Ann. Inst. Statist. Math.
Vol. 44, No. 1, 185–196 (1992)

A SERIES OF SEARCH DESIGNS FOR 2^m FACTORIAL DESIGNS OF RESOLUTION V WHICH PERMIT SEARCH OF ONE OR TWO UNKNOWN EXTRA THREE-FACTOR INTERACTIONS

TERUHIRO SHIRAKURA[1] AND SHINSEI TAZAWA[2]

[1] *Department of Mathematics, Kobe University, Nada-ku, Kobe 657, Japan*
[2] *Department of Mathematics, Kinki University, Higashi-Osaka 577, Japan*

(Received January 11, 1990; revised July 30, 1990)

Abstract. In the absence of four-factor and higher order interactions, we present a series of search designs for 2^m factorials ($m \geq 6$) which allow the search of at most k $(= 1, 2)$ nonnegligible three-factor interactions, and the estimation of them along with the general mean, main effects and two-factor interactions. These designs are derived from balanced arrays of strength 6. In particular, the nonisomorphic weighted graphs with 4 vertices in which two distinct vertices are assigned with integer weight ω $(1 \leq \omega \leq 3)$, are useful in obtaining search designs for $k = 2$. Furthermore, it is shown that a search design obtained for each $m \geq 6$ is of the minimum number of treatments among balanced arrays of strength 6. By modifying the results for $m \geq 6$, we also present a search design for $m = 5$ and $k = 2$.

Key words and phrases: Search design, minimum treatment, balanced array, strength 6, weighted graph, isomorphic graph.

1. Introduction

Consider factorial experiments with m factors each at the level 0 or 1 (i.e., 2^m factorial designs). These designs are vastly popular and important for practical usages. Although they are unable to explore fully a wide region in the factor space, they can indicate major trends and so determine a promising direction for further experimentations (see Box *et al.* (1978)). The main object of the designs is to estimate certain factorial effects of interest, assuming that the remaining effects are negligible. For example, a design of resolution V gives the estimation of the general mean, the main effects and the two-factor interactions, under the assumption that the three-factor and higher order interactions are zero.

Empirically, it has been justified that in most experiments the effects that are assumed negligible, are actually negligible. However it may also be true that in some experiments there do occur effects which are assumed negligible in advance, but which are not actually negligible. Of course, the number of such effects is very small. In view of the above, we need a *search design* such that the nonnegligible effects occurred in the experiments can be searched, and the inference about them can be given along with the effects of interest.

The concept of search designs was introduced by Srivastava (1975). Research on the constructions of search designs has been done by various authors. In particular, Srivastava and Gupta (1979), Ghosh (1980, 1981), Gupta and Carvajal (1984), Ohnishi and Shirakura (1985) and Gupta (1988) treated the problem of finding search designs for 2^m factorials which allow the search of at most one nonnegligible interaction, and the estimation of it along with the general mean and main effects. Srivastava and Ghosh (1976, 1977) also presented search designs for 2^m factorials such that at most one nonnegligible effect can be searched among the three-factor and higher order interactions, and it can be estimated along with the general mean, main effects and two-factor interactions. We are now interested in the construction of search designs which make it possible to search at most two nonnegligible effects. Very little work on such designs has been done so far because of combinatorial difficulties which the designs possess. In the absence of the three-factor and higher order interactions, Shirakura (1991), and Shirakura and Tazawa (1991) obtained search designs for 2^m factorials yielding the search of at most two nonnegligible two-factor interactions.

In this paper, we assume that the four-factor and higher order interactions are negligible. Then we present a series of search designs for 2^m factorials ($m \geq 6$) which allow the search of at most k ($= 1, 2$) nonnegligible three-factor interactions, and the estimation of them along with the general mean, main effects and two-factor interactions. That is, in setting of a design of resolution V, we here consider the situation where the effects to be searched may lie in lower order (three-factor) interactions. Resolution V designs for which the two-factor interactions are included in the effects of interest, have been widely used in various experimentations. Our search designs are derived from balanced arrays of strength 6. In particular, the consideration of the nonisomorphic weighted graphs with 4 vertices in which two distinct vertices are assigned with integer weight ω ($1 \leq \omega \leq 3$), is useful in obtaining search designs for $k = 2$. Furthermore, it is shown that the search design obtained here for each $m \geq 6$ is of the minimum number of treatments among all search designs derived from balanced arrays of strength 6. By modifying the results for $m \geq 6$, we also present a search design for $m = 5$ and $k = 2$.

2. Search designs

For 2^m factorial experiments, let μ, F_i, $F_{i_1 i_2}$, and $F_{i_1 i_2 i_3}$ be the general mean, the main effect of i-th factor, the two-factor interaction of i_1-th and i_2-th factors, and the three-factor interaction of i_1-th, i_2-th and i_3-th factors, respectively. Consider the vectors $\boldsymbol{\xi}_j(\nu_j \times 1)$, $j = 1, 2$, of effects:

$$\boldsymbol{\xi}_1 = (\mu; F_1, F_2, \ldots, F_m; F_{12}, F_{13}, \ldots, F_{m-1,m})',$$
$$\boldsymbol{\xi}_2 = (F_{123}, F_{124}, \ldots, F_{m-2,m-1,m})',$$

where $\nu_1 = (m^2 + m + 2)/2$ and $\nu_2 = m(m-1)(m-2)/6$. Throughout this paper, note that the four-factor and higher order interactions are assumed to be negligible.

Let T be a design of n treatments (treatment combinations or assemblies). For an observation vector $\boldsymbol{y}(n \times 1)$ of T, the following linear model can then be considered:

$$E(\boldsymbol{y}) = A_1\boldsymbol{\xi}_1 + A_2\boldsymbol{\xi}_2 \quad \text{and} \quad V(\boldsymbol{y}) = \sigma^2 I_n, \tag{2.1}$$

where $A_j(n \times \nu_j)$ are the design matrices for $\boldsymbol{\xi}_j$ $(j = 1, 2)$, σ^2 is a variance of the observations, and I_n is the identity matrix of order n. We assume that at most k $(= 1, 2)$, the effects of $\boldsymbol{\xi}_2$ are nonzero, but it is not known which effects these are. We want to find a search design T for each case of $k = 1, 2$ such that the nonzero effects of $\boldsymbol{\xi}_2$ can be searched and they can be estimated along with $\boldsymbol{\xi}_1$. The following theorem has been established by Srivastava and Ghosh (1977):

THEOREM 2.1. *Let T be a design of resolution* V *(simply,* design(V)*). In Model* (2.1)*, a necessary condition for T to be a search design is that for every $n \times 2k$ submatrix A_{20} of A_2,*

$$\operatorname{rank}(W) = 2k \tag{2.2}$$

holds, where

$$W = A'_{20}A_{20} - A'_{20}A_1M^{-1}A'_1A_{20}. \tag{2.3}$$

Here $M(\nu_1 \times \nu_1) = A'_1A_1$ is said to be the information matrix for $\boldsymbol{\xi}_1$ of T. It is noted that T is a design(V) if and only if M is nonsingular. In the case of $\sigma = 0$ (called a noiseless case), the above condition is also sufficient. The procedure for searching and estimating the extra nonnegligible effects has been discussed in Srivastava (1975).

DEFINITION 1. A design T is said to be a search design of resolution V.k (simply, search design(V.k)) if (2.2) holds for every submatrix A_{20}.

3. Balanced array and W matrix

Let T be a balanced array of strength 6, size n and m (≥ 6) constraints with index set $\mathcal{M} = \{\mu_0, \mu_1, \ldots, \mu_6\}$ (briefly, BA$(n, m; \mathcal{M})$). For the definition of a balanced array, e.g., see Srivastava (1972) and Yamamoto *et al.* (1975). Denote

$$\begin{aligned}
\gamma_0 &= \mu_6 + \mu_0 + 6(\mu_5 + \mu_1) + 15(\mu_4 + \mu_2) + 20\mu_3, \\
\gamma_1 &= \mu_6 - \mu_0 + 4(\mu_5 - \mu_1) + 5(\mu_4 - \mu_2), \\
\gamma_2 &= \mu_6 + \mu_0 + 2(\mu_5 + \mu_1) - (\mu_4 + \mu_2) - 4\mu_3, \\
\gamma_3 &= \mu_6 - \mu_0 - 3(\mu_4 - \mu_2), \\
\gamma_4 &= \mu_6 + \mu_0 - 2(\mu_5 + \mu_1) - (\mu_4 + \mu_2) + 4\mu_3, \\
\gamma_5 &= \mu_6 - \mu_0 - 4(\mu_5 - \mu_1) + 5(\mu_4 - \mu_2), \\
\gamma_6 &= \mu_6 + \mu_0 - 6(\mu_5 + \mu_1) + 15(\mu_4 + \mu_2) - 20\mu_3
\end{aligned}$$

and

$$\begin{aligned}
\kappa_0^{01} &= m^{1/2}\gamma_1, \quad \kappa_0^{02} = \{m(m-1)/2\}^{1/2}\gamma_2,\\
\kappa_0^{03} &= \{m(m-1)(m-3)/6\}^{1/2}\gamma_3,\\
\kappa_0^{11} &= \gamma_0 + (m-1)\gamma_2, \quad \kappa_0^{12} = \{(m-1)/2\}^{1/2}\{2\gamma_1 + (m-2)\gamma_3\},\\
\kappa_0^{13} &= \{(m-1)(m-2)/6\}^{1/2}\{3\gamma_2 + (m-3)\gamma_4\},\\
\kappa_0^{22} &= \gamma_0 + 2(m-2)\gamma_2 + \{(m-2)(m-3)/2\}\gamma_4,\\
\kappa_0^{23} &= \{(m-2)/3\}^{1/2}[3\gamma_1 + 3(m-3)\gamma_3 + \{(m-3)(m-4)/2\}\gamma_5],\\
\kappa_0^{33} &= \gamma_0 + 3(m-3)\gamma_2 + \{3(m-3)(m-4)/2\}\gamma_4\\
&\quad + \{(m-3)(m-4)(m-5)/6\}\gamma_6;\\
\kappa_1^{00} &= \gamma_0 - \gamma_2, \quad \kappa_1^{01} = (m-2)^{1/2}(\gamma_1 - \gamma_3),\\
\kappa_1^{02} &= \{(m-2)(m-3)/2\}^{1/2}(\gamma_2 - \gamma_4),\\
\kappa_1^{11} &= \gamma_0 + (m-4)\gamma_2 - (m-3)\gamma_4,\\
\kappa_1^{12} &= \{(m-3)/2\}^{1/2}\{2\gamma_1 + (m-6)\gamma_3 - (m-4)\gamma_5\},\\
\kappa_1^{22} &= \gamma_0 + (2m-9)\gamma_2 + \{(m-4)(m-9)/2\}\gamma_4 - \{(m-4)(m-5)/2\}\gamma_6;\\
\kappa_2^{00} &= \gamma_0 - 2\gamma_2 + \gamma_4, \quad \kappa_2^{01} = (m-4)^{1/2}(\gamma_1 - 2\gamma_3 + \gamma_5),\\
\kappa_2^{11} &= \gamma_0 + (m-7)\gamma_2 - (2m-11)\gamma_4 + (m-5)\gamma_6.
\end{aligned}$$

Further define

$$\underset{(3\times 3)}{K_0} = \begin{bmatrix} \gamma_0 & \kappa_0^{01} & \kappa_0^{02} \\ & \kappa_0^{11} & \kappa_0^{12} \\ (\text{Sym.}) & & \kappa_0^{22} \end{bmatrix}, \quad \underset{(2\times 2)}{K_1} = \begin{bmatrix} \kappa_1^{00} & \kappa_1^{01} \\ \kappa_1^{01} & \kappa_1^{11} \end{bmatrix};$$

$$\underset{(3\times 1)}{\boldsymbol{\kappa}_0} = (\kappa_0^{03}, \kappa_0^{13}, \kappa_0^{23})', \quad \underset{(2\times 1)}{\boldsymbol{\kappa}_1} = (\kappa_1^{02}, \kappa_1^{12})'; \quad \kappa_3 = 2^6\mu_3.$$

Now we denote the element of W in (2.3) corresponding to three-factor interactions $F_{t_1^1 t_2^1 t_3^1}$ and $F_{t_1^2 t_2^2 t_3^2}$ by $\epsilon(t_1^1 t_2^1 t_3^1 : t_1^2 t_2^2 t_3^2)$. The following lemma can be obtained from Theorem 2.1 of Shirakura and Ohnishi (1985) for $\ell = 2$:

LEMMA 3.1. *Let T be a* $\mathrm{BA}(n, m; \mathcal{M})$ *such that K_0 and K_1 are nonsingular, and κ_2^{00} is not zero. Then the element of W, $\epsilon(t_1^1 t_2^1 t_3^1 : t_1^2 t_2^2 t_3^2)$ depends only on the cardinality of set $|\{t_1^1, t_2^1, t_3^1\} \cap \{t_1^2, t_2^2, t_3^2\}|$. That is, for $\alpha = 3 - |\{t_1^1, t_2^1, t_3^1\} \cap \{t_1^2, t_2^2, t_3^2\}|$, the element can be given by π_α $(\alpha = 0, 1, 2, 3)$ such that*

$$
\begin{aligned}
\pi_0 &= \frac{6}{m(m-1)(m-2)}a_0 + \frac{6}{m(m-2)}a_1 \\
&\quad + \frac{3(m-3)}{(m-1)(m-2)}a_2 + \frac{m-5}{m-2}a_3, \\
\pi_1 &= \frac{6}{m(m-1)(m-2)}a_0 + \frac{2(2m-9)}{m(m-2)(m-3)}a_1 \\
&\quad + \frac{m-7}{(m-1)(m-2)}a_2 - \frac{m-5}{(m-2)(m-3)}a_3, \\
\pi_2 &= \frac{6}{m(m-1)(m-2)}a_0 + \frac{2(m-9)}{m(m-2)(m-3)}a_1 \\
&\quad - \frac{2(2m-11)}{(m-1)(m-2)(m-4)}a_2 + \frac{2(m-5)}{(m-2)(m-3)(m-4)}a_3, \\
\pi_3 &= \frac{6}{m(m-1)(m-2)}a_0 - \frac{18}{m(m-2)(m-3)}a_1 \\
&\quad + \frac{18}{(m-1)(m-2)(m-4)}a_2 - \frac{6}{(m-2)(m-3)(m-4)}a_3,
\end{aligned}
\tag{3.1}
$$

where

$$
\begin{aligned}
a_0 &= \kappa_0^{33} - \boldsymbol{\kappa}_0' K_0^{-1}\boldsymbol{\kappa}_0, & a_1 &= \kappa_1^{22} - \boldsymbol{\kappa}_1' K_1^{-1}\boldsymbol{\kappa}_1, \\
a_2 &= \kappa_2^{11} - (\kappa_2^{01})^2/\kappa_2^{00}, & a_3 &= \kappa_3.
\end{aligned}
\tag{3.2}
$$

It is noted that the array of Lemma 3.1 yields a balanced design(V) (see Yamamoto *et al.* (1975)).

4. Construction of search designs(V.1)

From Theorem 2.1 and Lemma 3.1, we establish the following theorem (see Shirakura and Ohnishi (1985)):

THEOREM 4.1. *The array of Lemma* 3.1 *is a* search design (V.1) *if and only if* $|\pi_\alpha| \neq |\pi_0|$ *hold for* $\alpha = 1, 2, 3$.

Using this theorem, we obtain a search design(V.1) for $m \geq 6$. In what follows, suppose $\Omega(m, s)$ is the set of all distinct treatments in which the number of 1-levels is exactly s $(0 \leq s \leq m)$.

THEOREM 4.2. *The designs*

(A1) $T = \Omega(6,0) \cup \Omega(6,3) \cup \Omega(6,5) \cup (6,6), \quad m = 6,$

(A2) $T = \Omega(6,1) \cup \Omega(6,4) \cup \Omega(6,5) \cup (6,6), \quad m = 6,$

(A3) $T = \Omega(6,0) \cup \Omega(6,1) \cup \Omega(6,4) \cup (6,5), \quad m = 6$

and

$$\text{(B)} \qquad T = \Omega(m,1) \cup \Omega(m,m-2) \cup \Omega(m,m-1), \qquad m \geq 7$$

are search designs(V.1) *with* $n = 28$ *and* $n = m(m+3)/2$ *treatments, respectively.*

PROOF. First consider the design of (B). It is clear that T is a BA$(n, m; \mathcal{M} = \{\mu_0 = m-6, \mu_1 = 1, \mu_2 = 0, \mu_3 = 0, \mu_4 = 1, \mu_5 = m-5, \mu_6 = (m-5)(m-6)\})$. In view of the previous section, we obtain $\det(K_0) = 16m(m-2)^2(m-3)^2 > 0$, $\det(K_1) = 64(m^2 - 5m + 8) > 0$ and $\kappa_2^{00} = 16$. We further get $a_1 = 32(m-2)(m-3)/(m^2 - 5m + 8)$ and $a_0 = a_2 = a_3 = 0$ in (3.2). It follows from (3.1) that $\pi_0 \neq \pi_1$, $\pi_0 \neq \pm\pi_2$ and $\pi_0 \neq -\pi_3$ hold for $m \geq 7$. Next consider the designs of (A1, A2, A3). Similarly, the design of (A1) gives $a_0 > 0$, $a_3 > 0$, $a_1 = a_2 = 0$, and the designs of (A2, A3) give $a_0 > 0$, $a_1 > 0$, $a_2 = a_3 = 0$. These yield $|\pi_0| \neq |\pi_\alpha|$, $\alpha = 1, 2, 3$. The proof follows from Theorem 4.1.

From Shirakura and Ohnishi (1985), it may be remarked that the design of (A1) and the designs of (B) for $m = 7, 8$ are optimal search designs with respect to a certain criterion due to Srivastava (1977). Also, the design $T = \Omega(m,1) \cup \Omega(m,m-2) \cup \Omega(m,m-1) \cup \Omega(m,m)$ is a search design of Srivastava and Ghosh (1976) which permits the search of at most one nonnegligible effect among three-factor and higher order interactions.

5. Representation of W matrix for $k = 2$

We are interested in constructing a search design(V.2). From Lemma 3.1, the 4×4 matrix W of (2.3) for four distinct three-factor interactions $F_{t_1^1 t_2^1 t_3^1}$, $F_{t_1^2 t_2^2 t_3^2}$, $F_{t_1^3 t_2^3 t_3^3}$ and $F_{t_1^4 t_2^4 t_3^4}$ can be given in terms of π_α's as follows:

$$(5.1) \qquad W = \begin{bmatrix} \pi_0 & \pi_{\alpha_{12}} & \pi_{\alpha_{13}} & \pi_{\alpha_{14}} \\ & \pi_0 & \pi_{\alpha_{23}} & \pi_{\alpha_{24}} \\ & & \pi_0 & \pi_{\alpha_{34}} \\ (\text{Sym.}) & & & \pi_0 \end{bmatrix},$$

where $\alpha_{ij} = 3 - |\{t_1^i, t_2^i, t_3^i\} \cap \{t_1^j, t_2^j, t_3^j\}|$, $1 \leq i < j \leq 4$. However, for every $m \geq 6$, it may be impossible to check the nonsingularities of W in (5.1) for all possible sets of four distinct effects in ξ_2.

Let $\mathfrak{M}$ be the class for all the 3-subsets of $\{1, 2, \ldots, m\}$. Then it follows that the determinant of W of (5.1) is dependent on $U_1, U_2, U_3, U_4 \in \mathfrak{M}$ only through the cardinalities of $U_i \cap U_j$, $1 \leq i < j \leq m$, i.e., there exists a function φ for which

$$\det(W) = \varphi(\alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{23}, \alpha_{24}, \alpha_{34}),$$

where $\alpha_{ij} = 3 - |U_i \cap U_j|$. In what follows, we consider φ as a function over $\mathcal{A} = \{\omega = (\omega_{12}, \omega_{13}, \omega_{14}, \omega_{23}, \omega_{24}, \omega_{34}) \mid \omega_{ij} = 1, 2, 3; 1 \leq i < j \leq m\}$. Note that for $\omega \in \mathcal{A}$, the (i,j)-th elements of W are π_0 or $\pi_{\omega_{ij}}$ according as $i = j$ or not, where $\omega_{ij} = \omega_{ji}$.

Let $\mathfrak{S}$ be the symmetric group of degree 4 on $\{1, 2, 3, 4\}$. For $\tau \in \mathfrak{S}$ and $\omega = (\omega_{12}, \omega_{13}, \ldots, \omega_{34}) \in \mathcal{A}$, suppose

$$\omega^\tau = (\omega_{\tau(12)}, \omega_{\tau(13)}, \ldots, \omega_{\tau(34)}),$$

where $\omega_{\tau(ij)} = \omega_{\tau(i)\tau(j)}$. Then it is easy to see that the function φ has the property

$$\varphi(\omega) = \varphi(\omega^\tau) \tag{5.2}$$

for all $\omega \in \mathcal{A}$ and $\tau \in \mathfrak{S}$. Since $\omega^\tau \in \mathcal{A}$, we can also introduce an equivalence relation as follows: Two elements ω and $\tilde{\omega}$ in $\mathcal{A}$ are equivalent, $\omega \sim \tilde{\omega}\ (\mathrm{mod}\, \mathfrak{S})$ if there exists a permutation $\tau \in \mathfrak{S}$ satisfying $\tilde{\omega} = \omega^\tau$. This means that the set $\mathcal{A}$ is partitioned by equivalence classes, i.e.,

$$\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \cdots \cup \mathcal{A}_R, \tag{5.3}$$

where $\omega_1, \omega_2 \in \mathcal{A}_u$ if and only if $\omega_1 \sim \omega_2\ (\mathrm{mod}\, \mathfrak{S})$, and $\mathcal{A}_u \cap \mathcal{A}_v = \emptyset\ (u \neq v)$. These $\mathcal{A}_u$ are so-called the orbits of $\mathcal{A}$.

Clearly, for any ω_1 and ω_2 in $\mathcal{A}_u$, $\varphi(\omega_1) = \varphi(\omega_2)$ holds. In obtaining a search design(V.2), therefore, we may calculate the matrix W and check its nonsingularity only for a *representative* 4-*plet* (U_1, U_2, U_3, U_4) of $\mathfrak{M}$ such that $\omega_{ij} = 3 - |U_i \cap U_j|$ for a *representative* ω in each orbit of $\mathcal{A}$.

Consider a weighted graph with the vertex-set $V = \{1, 2, 3, 4\}$ in which two distinct vertices are adjacent with weight ω, where $\omega = 1, 2, 3$. Between the elements $\omega = (\omega_{12}, \omega_{13}, \ldots, \omega_{34}) \in \mathcal{A}$ and the weighted graphs, there exists a one-to-one correspondence such that two vertices i and j are adjacent with weight ω_{ij} $(i \neq j)$. On the other hand, two weighted graphs are said to be isomorphic if there exists a one-to-one correspondence between their vertex-sets which preserves weights assigned to the pairs of vertices. It is observed that two elements of $\mathcal{A}$ belong to distinct orbits if and only if the corresponding two weighted graphs are nonisomorphic. As a result, the number of R in (5.3) is equal to that of nonisomorphic weighted graphs with vertex-set V. Moreover, each of the nonisomorphic graphs corresponds to a representative of some orbit of $\mathcal{A}$. From Shirakura and Tazawa (1986), it is known that $R = 66$. In fact, the 66 nonisomorphic graphs have been listed in Table 2 of their paper. In their paper, note that the weights assigned are equal to 0, 1 and 2. For a representative $\omega \in \mathcal{A}_u$, however, there does not always exist a representative 4-plet (U_1, U_2, U_3, U_4) of $\mathfrak{M}$. The following theorem can easily be established:

THEOREM 5.1. *For an element $\omega = (\omega_{12}, \omega_{13}, \omega_{14}, \omega_{23}, \omega_{24}, \omega_{34}) \in \mathcal{A}$, a necessary and sufficient condition for the existence of a 4-plet $(U_1, U_2, U_3, U_4) \in \mathfrak{M}$ satistying $|U_i \cap U_j| = 3 - \omega_{ij}$ $(1 \leq i < j \leq 4)$ is that the following conditions hold:*

(i) *For any integers i, j, ℓ $(1 \leq i < j < \ell \leq 4)$, there exist integers $r_{ij\ell}$ satisfying*

$$r_{ij\ell} \geq \max\{0, 3 - \omega_{ij} - \omega_{i\ell}, 3 - \omega_{ij} - \omega_{i\ell}, 3 - \omega_{i\ell} - \omega_{j\ell}\},$$
$$r_{ij\ell} \leq \min\{3 - \omega_{ij}, 3 - \omega_{i\ell}, 3 - \omega_{j\ell}, m - \omega_{ij} - \omega_{i\ell} - \omega_{j\ell}\},$$

(ii) *for the above integers $r_{il\ell}$, there exists an integer r satisfying*

$$r \geq \max\Big\{0, b_{ij}\ (1 \leq i < j \leq 4),$$
$$r_{123} + r_{124} + r_{134} + r_{234} + \sum_{1 \leq i < j \leq 4}\sum \omega_{ij} - m - 6\Big\},$$
$$r \leq \min\{r_{123}, r_{124}, r_{134}, r_{234}, c_i\ (1 \leq i \leq 4)\},$$

Table 1.

No.	ω_{12}	ω_{13}	ω_{14}	ω_{23}	ω_{24}	ω_{34}	U_2	U_3	U_4	m^*
1	1	1	1	1	1	1	124	134	234	4
2	1	1	1	1	1	2	124	234	125	5
3	1	1	1	1	2	2	124	134	135	5
4	1	1	2	2	1	1	124	135	145	5
5	1	1	1	2	2	2	124	134	236	6
6	1	1	2	1	2	2	124	125	345	5
7	1	1	2	2	1	2	124	135	245	5
8	1	1	2	1	3	3	124	125	367	7
9	1	1	3	1	2	2	124	125	456	6
10	1	1	2	2	1	3	124	235	146	6
11	1	1	3	1	3	3	124	234	567	7
12	1	1	2	2	2	2	124	235	256	6
13	1	2	2	2	2	1	124	256	156	6
14	1	1	3	2	2	2	124	135	456	6
15	1	1	2	2	3	2	124	236	357	7
16	1	2	2	3	2	1	124	356	156	6
17	1	2	2	3	3	1	124	367	357	7
18	1	1	3	2	2	3	124	135	467	7
19	1	2	3	3	2	1	124	356	456	6
20	1	1	3	2	3	3	124	135	678	8
21	1	3	3	2	3	1	124	456	567	7
22	1	3	3	3	3	1	124	567	568	8
23	1	2	2	2	2	2	124	345	156	6
24	1	3	2	2	2	2	124	456	167	7
25	1	2	2	2	2	3	124	345	167	7
26	1	3	2	3	2	2	124	567	345	7
27	1	3	3	2	2	2	124	456	478	8
28	1	2	3	2	2	3	124	156	478	8
29	1	2	3	3	2	2	124	356	457	7
30	1	2	3	2	3	3	124	348	567	8
31	1	3	2	3	3	2	124	567	358	8
32	1	3	2	2	3	3	124	456	378	8
33	1	2	3	3	3	3	124	356	789	9
34	1	3	3	3	3	2	124	567	589	9
35	1	3	3	3	3	3	124	567	89, 10	10
36	2	2	2	2	2	2	145	246	356	6
37	2	2	2	2	2	3	145	246	357	7
38	2	2	2	2	3	3	145	246	378	8
39	2	2	3	3	2	2	145	267	468	8
40	2	2	2	3	3	3	145	267	389	9
41	2	2	3	2	3	3	145	246	789	9
42	2	2	3	3	2	3	145	267	489	9
43	2	2	3	3	3	3	145	267	89, 10	10
44	2	3	3	3	3	2	145	678	69, 10	10
45	2	3	3	3	3	3	145	678	9, 10, 11	11
46	3	3	3	3	3	3	456	789	10, 11, 12	12

where $b_{ij} = r_{ijp} + r_{ijq} + \omega_{ij} - 3$ *for* $\{p,q\} = \{1,2,3,4\} \backslash \{i,j\}$ *and* $c_i = r_{ij_1j_2} + r_{ij_1j_3} + r_{ij_2j_3} + \omega_{ij_1} + \omega_{ij_2} + \omega_{ij_3} - 6$ *for* $\{j_1, j_2, j_3\} = \{1,2,3,4,\} \backslash \{i\}$ *(here* $r_{ij\ell} = r_{i\ell j} = r_{ji\ell} = \cdots = r_{\ell ji}$*).*

Using Theorem 5.1, we can obtain representative 4-plets $(U_1, U_2, U_3, U_4) \in \mathfrak{M}$ for representatives of ω. Without loss of generality, $U_1 = \{1,2,3\}$ may be assumed. Table 1 gives the sets U_2, U_3 and U_4 of $\mathfrak{M}$ for ω. The values of m^* in this table denote the smallest values of m for which (U_1, U_2, U_3, U_4) exist for ω. In Table 2, we present the number of representative 4-plets for each $m \geq 6$.

Table 2.

6	7	8	9	10	11	$12 \leq m$
16	27	36	41	44	45	46

6. Construction of search designs(V.2)

From Section 5, all possible matrices of W in (2.3) may be reduced to the matrices for ω in Table 1. Denote the matrices of W according to Nos. i in Table 1 by W_i $(i = 1, 2, \ldots, 46)$. For example, W_4 and W_{19} become

$$\underset{(m \geq 6)}{W_4} = \begin{bmatrix} \pi_0 & \pi_1 & \pi_1 & \pi_2 \\ & \pi_0 & \pi_2 & \pi_1 \\ & & \pi_0 & \pi_1 \\ \text{(Sym.)} & & & \pi_0 \end{bmatrix}, \quad \underset{(m \geq 6)}{W_{19}} = \begin{bmatrix} \pi_0 & \pi_1 & \pi_2 & \pi_3 \\ & \pi_0 & \pi_3 & \pi_2 \\ & & \pi_0 & \pi_1 \\ \text{(Sym.)} & & & \pi_0 \end{bmatrix}.$$

The determinants of these matrices are given by

$$\begin{aligned} \det(W_4) &= (\pi_0 - \pi_2)^2(\pi_0 + 2\pi_1 + \pi_2)(\pi_0 - 2\pi_1 + \pi_2), \\ \det(W_{19}) &= (\pi_0 + \pi_1 + \pi_2 + \pi_3)(\pi_0 + \pi_1 - \pi_2 - \pi_3) \\ &\quad \cdot (\pi_0 - \pi_1 + \pi_2 - \pi_3)(\pi_0 - \pi_1 - \pi_2 - \pi_3). \end{aligned} \tag{6.1}$$

THEOREM 6.1. *The designs*

(A1) $T = \Omega(6,1) \cup \Omega(6,2) \cup \Omega(6,4), \quad m = 6,$

(A2) $T = \Omega(6,0) \cup \Omega(6,3) \cup \Omega(6,4), \quad m = 6,$

(B) $T = \Omega(7,1) \cup \Omega(7,4) \cup \Omega(7,7), \quad m = 7$

and

(C) $T = \Omega(m,2) \cup \Omega(m,m-2) \cup \Omega(m,m), \quad m \geq 7$

are search designs(V.2) *with* $n = 36$, $n = 43$ *and* $n = m(m-1) + 1$ *treatments, respectively.*

PROOF. Consider the design of (A1). This design is a BA$(6, n; \mathcal{M} = \{\mu_0 = 0, \mu_1 = 1, \mu_2 = 1, \mu_3 = 0, \mu_4 = 1, \mu_5 = 0, \mu_6 = 0\})$. By Section 4, this leads to $a_1 = 192/13$, $a_2 = 64$ and $a_0 = a_3 = 0$. We thus have $\pi_0 = 2112/65$, $\pi_1 = -128/65$, $\pi_2 = -288/65$ and $\pi_3 = 4896/65$, for which the determinants of the matrices $W_1 \sim W_7$, W_9, W_{10}, $W_{12} \sim W_{14}$, W_{16}, W_{19} and W_{23} are nonzero. For the design of (A2), similarly, we observe that the above matrices are also nonsingular. Next consider the design of (B). This gives $a_3 = 64$, $a_0 = a_1 = a_2 = 0$, and hence, $\pi_0 = 128/5$, $\pi_1 = \pi_3 = -32/5$, $\pi_2 = 64/15$. It can be shown that the 27 matrices W_i defined for $m = 7$ are nonsingular.

Consider the design of (C). This design is a BA$(m, n; \mathcal{M} = \{\mu_0 = (m-6)(m-7)/2, \mu_1 = m-6, \mu_2 = 1, \mu_3 = 0, \mu_4 = 1, \mu_5 = m-6, \mu_6 = (m-6)(m-7)/2+1\})$, which leads to $a_1 = 32(m-4)$ and $a_0 = a_2 = a_3 = 0$. Therefore, by (3.1) we get $\pi_0 = 96(m-3)(m-4)/\{(m-1)(m-2)\}$, $\pi_1 = 32(m-4)(m-7)/\{(m-1)(m-2)\}$, $\pi_2 = -64(2m-11)/\{(m-1)(m-2)\}$ and $\pi_3 = 576/\{(m-1)(m-2)\}$. Now the determinants of $W_1 \sim W_{46}$ can be calculated and it can be shown that each determinant of W_i is positive for all values of m defined for W_i. For example,

$$\begin{aligned}
\det(W_{10}) &= \frac{2^{20}(m-6)[(m-5)(m-6)\{11(m-7)+59\}+88(m-5)+24]}{(m-1)(m-4)^4} \\
&> 0, \quad m \geq 7, \\
\det(W_{22}) &= \frac{2^{26}(m-7)\{(m-5)(m-6)+3m-5\}}{(m-1)(m-4)^2} > 0, \qquad m \geq 8.
\end{aligned}$$

(It may be remarked that the calculation of the determinants of matrices dependent on m was carried out using the computer soft "REDUCE".) This completes the proof.

7. Minimum treatments for search designs

If there exists a search design with n' treatments, then for any n $(\geq n')$, we obtain a search design with n treatments by adding any $(n-n')$ treatments to it. Therefore, we are interested in search designs of smaller values of treatments. In view of Section 3, define the following matrices:

$$\underset{(4\times4)}{K_0^*} = \begin{bmatrix} K_0 & \kappa_0 \\ \kappa_0' & \kappa_0^{33} \end{bmatrix}, \qquad \underset{(3\times3)}{K_1^*} = \begin{bmatrix} K_1 & \kappa_1 \\ \kappa_1' & \kappa_1^{22} \end{bmatrix},$$

$$\underset{(2\times2)}{K_2^*} = \begin{bmatrix} \kappa_0^{00} & \kappa_2^{01} \\ \kappa_2^{01} & \kappa_2^{11} \end{bmatrix}, \qquad \underset{(1\times1)}{K_3^*} = \kappa_3.$$

Then it is known from Yamamoto *et al.* (1976) that for a BA$(n, m; \mathcal{M})$, the rank of the information matrix M is written by

$$\text{(7.1)} \quad \operatorname{rank}(M) = \rho_0 + (m-1)\rho_1 + \{m(m-3)/2\}\rho_2 + \{m(m-1)(m-5)/6\}\rho_3,$$

where $\rho_\beta = \operatorname{rank}(K_\beta^*)$, $\beta = 0, 1, 2, 3$. The following lemma can easily be shown:

LEMMA 7.1. *Consider the array of Lemma* 3.1. *Then* $3-\beta \leq \rho_\beta \leq 4-\beta$ *hold for* $\beta = 0,1,2,3$. *Moreover, for each* $\beta = 0,1,2,3, \rho_\beta = 3-\beta$ *if and only if* $a_\beta = 0$ *in* (3.2).

Let $\mathcal{B}_m^k$ be the collection of all possible BA$(n, m; \mathcal{M})$ which yield search designs(V.k).

THEOREM 7.1. *For each* $m \geq 6$, *the* search designs(V.2) *of Theorem* 6.1 *are of the minimum value of* n *treatments in* $\mathcal{B}_m^2$.

PROOF. Assume there exists a search design(V.2) in $\mathcal{B}_m^2$ with $n \leq 35$ for $m = 6$ and $n \leq m(m-1)$ for $m \geq 7$. First consider the case of $m \geq 7$. Since rank$(M) \leq n$, Lemma 7.1 and (7.1) imply

$$\begin{aligned} 0 \leq & (\rho_0 - 3) + (m-1)(\rho_1 - 2) + \{m(m-3)/2\}(\rho_2 - 1) \\ & + \{m(m-1)(m-5)/6\}\rho_3 \\ \leq & \, m(m-3)/2 - 1. \end{aligned}$$

Since $m(m-1)(m-5)/6 > m(m-3)/2-1$ for $m \geq 7$, we have $\rho_2 = 1$ and $\rho_3 = 0$. Again from Lemma 7.1, $a_2 = a_3 = 0$, and from (3.1), $\pi_0 + \pi_2 = 2\pi_1$ must hold. This leads to $\det(W_4) = 0$ in (6.1), a contradiction. Next consider the case of $m = 6$. Similarly,

$$0 \leq (\rho_0 - 3) + 5(\rho_1 - 2) + 9(\rho_2 - 1) + 5\rho_3 \leq 13.$$

Suppose $\rho_2 = 1$ and hence $a_2 = 0$. Then (3.1) gives $\pi_0 - \pi_1 - \pi_2 + \pi_3 = 0$, implying $\det(W_{19}) = 0$ in (6.2). Therefore $a_2 \neq 0$ (i.e., $\rho_2 = 2$) must hold. Hence, we get $\rho_1 = 2$ and $\rho_3 = 0$, i.e., $a_1 = a_3 = 0$, which leads to $\pi_0 = \pi_3$. This contradicts Theorem 4.1. The proof is now completed.

By arguments similar to the above, we eatablish

THEOREM 7.2. *For each* $m \geq 6$, *the* search designs(V.1) *of Theorem* 4.2 *are of the minimum value of* n *treatments in* $\mathcal{B}_m^1$.

8. Search design(V.2) for $m = 5$

Consider a balanced array of strength 5 and $m = 5$ constraints as a design T. Then the results discussed in Section 3 can easily be modified to the case of $m = 5$. As a result, the matrix W may be expressed by π_0, π_1 and π_2 in (3.1), where a_0, a_1 and a_2 depend on the indices of a balanced array of strength 5. A search design(V.2) is derivable by checking the nonsingularities of the matrices $W_1 \sim W_4$, W_6 and W_7 in Table 1. The proof for the minimum treatments of a search design may be given in the same way as in Section 7.

THEOREM 8.1. *The design*

$$T = \Omega(5,0) \cup \Omega(5,2) \cup \Omega(5.3) \cup \Omega(5,5)$$

is a search design(V.2) *with* $n = 22$ *treatments. This design is of the minimum treatments among all possible* search designs(V.2) *derived from balanced arrays of strength* 5 *and* 5 *constraints.*

REFERENCES

Box, G. E. P., Hunter, W. S. and Hunter, J. S. (1978). *Statistics for Experiments*, Wiley, New York.

Ghosh, S. (1980). On main effect plus one plans for 2^m factorials, *Ann. Statist.*, **8**, 922–930.

Ghosh, S. (1981). On some new search designs for 2^m factorial experiments, *J. Statist. Plann. Inference*, **5**, 381–389.

Gupta, B. C. (1988). A bound connected with 2^6 factorial search designs of resolution 3.1, *Comm. Statist. Theory Methods*, **17**, 3137–3144.

Gupta, B. C. and Carvajal, S. S. R. (1984). A necessary condition for the existence of main effect plus one plan for 2^m factorials, *Comm. Statist. Theory Methods*, **13**, 567–580.

Ohnishi, T. and Shirakura, T. (1985). Search designs for 2^m factorial experiments, *J. Statist. Plann. Inference*, **11**, 241–245.

Shirakura, T. (1991). Main effect plus one or two plans for 2^m factorials, *J. Statist. Plann. Inference*, **27**, 65–74.

Shirakura, T. and Ohnishi, T. (1985). Search designs for 2^m factorials derived from balanced arrays of strength $2(\ell+1)$ and AD-optimal search designs, *J. Statist. Plann. Inference*, **11**, 247–258.

Shirakura, T. and Tazawa, S. (1986). Enumeration and representation of nonisomorphic weighted graphs, *Kobe J. Math.*, **3**, 229–235.

Shirakura, T. and Tazawa, S. (1991). Series of main effect plus one or two plans for 2^m factorials when three-factor and higher order interactions are negligible, *J. Japan Statist. Soc.* (to appear).

Srivastava, J. N. (1972). Some general existence conditions for balanced arrays of strength t and 2 symbols, *J. Combin. Theory Ser. A*, **13**, 198–206.

Srivastava, J. N. (1975). Designs for searching non-negligible effects, *A Survey of Statistical Design and Linear Models* (ed. J. N. Srivastava), 507–519, North-Holland, Amsterdam.

Srivastava, J. N. (1977). Optimal search design, or designs optimal under bias free optimality criteria, *Statistical Decision Theory and Related Topics II* (eds. S. S. Gupta and D. S. Moore), 375–409, Academic Press, New York.

Srivastava, J. N. and Ghosh, S. (1976). A series of balanced 2^m factorial designs of resolution V which allow search and estimation of one extra unknown effect, *Sankyā Ser. B*, **38**, 280–289.

Srivastava, J. N. and Ghosh, S. (1977). Balanced 2^m factorial designs of resolution V which allow search and estimation of one unknown effect, $4 \le m \le 8$, *Comm. Statist. Theory Methods*, **6**, 141–166.

Srivastava, J. N. and Gupta, B. C. (1979). Main effect plan for 2^m factorials which allow search and estimation of one unknown effect, *J. Statist. Plann. Inference*, **3**, 259–265.

Yamamoto, S., Shirakura, T. and Kuwada, M. (1975). Balanced arrays of strength 2ℓ and balanced fractional 2^m factorial designs, *Ann. Inst. Statist. Math.*, **27**, 143–157.

Yamamoto, S., Shirakura, T. and Kuwada, M. (1976). Characteristic polynomials of the information matrices of balanced fractional 2^m factorial designs of higher $(2\ell+1)$ resolution, *Essays in Probability and Statistics*, Birthday volume in honor of Prof. Ogawa (eds. S. Ikeda, T. Hayakawa, H. Hudimoto, M. Okamono, M. Siotani and S. Yamamoto), 73–94, Shinko-Tsusho, Tokyo.