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A NOTE ON THE UNIFORMLY MOST POWERFUL TESTS IN THE PRESENCE OF NUISANCE PARAMETERS

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Abstract. When a testing problem has nuisance parameters, the uniformly most powerful (UMP) tests do not generally exist. Exceptional examples were given by Dubey (1962, *Skand. Aktuarietidskr.*, **45**, 25–38; 1963, *Skand. Aktuarietidskr.*, **46**, 1–24) and Takeuchi (1968, *Ann. Math. Statist.*, **40**, 1838–1839) for the exponential distributions. What is essential for proving the existence of UMP tests lies in a special relationship between null hypothesis and the alternative. Assuming a similar relationship between them, a similar kind of result can be shown under more general situation.

Key words and phrases: Uniformly most powerful test, nuisance parameter, exponential distribution, uniform distribution.

1. Introduction

On a one-sided testing problem as follows

$$H_0 : \theta = 0 \qquad \text{versus} \qquad H_1 : \theta < 0,$$

about the probability density function

$$f(x) = \tau^{-1} \exp\{-(x - \theta)/\tau\}, \qquad \theta < x < \infty, \quad \tau > 0,$$

Dubey (1962, 1963) and Takeuchi (1968) have constructed UMP tests even though there is the nuisance parameter τ. This result is essentially a consequence of the well-known lack of memory property of the exponential distribution. Such a peculiarity holds true in more general situations. Section 2 characterizes UMP tests and shows the existence of a very simple UMP test. Section 3 examines the uniform distribution as another example. Note that Kabe and Laurent (1981) have constructed the UMP tests in the family of tests whose power functions are independent of the nuisance parameters, while those of our UMP tests, of course, depend upon them.

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2. Main result

Let S be a sample space and μ a measure on it. Let Θ be a parameter space and, for $\theta \in \Theta$, $f(x;\theta)$ be the density function with respect to μ. The testing problem considered is

$$H_0 : \theta \in \Theta_0 \quad \text{versus} \quad H_1 : \theta \in \Theta_1,$$

where Θ_0 and Θ_1 are disjoint subsets of Θ. Let $S(\theta)$ be a support of $f(\cdot\,;\theta)$, and let S_0 be the union of $S(\theta)$ for $\theta \in \Theta_0$. We assume that S_0 is measurable, and let

$$c(\theta) = \int_{S_0} f(x;\theta)d\mu(x).$$

The following assumption about the hypotheses plays an important role in this paper.

ASSUMPTION. For any $\theta \in \Theta_1$, if $c(\theta) > 0$ then there exists $\theta_0 \in \Theta_0$ such that

$$f(x;\theta_0) = (c(\theta))^{-1} f(x;\theta) I_{S_0}(x),$$

where $I_{S_0}(\cdot)$ be the indicator function of S_0. Namely, the conditional density of $f(\cdot\,;\theta)$ on S_0 belongs to the class corresponding to the null hypothesis.

Let Θ_{01} be the collection of $\theta_0 \in \Theta_0$ that satisfies the condition in the Assumption for some $\theta \in \Theta_1$, and let Θ_{00} be its complement in Θ_0. From the Assumption, the family of the conditional (on S_0) distributions with a parameter in $\{\theta \in \Theta_1 : c(\theta) > 0\}$ is equivalent to the one in Θ_{01}. Therefore, though conditionally on S_0, the tests must satisfy the level conditions even for the alternatives, which strongly restricts the behavior of the power function on Θ_1.

THEOREM 2.1. *Under the Assumption, the test function*

$$\phi^*(x) = \begin{cases} \alpha & \textit{when} \quad x \in S_0 \\ 1 & \textit{otherwise} \end{cases}$$

is UMP with the level α, and has the power $1-(1-\alpha)c(\theta)$ in $\theta \in \Theta_1$. In general, the UMP test ϕ with a level α is characterized by the following

$$\int \phi(x) f(x;\theta)d\mu(x) \le \alpha \quad \textit{for} \quad \theta \in \Theta_{00}, \tag{2.1}$$

$$\int \phi(x) f(x;\theta)d\mu(x) = \alpha \quad \textit{for} \quad \theta \in \Theta_{01} \quad \textit{and} \tag{2.2}$$

$$\phi(x) = 1 \quad (\textit{a.e. } f(\cdot\,;\theta) \textit{ for } \theta \in \Theta_1) \quad \textit{on } S_0^c. \tag{2.3}$$

PROOF. Let ϕ be any test function with the level α, and let $\theta \in \Theta_1$. There are two cases, $c(\theta) = 0$ and > 0. If $c(\theta) = 0$, then ϕ^* is better than ϕ at θ, because the power of ϕ^* is 1 at θ. In this case, the best test must satisfy the condition (2.3). On the other hand, if $c(\theta) > 0$ then the Assumption yields

$$\begin{aligned}\int \phi(x)f(x;\theta)d\mu(x) &= \int_{S_0^c} \phi(x)f(x;\theta)d\mu(x) + \int_{S_0} \phi(x)f(x;\theta)d\mu(x)\\ &\leq \int_{S_0^c} f(x;\theta)d\mu(x) + c(\theta)\int_{S_0} \phi(x)f(x;\theta_0)d\mu(x)\\ &\leq (1-c(\theta)) + \alpha c(\theta) = 1-(1-\alpha)c(\theta).\end{aligned}$$

It is clear that the maximum power $1-(1-\alpha)c(\theta)$ is attained by ϕ^*, hence ϕ^* is UMP, that is, there exists at least one UMP test. The equalities in the above inequalities hold true only when ϕ satisfies the conditions (2.2) and (2.3). Considering the level condition, these results prove the assertion.

Remark 1. The ϕ^* is a randomized test function. Practical statisticians who do not like randomization might set α to 0, but they become to lose the power $\alpha c(\theta)$.

Remark 2. From Theorem 2.1, the power functions of these UMP tests take same value in $\theta \in \Theta_{01} \cup \Theta_1$, while they may differ in $\theta \in \Theta_{00}$ up to a maximum of α. Since the power of ϕ^* is constant α in $\theta \in \Theta_{00}$, the power comparison (e.g. by admissibility) on Θ_0 might produce a better test than ϕ^* if Θ_{00} is not empty.

Remark 3. Theorem 2.1 is applicable to testing problems with nuisance parameters, such as testing the location of an exponential distribution with a nuisance scale. We usually need a random sample of size more than one to estimate the nuisance parameters, but the test ϕ^* may be based on a sample of size one. Hence, ϕ^* is UMP even when the values of the nuisance parameters are allowed to differ from one observation to another.

3. An example: test on uniform distributions

Let $x_1, \ldots, x_n$ be a random sample of size n from the uniform distribution on the interval $\theta = (\theta_1, \theta_2)$. For any testing problem considered below, let Φ_α be the set consisting of the tests satisfying the conditions of Theorem 2.1. If the Assumption is satisfied, the set Φ_α becomes the collection of all UMP tests. Let MAX and MIN be the maximum and the minimum of the sample, respectively. The pair MAX and MIN is sufficient for θ. We use the term "unique" for a test in the sense that the test of interest is unique among the tests defined only through the sufficient statistics.

The one-sided testing problems

$$H_0 : \theta_1 = 0 \text{ and } \theta_2 = 1 \quad \text{versus} \quad H_1 : \theta_1 = 0 \text{ and } \theta_2 > 1, \tag{3.1}$$

$$H_0 : \theta_1 = 0 \text{ and } \theta_2 \leq 1 \quad \text{versus} \quad H_1 : \theta_1 = 0 \text{ and } \theta_2 > 1, \tag{3.2}$$

clearly satisfy the Assumption, hence Theorem 2.1 holds true (see Lehmann (1986), p. 111, Problem 1). Since the null hypothesis H_0 of (3.1) is simple, any test in Φ_α is admissible for testing (3.1). For testing (3.2), the test ϕ_1 in Φ_α, defined by

$$\phi_1(x) = \begin{cases} 1 & \text{when } \ \text{MAX} > (1-\alpha)^{1/n} \\ 0 & \text{otherwise} \end{cases}$$

is uniquely admissible as a function of the sufficient statistic MAX. On the other hand, the two-sided testing problem

$$H_0 : \theta_1 = 0 \text{ and } \theta_2 = 1 \quad \text{versus} \quad H_1 : \theta_1 = 0 \text{ and } \theta_2 \neq 1, \tag{3.3}$$

does not satisfy the Assumption, hence the tests in Φ_α are not necessarily UMP, but the test ϕ_2 in Φ_α, defined by

$$\phi_2(x) = \begin{cases} 1 & \text{when } \ \text{MAX} > 1 \ \text{ or } \ < \alpha^{1/n} \\ 0 & \text{otherwise} \end{cases}$$

is admissible and uniquely UMP (see Lehmann (1986)).

Now consider testing problems when θ_2 is regarded as a nuisance parameter. The one-sided testing problems

$$H_0 : \theta_1 = 0 \quad \text{versus} \quad H_1 : \theta_1 < 0, \tag{3.4}$$

$$H_0 : \theta_1 \geq 0 \quad \text{versus} \quad H_1 : \theta_1 < 0, \tag{3.5}$$

also satisfy the Assumption. For both testing (3.4) and (3.5), the tests in Φ_α are all UMP and particularly ϕ^* is the same. The tests in Φ_α are also admissible for testing (3.4), but this is not true for testing (3.5). The unique admissible test $\phi_3(x)$ in Φ_α is given by

$$\phi_3(x) = \begin{cases} 1 & \text{when } \ \text{MIN/MAX} < 1 - (1-\alpha)^{1/(n-1)} \ \text{ or } \ \text{MIN} < 0 \\ 0 & \text{otherwise.} \end{cases}$$

The critical point is determined by the fact that the distribution of MIN/MAX under $\theta_1 = 0$ is equal to the one of MIN based on a random sample of size $n-1$ from the uniform distribution on $(0, 1)$.

Next, consider the two-sided testing problem

$$H_0 : \theta_1 = 0 \quad \text{versus} \quad H_1 : \theta_1 \neq 0, \tag{3.6}$$

which does not satisfy the Assumption, hence tests in Φ_α are not necessarily UMP. It is easy to see that the test ϕ_4 in Φ_α, defined by

$$\phi_4(x) = \begin{cases} 1 & \text{when } \ \text{MIN/MAX} > 1 - \alpha^{1/(n-1)} \ \text{ or } \ \text{MIN} < 0 \\ 0 & \text{otherwise,} \end{cases}$$

is UMP unbiased, but we will show that the UMP tests do not exist.

For any $s > 0$, let $C_s = \{x \in R^n : \text{MAX} < s \text{ and MIN} > s(1 - \alpha^{(1/n)})\}$. The indicator function I_{C_s} is a level α test and has power 1 for $\theta = (\theta_1, \theta_2) = (s(1 - \alpha^{1/n}), s)$. If there exists a UMP test, say ϕ', with level α for (3.6) then we have $\phi' \geq I_{C_s}$. Since $s > 0$ is free, ϕ' should be 1 on $\{x \in R^n : \text{MIN/MAX} > 1 - \alpha^{1/n}\}$. But this region has probability $\alpha^{1-1/n}$ for any θ in H_0, which contradicts the level condition of ϕ'.

On the other hand, for the two-sided testing problem such as

$$H_0 : \theta_1 \geq 0 \text{ and } \theta_2 \leq 1 \quad \text{versus} \quad H_1 : \theta_1 < 0 \text{ and } \theta_2 > 1, \tag{3.7}$$

the test ϕ^* is UMP, since the Assumption is satisfied.

In general, when a family of distributions is parametrized by the left (or right) extreme point of the support of a distribution, the one-sided testing problems like (3.3) and (3.4) about the end point of the support of distributions usually satisfy the Assumption. Other examples include the geometric, discrete uniform, trapezoidal and triangular distributions.

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References

Dubey, S. D. (1962). A simple test function for guarantee time associated with the exponential failure law, *Skand. Aktuarietidskr.*, **45**, 25–38.

Dubey, S. D. (1963). A generalization of a simple test function for guarantee time associated with the exponential failure law, *Skand. Aktuarietidskr.*, **46**, 1–24.

Kabe, D. G. and Laurent, A. G. (1981). On some nuisance parameter free uniformly most powerful tests, *Biometrical J.*, **23**, 245–250.

Lehmann, E. L. (1986). *Testing Statistical Hypothesis*, Wiley, New York.

Takeuchi, K. (1968). A note on the test for the location parameter of an exponential distribution, *Ann. Math. Statist.*, **40**, 1838–1839.