \gamma + \frac{1}{2}u_\alpha^2 + 1\right]\right).$$

In this example, the result is particularly easy to explain. In the first place, $\bar{S}_m^2$ is modified according to (3.9) to correct for bias and variance effects. What remains is essentially the deficiency of the normal scores test with respect to the test based on the sample mean, which exactly produces π_1 if $n = m(u_\alpha - u_\pi)^2\sigma^2/\kappa^2$ (cf. ABZ, (6.8)). It only remains to correct this deficiency by an amount $(1/2)\log\log m + (1/2)\log\log([N_1] - m) - (1/2)\log\log[N_1] + \gamma/2$ to account for the splitting (cf. Albers (1992), (3.20)) and by 1 to account for the interaction term $(\int L\tilde{\Psi}_1^2/\int \tilde{\Psi}_1^2 - 1)/2$.

Next let once more $\tilde{F} = \Phi$, but now use Wilcoxon scores $J(t) = t$, rather than the optimal normal scores. Then we have to resort to Theorem 3.1 itself, instead of the more simple corollary. To apply (3.7), we evaluate that $\tilde{b}_{0,m}$ and $\tilde{b}_{0,[r]-m}$ equal $7/2 - 2\sqrt{2}$ to first order, while $b_0 = \pi/9$, $b_1 = 2 - 2/\sqrt{3}$, $b_2 = (12\arctan\sqrt{2}\,)/\pi - 3$, $b_3 = 3/20$ and $\{\int LM - \int L \int M\}/\int J\tilde{\Psi}_1 = 1$. Together with (3.8) and (3.9) this leads to

$$(4.2)\qquad N = \max\left(m, \left[\frac{m(u_\alpha - u_\pi)^2}{\kappa^2}\frac{\pi}{3}\left(\bar{S}_m^2 - \frac{\kappa^2}{m}\right)\left(1 + \frac{1 - u_\alpha u_\pi + u_\pi^2}{2m}\right)\right.\right.$$
$$+ u_\alpha^2\left(\frac{2\pi}{9} - \frac{3}{10}\right) + u_\alpha u_\pi\left(-\frac{4\pi}{9} - \frac{4}{3}\sqrt{3} + \frac{37}{10}\right)$$
$$+ u_\pi^2\left(\frac{2\pi}{9} + \frac{4}{3}\sqrt{3} - \frac{103}{10} + \frac{24\arctan\sqrt{2}}{\pi}\right)$$
$$\left.\left. + \left(\frac{154}{10} - 4\sqrt{2} - \frac{24\arctan\sqrt{2}}{\pi}\right)\right]\right).$$

Compared to the corresponding term of (4.1), the leading term of (4.2) contains the additional factor $\pi/3$, which reflects the well-known fact that the ARE of Wilcoxon's test with respect to the normal scores test under normal alternatives equals $3/\pi$. However, it is nice to observe that the second order terms are also remarkably close to their counterparts from (4.1). In fact, the coefficients of u_α^2, $u_\alpha u_\pi$ and u_π^2 are (to three decimal places) 0.578, -0.006 and 0.006, respectively, whereas in (4.1) the corresponding values are 1/2, 0 and 0. The constant term equals 2.445, of which $7/2 - 2\sqrt{2} = 0.672$ is due to the splitting and 1 to the interaction term. This value as well is close to the corresponding expression $(1/2)\log\log m + (1/2)\log\log([N_1] - m) + \gamma + 1$ from (4.1).

As our second choice for the underlying distribution we consider the logistic case $\tilde{F}(x) = (1 + \exp(\pi 3^{-1/2}x))^{-1}$. First we again deal with the locally most powerful case, where T^* becomes a combination of two Wilcoxon statistics. Here $\int \tilde{\Psi}_1^2 = (\pi/3)^2$, $\kappa_4 = 6/5$, $\zeta_1 = \zeta_2 = 9/5$, $(\int \tilde{\Psi}_1^2 / \int L\tilde{\Psi}_1^2 - 1)/2 = 6/\pi^2$ and $\sum_{j=1}^n \sigma^2(\tilde{\Psi}_1(U_{j:n}))/\int \tilde{\Psi}_1^2 = 1/2 + o(1)$, which leads through (3.8) and (3.11) to

$$N = \max\left(m, \left[\frac{m(u_\alpha - u_\pi)^2}{\kappa^2}\left(\frac{3}{\pi}\right)^2\left(\bar{S}_m^2 - \frac{\kappa^2}{m}\right)\times\left(1 + \frac{4(1 - u_\alpha u_\pi + u_\pi^2)}{5m}\right) + \frac{1}{10}(5u_\alpha^2 - 3u_\alpha u_\pi + u_\pi^2 + 2) + \frac{6}{\pi^2} + 1\right]\right). \tag{4.3}$$

Of the amount in (4.3), we can ascribe 1/2 to the splitting and $6/(\pi^2)$ to the interaction term.

Our fourth and final example is the counterpart of the second one: the distribution remains logistic, but we return to normal scores. Then $\tilde{b}_{0,m}$ to first order equals $4 - 2\sqrt{2} - (1/2)\log\log m - \gamma/2$, while $b_0 = 6\arctan\sqrt{2} - 5\pi/3$, $b_1 = \pi/6 + 1/2$, $b_2 = 5/6$, $b_3 = 1/4$ and $\{\int LM - \int L \int M\}/\int J\tilde{\Psi}_1 = 0.559$ (which value has been obtained numerically). Together with (3.8) and (3.9) we arrive at

$$\begin{aligned} N = \max\Bigg(m, \Bigg[&\frac{m(u_\alpha - u_\pi)^2}{\kappa^2}\frac{3}{\pi}\left(\bar{S}_m^2 - \frac{\kappa^2}{m}\right)\left(1 + \frac{4(1 - u_\alpha u_\pi + u_\pi^2)}{5m}\right) \\ &+ u_\alpha^2\left(12\arctan\sqrt{2} - \frac{10_\pi}{3} - \frac{1}{2}\right) \\ &+ u_\alpha u_\pi\left(-24\arctan\sqrt{2} + 7\pi + \frac{1}{2}\right) \\ &+ u_\pi^2\left(12\arctan\sqrt{2} - \frac{11\pi}{3} + \frac{1}{6}\right) \\ &+ \Bigg(\frac{25}{3} - 4\sqrt{2} + 0.559 - \gamma \\ &- \frac{1}{2}\log\log m - \frac{1}{2}\log\log([N_1] - m)\Bigg)\Bigg]\Bigg). \end{aligned} \tag{4.4}$$

The factor $3/\pi$ in the leading term of (4.4), as compared to $(3/\pi)^2$ in (4.3) reflects the fact that the ARE of the normal scores test w.r.t Wilcoxon's test equals $3/\pi$ under logistic alternatives. The coefficients of u_α^2, $u_\alpha u_\pi$ and u_π^2 are 0.492, -0.436 and 0.111 respectively, which is again close to the corresponding values 5/10, $-3/10$ and 1/10 in (4.3). A similar observation holds for the constant terms in (4.4) and (4.3).

After these four examples, let us now return to Stein's procedure and compare it to the test in example 1. To avoid confusion, denote N from (1.1) by N_{ST} and N from (4.1) by N_{NS}. From Lehmann ((1986), p. 260), we obtain that c^{-1} in (1.1) should be chosen equal to $m(t_{m-1,\alpha} - t_{m-1,\pi})^2/\kappa^2$, where $t_{m-1,\alpha}$ is the upper α-point of the t-distribution with $(m-1)$ degrees of freedom. It is easy to verify (cf. e.g. Hodges and Lehmann (1970), p. 792) that

$$t_{m-1,\alpha} = u_\alpha + \frac{u_\alpha^3 + u_\alpha}{4m} + O(m^{-3/2}). \tag{4.5}$$

A straightforward computation shows that N_{ST} satisfies to $o(1)$

$$(4.6)\quad N_{ST} = \max\left(m, \left[\frac{m(u_\alpha - u_\pi)^2}{\kappa^2} S_m^2 \left(1 + \frac{(1 + u_\alpha^2 + u_\alpha u_\pi + u_\pi^2)}{2m}\right) + 1\right]\right).$$

Before comparing N_{ST} to N_{NS}, observe that N_{ST} is the smallest integer for which the power at θ_1 exceeds π_1. If, as before, our objective is to achieve π_1 to $o(m^{-1})$, it is easily verified that the final 1 in (4.6) can be replaced by 1/2 (cf. the proof of Theorem 3.1). It seems more fair to use this modified version $\tilde{N}_{ST}$ of N_{ST} in the comparison. We then obtain

$$(4.7)\quad E(\tilde{N}_{NS} - N_{ST}) = \frac{1}{2}\log\log m + \frac{1}{2}\log\log([N_1] - m) + \gamma + \frac{1}{2}u_\alpha^2 + \frac{1}{2} - \frac{(u_\alpha - u_\pi)^2}{\kappa^2}\sigma^2\left(\frac{1}{2}u_\alpha^2 + u_\alpha u_\pi\right).$$

Typically, the amount in (4.7) will be quite small. To be a little more specific, $\log\log n + \gamma$ attains the values 1.411, 1.674, 1.941 and 2.104 for $n = 10$, 20, 50 and 100, respectively. Moreover, in view of (3.2) the factor $(u_\alpha - u_\pi)^2\sigma^2/\kappa^2$ exceeds 1, while $(u_\alpha^2/2 + u_\alpha u_\pi)$ will usually be positive, unless high values of α are combined with high values of π_1 (for $\alpha = 0.05$ we still have $u_\alpha^2/2 + u_\alpha u_\pi \geq 0$ as long as $\pi_1 \leq 0.791$).

To conclude this section, we shall by way of illustration present a small simulation study. Consider once more the situation of example 3, i.e. a combination of two Wilcoxon statistics under logistic alternatives. We shall consider initial sample sizes $m = 10$, 15 and 20 and levels of significance $\alpha = .05$, .025 and .01. The desired powers π_1 are chosen in the region of practical interest, say (.5, .9). Next, the alternatives $\theta = \kappa m^{-1/2}$ are selected such that $\lambda = r/m$ (cf. (3.1)) satisfies $\lambda \in (1.5, 3.5)$. Now for each simulation step, we first draw a sample from $F(x) = 1/(1 + e^{-x})$, shift it over θ and compute T_1 and $\bar{S}_m^2$ from (3.3), and subsequently N from (4.3). Then the additional sample is drawn and T_2 is obtained, after which H_0 is rejected if $T^* = T_1 + T_2$ exceeds the approximate critical value $N/4 + ((N-1)/3)^{1/2}\{u_\alpha - 3(u_\alpha^3 - 3u_\alpha)/(20N)\}$ (cf. (2.21)). For each configuration $(m, \alpha, \pi_1, \theta)$ we use 10^4 simulations, thus obtaining power estimators $\hat{\pi}$ with standard deviation at most 1/2%. The results are collected in Table 1.

Inspection of Table 1 reveals that the agreement between the prescribed π_1 and the estimated $\hat{\pi}$ is quite satisfactory: on the average, $\hat{\pi}$ falls short of π_1 by about 1%, which is negligible for most practical purposes. Moreover, simulations using $\theta = 0$ show that the approximate critical values used are reasonably accurate, but typically slightly conservative. Upon correction for this effect, the agreement between π_1 and $\hat{\pi}$ is still better. By way of contrast, let us neglect the refinements offered by our second order analysis and simply use $\tilde{N} = \max(m, [m(u_\alpha - u_\pi)^2(3/\pi)^2(\bar{S}_m^2 - \kappa^2/m)/\kappa^2])$ (cf. (4.3)). The final column of Table 1 gives the results for the corresponding estimator $\hat{\hat{\pi}}$, computed on the same run as on which $\hat{\pi}$ is based. This first order approach is clearly inadequate: $\hat{\hat{\pi}}$ typically is about 10% below the prescribed value π_1.

Table 1. The realized power $\hat{\pi}$ ($\hat{\hat{\pi}}$) using second (first) order methods for the Wilcoxon two-stage procedure under $F(x) = 1/(1+e^{-x})$. For each initial sample size m, level α, shift θ and prescribed power π_1, the number of simulations used is 10^4. (As concerns $\lambda = r/m$, cf. (3.1).)

α	θ	λ	π_1	$\hat{\pi}$	$\hat{\hat{\pi}}$
		(i) $m = 10$			
.05	.75	1.93	.600	.603	.531
.05	1.00	1.62	.750	.742	.658
.05	1.25	1.65	.900	.866	.785
.025	.75	3.30	.700	.686	.581
.025	1.00	1.85	.700	.691	.583
.025	1.00	2.36	.800	.767	.653
.01	.75	2.89	.500	.492	.412
.01	1.00	2.00	.600	.595	.478
.01	1.00	3.01	.800	.770	.618
		(ii) $m = 15$			
.05	.50	3.77	.700	.684	.635
.05	.75	2.56	.850	.833	.752
.025	.75	2.47	.750	.731	.640
.025	1.00	1.57	.800	.789	.704
.01	.75	2.37	.600	.594	.509
.01	.75	2.14	.550	.540	.465
		(iii) $m = 20$			
.05	.50	2.48	.650	.642	.590
.05	.75	2.29	.900	.886	.813
.025	.75	1.85	.750	.747	.662
.025	.75	2.10	.800	.781	.705
.01	.75	2.17	.700	.690	.597
.01	.75	1.78	.600	.600	.522

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Appendix

PROOF OF LEMMA 2.2. To begin with we note that the conditions of this lemma are those of Theorem 4.1 of ABZ, and as such imply those of the previous theorems and lemmas of that paper except possibly on sets of probability of order $n^{-5/4}$. Let

$$K(x) = \Phi(x) + \phi(x)\left\{\frac{1}{2}\left(\sum a_j^2(2P_j-1)^2 \Big/ \sum a_j^2\right) H_1(x) + \frac{1}{3}\left(\sum a_j^3(2P_j-1) \Big/ \left(\sum a_j^2\right)^{3/2}\right) H_2(x) + \frac{1}{12}\left(\sum a_j^4 \Big/ \left(\sum a_j^2\right)^2\right) H_3(x)\right\}. \tag{A.1}$$

(Here and in the sequel of this proof we assume that summation is from 1 to n, unless stated otherwise.) Taking steps similar to those leading in ABZ from Theorem 2.1 to Theorem 2.3, we obtain that (2.15) holds if $\tilde{K}(x-\eta)$ is replaced by $K(x-(\sum a_j(2P_j-1))/(\sum a_j^2)^{1/2})$. In fact, at this point a remainder of the form $A\{n^{-5/4}+\sum|2P_j-1|^5\}$ suffices.

Next the argument of K is changed into $\tilde{x}=x-(\sum a_j(2\bar{\pi}_j-1))/(\sum a_j^2)^{1/2}$, which involves an expansion in powers of U from (2.13). Thus the U- and U^2-terms in (2.14) and the $|U|^3$-term in (2.15). Note by way of check that if we take the expectation with respect to $Z_{(n)}$ at this point, we get back Theorem 2.3 of ABZ: $EU=0$, $EU^2=\operatorname{var}(\sum a_jP_j)/\sum a_j^2$, and $E|U|^3$ leads to the complicated term involving $E|P_j-\bar{\pi}_j|^3$. However, for the application of the present paper, we need to keep track of the dependence on $Z_{(n)}$ till the end. Consequently, we divide our expansion into a deterministic and a stochastic part in a very simple manner. Replacing each term in the expansion by its expectation produces the first part, whereas the collection of terms needed to correct this change constitutes the second part. It is straightforward from (A.1) and the expansion in powers of U that the stochastic part precisely equals $\tilde{K}(\tilde{x})-\tilde{G}(\tilde{x})$ in (2.14), which agrees to the desired order with $\tilde{K}(x-\eta)-\tilde{G}(x-\eta)$. Hence it remains to show that the deterministic part agrees to order n^{-1} with $\tilde{G}(x-\eta)$ from (2.10). But that is rather trivial: as we observed before, taking the expectation not only produces the deterministic part, but also the expansion from Theorem 2.3 in ABZ. The development from Theorem 2.3 to Theorem 4.1 of ABZ is devoted to demonstrating that the expansion from the first theorem can be simplified to that of the latter, which is nothing but our $\tilde{G}(x-\eta)$. □

PROOF OF LEMMA 2.3. First we collect some results from Albers (1992). Let W_ν, $\nu=1,2$, be independent statistics with continuous df's, admitting expansions of the form

$$\sup_x |P((W_\nu-\zeta_\nu)/\beta_\nu\le x)-\tilde{G}_\nu(x-\eta_\nu)|\le\delta_\nu, \tag{A.2}$$

where

$$\tilde{G}_\nu(x)=\Phi(x)+\phi(x)\sum_{k=0}^{P}b_{k\nu}H_k(x), \tag{A.3}$$

in which all $|b_{k\nu}|\le 1$. Then in Lemma 2.1 of Albers (1992) the following special case is contained: for certain $C_1,C_2>0$,

$$\begin{aligned}\sup_x\left|P\left(\sum_{\nu=1}^{2}(W_\nu-\zeta_\nu)\Big/\left(\sum_{\nu=1}^{2}\beta_\nu^2\right)^{1/2}\le x\right)-\tilde{G}^*\left(x-\sum_{\nu=1}^{2}\gamma_\nu\eta_\nu\right)\right|\\ \le\sum_{\nu=1}^{2}\delta_\nu\left(1+C_1\sum_{k=0}^{P}\sum_{\nu=1}^{2}|b_{k\nu}|\right)+C_2\sum_{k=0}^{P}\sum_{\nu=1}^{2}b_{k\nu}^2,\end{aligned} \tag{A.4}$$

where $\gamma_\nu = \beta_\nu/(\beta_1^2+\beta_2^2)^{1/2}$, $\nu = 1, 2$ and

$$\tilde{G}^*(x) = \Phi(x) + \phi(x) \sum_{k=0}^{P} \sum_{\nu=1}^{2} b_{k\nu} \gamma_\nu^{k+1} H_k(x). \tag{A.5}$$

If we specialize to the case where W_ν is the rank statistic from (2.3), the formal ζ_ν, β_ν, η_ν, δ_ν, $b_{k\nu}$ and p in (A.2) and (A.3) can be made specific through (2.9)–(2.11), using n_ν rather than n. Moreover, it follows from Theorem 3.2 of Albers (1992) that in this particular application the expansion $\tilde{G}^*(x - \sum_{\nu=1}^2 \gamma_\nu \eta_\nu)$ from (A.4) and (A.5) boils down to

$$\tilde{G}(x-\eta_n) + \phi(x-\eta_n) \left\{ \sum_{\nu=1}^{2} \tilde{b}_{0,n_\nu} - \tilde{b}_{0,n} \right\} \eta_n/(2n) \tag{A.6}$$

with $n = \sum_{\nu=1}^2 n_\nu$ and $\tilde{G}$, η_n and $\tilde{b}_{0,n}$ as in (2.9). In words, if we compare the df of the standardized version of the sum $W_1 + W_2$ to that of the standardized version of T from (2.3) with corresponding sample size $n = \sum_{\nu=1}^2 n_\nu$, surprisingly the only difference to $o(n^{-1})$ is the second term in (A.6).

Now we are in a position to prove the result of the present lemma. Conditional on $Z_{(m)}$, the statistics T_1 and T_2 are independent, while expansions of the form (A.3) for their df's are given by (2.14) and (2.10), respectively. Hence, through (A.4) this leads to a conditional expansion for the df of T^* from (2.4). Moreover, if the expansions in (A.3) are based on $\tilde{G}$ from (2.10), a greatly simplified expansion results (cf. (A.6)). But here T_2 indeed leads to $\tilde{G}$ (cf. (2.10)), while T_1 leads to $\tilde{K}$, which equals $\tilde{G}$ plus a stochastic part (cf. (2.14)). Consequently, the deterministic part of the conditional expansion for the df of T^* is given by (A.6), for $n_1 = m$, $n_2 = N(Z_{(m)}) - m$, and hence $n = N(Z_{(m)})$. The additional contribution due to the stochastic component is easily determined from (2.14) and (A.5). Finally, from (A.4) it is clear that the new remainder will be as in (2.15), with the appropriate change in sample size. □

PROOF OF THEOREM 2.1. We have to show that taking the expectation of $\tilde{H}(x-\eta_N)$ in (2.20) leads to $\bar{H}(x-\eta_r)$ in (2.25). First consider the leading term $E\Phi(x-\eta_N)$. As $\eta_N = \eta_r(1+\bar{U})$, with $\bar{U}$ as in (2.23), the corresponding expansion clearly produces $\Phi(x)$ in (2.24), as well as the terms involving $E\bar{U}$ and $E\bar{U}^2$. The contribution $E|\bar{U}|^3$ to the remainder is $o(m^{-1})$, as $\bar{U}$ in view of (2.5) and (2.22) is bounded and satisfies $E|\bar{U}|^{2\beta} = O(m^{-\beta})$ for some $\beta > 1$.

Next replacing η_N by η_r everywhere in $\phi(x-\eta_N)$ and $H_k(x-\eta_N)$, $k = 0, 1, 2, 3$, causes differences which are $O(m^{-1}|\bar{U}|) = O(m^{-3/2} + |\bar{U}|^3)$ and hence regligible. As $\tilde{b}_{0,m} = o(m^{1/2})$, substitution of $[r]$ for N in the corresponding term in (2.19) results in a remainder term $o(m^{-1/2}|\bar{U}|) = o(m^{-1} + \bar{U}^2)$, which again is negligible. A similar argument applies to the terms involving $N^{-1}\eta_N^{(3-k)}$. The remaining terms in (2.19) all have a numerator with expectation 0 and a denominator of the form $(\sum a_{1j}^2 + \sum a_{2j}^2)^{k/2}$, $k = 1, 2, 3$. In the first place, note that these denominators can be replaced by $(N \int J^2)^{k/2}$, since this change leads to a factor $1 + o(m^{-1/2})$, which can be neglected. Let Y denote the numerator of such a term, then $EY = 0$ implies that $EYN^{-k/2} = EY(N^{-k/2} - r^{-k/2}) = O(m^{-k/2}E|Y||\bar{U}|)$, which is easily shown to be $o(m^{-1})$ for those Y that satisfy $m^{-k/2}E|Y| = O(m^{-1})$. This latter relation holds for all terms in (2.19), besides $(-2\tilde{U})$. Hence, as already indicated after Lemma 2.2, the contribution of those terms is negligible.

As $\tilde{U}$ is $O_P(m^{-1/2})$, however, this argument fails and we need to take an additional term into account. In fact, $E(-2\tilde{U}) = 2E(\bar{U}\sum a_{1j}(P_{1j} - \pi_{1j}))/(r\int J^2)^{1/2} + o(m^{-1}) + O(E|U|\bar{U}^2)$ with U as in (2.13), with $n = m$. This last remainder is easily shown to be $o(m^{-1})$. Next observe that $2(P_{1j} - \pi_{1j})$ can be replaced by $-\theta(\psi_1(Z_{1j}) - E\psi_1(Z_{1j}))$, to arrive at the mixed term involving $\bar{U}$ in (2.24), which was the last term to be explained. Moreover, note that it is straightforward from ABZ that the expectation of the remainder on (2.20) is $o(m^{-1})$ as well.

It remains to deal with the set E of probability $o(m^{-1})$, on which the conditions for the conditional expansion and (2.5) are not fulfilled. If $\tilde{H}$ in (2.19) is bounded, or if it at least is sufficiently close on E^c to bounded function, we simply argue that the contribution over E to the expectation of the left-hand side in (2.20) is $O(P(E)) = o(m^{-1})$. To verify this (near-)boundedness of $\tilde{H}$ in (2.19), we note in the first place that $N \geq m$ will always hold. Hence using the obvious definition $\tilde{b}_{0,0} = 0$ ensures that $N^{-1}\eta_N(\tilde{b}_{0,m} + \tilde{b}_{0,N-m})$ is bounded. Next we observe that the $(\sum a_{1j}^2 + \sum a_{2j}^2)$ parts in the denominator cause no trouble, as $\sum a_{1j}^2 + \sum a_{2j}^2 \geq \sum a_{1j}^2 = m\int J^2(1+o(1))$. Of the terms involving $N^{(1-k)/2}\theta^{(3-k)}$, only the one with $N^{1/2}\theta^3$ can become unbounded. However, replacement of $N^{1/2}$ by $r^{1/2}$ removes this obstacle, while leading to a difference $r^{1/2}\theta^3\bar{U}$, which on E^c is negligible. Likewise, $\tilde{U}$ and $\tilde{U}^2$ could cause problems. But modifying $\tilde{H}$ by using in (2.20) as an argument $(x - \eta_N - 2\tilde{U})$, rather than $x - \eta_N$, again removes the difficulty: the difference involved is $|\tilde{U}|^3$, while $\Phi(x - \eta_N - 2\tilde{U})$ and $\phi(x - \eta_N - 2\tilde{U})$ are bounded. □

PROOF OF THEOREM 3.1. According to the Marcinkievitz-Zygmund-Chung inequality (see Chung (1951)), we have for all $p \geq 1$

$$E\left|\sum_{j=1}^{m} W_j\right|^{2p} \leq C_p m^{p-1}\sum_{j=1}^{m} E|W_j|^{2p}, \tag{A.7}$$

where C_p is a constant depending only on p and $W_1, \ldots, W_m$ are independent rv's with $EW_j = 0$, $j = 1, \ldots, m$. From (A.7) it follows in view of (3.3) and the condition $\int_{-\infty}^{\infty} x^6 d\tilde{F}(x) < \infty$ that $E|\bar{S}_m^2 - E\bar{S}_m^2|^3 = O(m^{-3/2})$. As $E\bar{S}_m^2 = \sigma^2 + \theta^2$, with $0 \leq \theta \leq Cm^{-1/2}$, we also have

$$E|(\bar{S}_m^2/\sigma^2) - 1|^3 = O(m^{-3/2}). \tag{A.8}$$

Application of Chebyshev's inequality together with (A.8) then gives that for $0 < \delta < 1/6$

$$P(|(\bar{S}_m^2/\sigma^2) - 1| > m^{-\delta}) = O(m^{-(3/2)(1-2\delta)}) = o(m^{-1}). \tag{A.9}$$

The results (A.8) and (A.9) for $\bar{S}_m^2$ now enable us to show that for r from (3.1) and N_1 from (3.4) the conditions of Theorem 2.1 are fulfilled. First recall that condition (3.2) on κ has been chosen such that $r \geq (1+2\epsilon)m$ for some $\epsilon > 0$. Together with (A.9) this implies that N_1 satisfies (2.5). Moreover, as $EN_1 = r + r\theta^2/\sigma^2$, we obviously have that $r = EN_1 + o(m^{1/2})$. This fact in combination with (A.8) then implies (2.22). Since N in (3.8) is related to N_1 by

$$N = \max(m, [N_1 + \hat{f}_r + 1/2]), \tag{A.10}$$

while f_r from (3.7) is $o(m^{1/2})$, it immediately follows that the conditions hold for r and N as well.

Hence we can now apply Theorem 2.1 and evaluate $\bar{H}(x)$ from (2.24) for r from (3.1) and N from (3.8). To begin with, we note that working with $[N_1 + f_r + 1/2]$ rather than with N (cf. (A.10)) will clearly cause differences of $o(m^{-1})$. The same holds if we replace $[N_1 + f_r + 1/2]$ in its turn by $N_1 + f_r$. To see this, note the density of $\tilde{N}_1 = (N_1 - EN_1)/\sigma(N_1)$ tends to ϕ. Hence $E\{[N_1 + f_r + 1/2] - (N_1 + f_r)\}$ tends in its turn to

$$\int \{[EN_1 + f_r + \sigma(N_1)x + 1/2] - (EN_1 + f_r + \sigma(N_1)x)\}\phi(x)dx. \tag{A.11}$$

Now $\int\{[a + bx + 1/2] - (a + bx)\}dx = 0$ over intervals $(k/b - a, (k+1)/b - a)$ if k is an integer. Hence if we replace ϕ in (A.11) by a step function on a suitable lattice of width $\sigma^{-1}(N_1) = O(m^{-1/2})$, the integral can be made equal to zero. But this replacement clearly can be achieved such that the difference caused in (A.11) is $O(m^{-1/2})$. Hence $E\{[N_1 + f_r + 1/2] - (N_1 + f_r)\} = o(1)$, which is indeed a negligible difference. In terms containing second and third powers of N it is trivial that substitution of $N_1 + f_r$ for $[N_1 + f_r + 1/2]$ is allowed.

It follows that in our calculations $\bar{U}$ from (2.23) can be replaced by $\{r^{-1}(N_1 + f_r)\}^{1/2} - 1 = \bar{S}_m/\sigma - 1 + (1/2)r^{-1}f_r + o(m^{-1})$. As $E(\bar{S}_m^2/\sigma^2 - 1) = \theta^2/\sigma^2 = r^{-1}\eta_r^2 \int J^2/(\int J\tilde{\Psi}_1)^2$ and $E(\bar{S}_m^2/\sigma^2 - 1)^2 = \sigma^{-4}\operatorname{var}(\bar{S}_m^2)(1 + o(1)) = m^{-1}(2 + \kappa_4) + o(m^{-1})$, we obtain that

$$\begin{aligned} E\bar{U} &= \frac{1}{2}r^{-1}\int J^2 \Big/ \left(\int J\tilde{\Psi}_1\right)^2 - \frac{1}{8}m^{-1}(2+\kappa_4) + \frac{1}{2}r^{-1}f_r + o(m^{-1}), \\ E\bar{U}^2 &= \frac{1}{4}m^{-1}(2+\kappa_4) + o(m^{-1}). \end{aligned} \tag{A.12}$$

To deal with the last term in (2.24), we begin by noting that in this term $\bar{U}$ can be replaced by $(\bar{S}_m^2/\sigma^2 - 1)/2 = (1/2)m^{-1}\sum\{(Z_{1j}/\sigma)^2 - E(Z_{1j}/\sigma)^2\}$. Using results like $n^{-1}\operatorname{Cov}(\sum Ew_1(U_{j:n})w_3(U_{j:n}), \sum Ew_2(U_{j:n})w_4(U_{j:n})) = \int_0^1\int_0^1 w_1(s)w_2(t) \cdot w_3'(s)w_4'(t)(s \wedge t - st)dsdt + o(1)$ (see Albers (1980), Lemma 5.5) and $\int_0^1\int_0^1 v_1(s) \cdot v_2(t)(s \wedge t - st)dsdt = \int V_1V_2 - \int V_1 V_2$, where $V_i(t) = \int_0^t v_i(u)du$, $i = 1, 2$ (see Albers (1980), p. 139), we arrive through (3.6) at

$$\begin{aligned} &E\left\{\bar{U}\sum a_{1j}(\psi_1(Z_{1j}) - E\psi_1(Z_{1j}))\right\} \Big/ \int J\Psi_1 \\ &= \frac{1}{2}\left\{\int LM - \int L\int M\right\} \Big/ \int J\tilde{\Psi}_1 + o(1). \end{aligned} \tag{A.13}$$

Together (A.12) and (A.13) provide an explicit expression for $\bar{H}$ from (2.24). As $\pi^*(\theta) = 1-\bar{H}(\bar{\xi}_\alpha - \eta_r) + o(m^{-1}) = 1-\bar{H}(u_\alpha - \eta_r) + r^{-1}b_3H_3(u_\alpha) + o(m^{-1})$ (cf. (2.26), while η_r for $\theta = \theta_1$ obviously attains the value $(u_\alpha - u_\pi)$, a straightforward calculation now shows that f_r from (3.7) indeed leads to $\pi^*(\theta_1) = 1 - \Phi(u_\alpha - (u_\alpha - u_\pi)) + o(m^{-1}) = \pi_1 + o(m^{-1})$, as desired. □

Proof of Corollary 3.1. Using ABZ, p. 128, we obtain in (2.9) that $\eta_n = n^{1/2}\theta(\int \Psi_1^2)^{1/2}$ and $b_0 = (\zeta_1 + 3\zeta_2)/18$, $b_1 = (3\zeta_1+1)/8$, $b_2 = \zeta_1/3$, $b_3 = \zeta_1/12$ and $\tilde{b}_{0,n} = \sum_{i=1}^n \sigma^2(\Psi_1(U_{j:n}))/\int \Psi_1^2$ with ζ_i, $i = 1, 2$ as given in (3.10). Moreover, $\int L = \int_{-\infty}^{\infty} x^2 d\tilde{F}(x) = 1$, $\int M / \int J\tilde{\Psi}_1 = (1/2)\int \tilde{\Psi}_1^2 / \int \tilde{\Psi}_1^2 = 1/2$ for $J = -\Psi_1$. Insertion of these results leads from (3.8) to (3.11). □

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