

ON LAPLACE CONTINUED FRACTION FOR THE NORMAL INTEGRAL

CHU-IN CHARLES LEE*

*Department of Mathematics and Statistics, Memorial University of Newfoundland,
St. John's, Newfoundland, Canada A1C 5S7*

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Abstract. The Laplace continued fraction is derived through a power series. It provides both upper bounds and lower bounds of the normal tail probability $\bar{\Phi}(x)$, it is simple, it converges for $x > 0$, and it is by far the best approximation for $x \geq 3$. The Laplace continued fraction is rederived as an extreme case of admissible bounds of the Mills' ratio, $\bar{\Phi}(x)/\phi(x)$, in the family of ratios of two polynomials subject to a monotone decreasing absolute error. However, it is not optimal at any finite x . Convergence at the origin and local optimality of a subclass of admissible bounds are investigated. A modified continued fraction is proposed. It is the sharpest tail bound of the Mills' ratio, it has a satisfactory convergence rate for $x \geq 1$ and it is recommended for the entire range of x if a maximum absolute error of 10^{-4} is required.

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1. Introduction

Let $\phi(x)$, $\Phi(x)$ and $\bar{\Phi}(x)$ be the normal density function, the normal distribution function and the normal tail probability function respectively. The purpose of this article is to find bounds of the Mills' ratio $R(x) = \bar{\Phi}(x)/\phi(x)$, or equivalently of the normal distribution function $\Phi(x)$. Earlier published results for bounds and approximations can be found in Mitrinović (1970), Abramowitz and Stegun (1972), Kendall and Stuart (1977) and Patel and Read (1982). The Laplace (1805) continued fraction,

$$(1.1) \quad L_n = \frac{1}{x+} \frac{1}{x+} \frac{2}{x+} \frac{3}{x+} \cdots \frac{n}{x},$$

is the best approximation to $R(x)$ for $x \geq 3$. However, it has seldom been recognized in the literature as providing an alternating sequence of upper and lower

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bounds of the Mills' ratio. It is known that L_n converges to $R(x)$ for each $x > 0$. The derivation of L_n as a power series can be found in Kendall and Stuart ((1977), p. 145) and they said that "the most useful forms for the calculation of $\Phi(x)$, or equivalently of $R(x)$, however, are continued fractions." Kerridge and Cook (1976) stated that "If accuracy is the only consideration, and speed unimportant, Laplace's continued fraction can hardly be improved on, although many thousand terms may be needed if x is small." The Laplace continued fraction was used even for small x by Sheppard (1939) to produce his highly accurate tables of the normal integral.

The Laplace continued fraction can be expressed as a ratio of two polynomials with degrees n and $n + 1$ respectively. In the family of ratios of two polynomials, there is no uniformly best bound of the Mills' ratio. A subclass having the best tail bound is the continued fraction

$$(1.2) \quad f_n(x; a_n, b_n) = \frac{1}{x+} \frac{1}{x+} \frac{2}{x+} \frac{3}{x+} \cdots \frac{n-1}{x+} \frac{b_n}{x+} a_n$$

with admissible coefficients a_n and b_n , and it shall be shown that

$$(1.3) \quad \bar{\Phi}(x) = \phi(x) \left\{ \frac{1}{x+} \frac{1}{x+} \frac{2}{x+} \frac{3}{x+} \cdots \frac{n-1}{x+} \frac{b_n}{x+} a_n \right\} + \epsilon_n$$

even at $x = 0$ where ϵ_n represents a quantity converging to zero as $n \rightarrow \infty$. The expression (1.3) is very simple to compute. It requires $n + 1$ additions, $n + 1$ reciprocals and $n + 1$ multiplications. For example, $\{(a_1 + x)^{-1}b_1 + x\}^{-1}\phi(x)$ at $x = 2$ has an error 0.000004 in approximating $\bar{\Phi}(x)$ if the optimal coefficients $a_1 = 1.252$ and $b_1 = 1.215$ (see Section 5) at $x = 2$ and $n = 1$ are used.

The admissibility of the parameters a_n and b_n in (1.2) is investigated in Section 2 where the Laplace continued fraction L_n with $a_n = 0$ and $b_n = n$ is found to be an extreme case. Convergence at the origin and properties of f_n are presented in Section 3. Bounds of the Mills' ratio are compared in Section 4. It is found that f_3 is superior to bounds of the Mills' ratio in the literature if $x \geq 0.7$. Asymptotic optimality of f_n is discussed in Section 5. It is remarkable to observe that the simple expression (1.3) with $n = 12$ has a maximum absolute error no more than 10^{-4} if $a_n = 2\{(b_n + 1)(b_n - n)/b_n\}^{1/2}$ and $b_n = 2n - xn^{1/2} + (x^2 - 1)/2$. In the last section, the continued fraction f_n is compared with other well-known approximations to $\Phi(x)$. The continued fraction f_n is recommended for desk computation in the range $x \geq 1$.

2. Admissible bounds

Let $P(x)/Q(x)$ be a ratio of two polynomials with degrees n and m respectively. The Laplace continued fraction L_n is such a ratio. The ratio is a lower (upper) bound of $\bar{\Phi}(x)/\phi(x)$ if the integral $\int_t^\infty \Delta_n(x)\phi(x)dx$ is nonnegative (non-positive) for each $t > 0$ where $\Delta_n(x) = 1 + d\{P(x)/Q(x)\}/dx - xP(x)/Q(x)$. The integral constraint is very complicated and it can at best achieve local optimality. We consider in this article a simpler constraint that requires $\Delta_n(x)$ itself to be

nonnegative or nonpositive for each $x > 0$ as in Feller ((1968), p. 179). Under this requirement, the bound $P(x)/Q(x)$ has a monotone decreasing absolute error in x and hence it is least favorable at the origin. The lower bound $P(x)/Q(x)$ is said to be admissible if there does not exist a ratio $p(x)/q(x)$ of two polynomials with the degree of the numerator no more than n such that

$$P(x)/Q(x) \leq p(x)/q(x) \leq R(x).$$

The admissible upper bound is defined similarly. The purpose of this section is to establish the following theorem.

THEOREM 2.1. *Let n be an odd (even) integer. Then a continued fraction $f_n(x; a_n, b_n)$ is an admissible lower (upper) bound of $\bar{\Phi}(x)/\phi(x)$ subject to a monotone decreasing absolute error in the family of ratios of two polynomials with the degree of the numerator no more than n if*

$$(2.1) \quad a_n = 2\{(b_n + 1)(b_n - n)/b_n\}^{1/2} \quad \text{and} \quad n \leq b_n \leq (n^2 + n + 1)^{1/2} + (n - 1).$$

It is an admissible upper (lower) bound of $\bar{\Phi}(x)/\phi(x)$ under the same conditions if $a_n = (n + 1)^{1/2}$ and $b_n = n$.

We shall first introduce the following notation. For nonnegative integers n and m with $m \leq n/2$, let

$$\begin{aligned} c_{n,n-2m} &= \{(2m)!/(2^m m!)\} \binom{n}{2m}, \\ d_{n,n-2m} &= \sum_{t=0}^m (-1)^t \{(2t)!/(2^t t!)\} c_{n,n-2m+2t}, \\ C_n(x) &= \sum_{m=0}^{[n/2]} c_{n,n-2m} x^{n-2m} \quad \text{and} \\ D_n(x) &= \sum_{m=0}^{[(n-1)/2]} d_{n,n-2m} x^{n-2m-1} \end{aligned}$$

where the symbol $[\alpha]$ represents the largest integer less than or equal to α . It can be shown that $c_{n,n} = 1$, $d_{n,n} = 1$,

$$\begin{aligned} c_{n,n-2m} &= c_{n-1,n-1-2m} + (n - 1)c_{n-2,n-2m} \quad \text{and} \\ d_{n,n-2m} &= d_{n-1,n-1-2m} + (n - 1)d_{n-2,n-2m}. \end{aligned}$$

Furthermore, $c_{2m,0} = (2m - 1)c_{2m-2,0}$, $d_{2m,0} = 0$ and $d_{2m+1,1} = 2md_{2m-1,1}$ if $m \geq 1$. It follows that

$$(2.2) \quad C_n(x) = xC_{n-1}(x) + (n - 1)C_{n-2}(x)$$

and

$$D_n(x) = xD_{n-1}(x) + (n - 1)D_{n-2}(x).$$

The polynomial $C_n(x)$ can also be expressed by $H_n(ix)/i^n$ where i is the imaginary number and $H_n(x)$ is the n -th Hermite polynomial (see Kendall and Stuart ((1977), p. 167)).

LEMMA 2.1. *Let a_n and b_n be nonnegative real numbers and let*

$$\begin{aligned} P_n(x) &= (x + a_n)D_n(x) + b_n D_{n-1}(x) \quad \text{and} \\ Q_n(x) &= (x + a_n)C_n(x) + b_n C_{n-1}(x). \end{aligned}$$

Then the continued fraction $f_n(x; a_n, b_n)$ in (1.2) can be expressed by $P_n(x)/Q_n(x)$.

PROOF. It is trivial that $f_1 = P_1(x)/Q_1(x)$. By (2.2),

$$(2.3) \quad Q_{n+1}(x; a_{n+1}, b_{n+1}) = (x + a_{n+1})Q_n\{x; b_{n+1}/(x + a_{n+1}), n\}$$

and the identity also holds for $P_{n+1}(x)$. The proof is completed by induction. \square

LEMMA 2.2. *Let a_n and b_n be nonnegative real numbers and let*

$$\Delta_n(x) = 1 + df_n/dx - x f_n.$$

Then

$$\begin{aligned} \Delta_n(x) &= (-1)^{n-1}(n-1)! \{ (b_n - n)x^2 + a_n(b_n - 2n)x \\ &\quad + b_n(b_n + 1) - na_n^2 \} / Q_n(x)^2. \end{aligned}$$

PROOF. Suppose instead that a_n is a function of x . It shall be established that $\Delta_n(x; a_n, b_n)$ is given by the above identity except we add another term $(-1)^{n-1}(n-1)!b_n(da_n/dx)/Q_n(x)^2$ to the right-hand side. The new identity holds when $n = 1$. By (2.3) and

$$\Delta_{n+1}(x; a_{n+1}, b_{n+1}) = \Delta_n(x; b_{n+1}/(x + a_{n+1}), n),$$

the proof is completed by induction. \square

PROOF OF THEOREM 2.1. We shall consider only the case that n is an odd integer. The case of even n follows similarly. The continued fraction $f_n(x; a_n, b_n)$ with nonnegative coefficients a_n and b_n is a lower bound of the Mills' ratio subject to a monotone decreasing absolute error if $\Delta_n(x) \geq 0$ for each $x > 0$. By Lemma 2.2, it is required that $b_n \geq n$, $b_n(b_n + 1) \geq na_n^2$, and $a_n^2 b_n \leq 4(b_n + 1)(b_n - n)$ if $b_n \leq 2n$. Let a, b, A and B be any nonnegative real numbers. Then

$$f_n(x; a, b) \leq f_n(x; A, B)$$

if and only if

$$(2.4) \quad (b - B)x + (bA - aB) \geq 0.$$

Consider a specific lower bound $f_n(x; A, B)$ with $A = (4n + 2)^{1/2}$ and $B = 2n$. If $b \geq 2n$ and $b(b + 1) \geq na^2$ then (2.4) holds and the lower bound $f_n(x; a, b)$ is not admissible. It follows from (2.4) that admissibility for fixed b requires the largest a and it suffices to consider $n \leq b \leq 2n$ and $a^2b = 4(b + 1)(b - n)$. Since a/b as a function of b has a unique mode $B = (n^2 + n + 1)^{1/2} + (n - 1)$, it follows from (2.4) again that admissibility requires $n \leq b \leq B$. On the other hand, a continued fraction $f_n(x; a_n, b_n)$ with nonnegative coefficients a_n and b_n is an upper bound of the Mills' ratio subject to the same constraint if $\Delta_n(x) \leq 0$ for each $x > 0$ or equivalently $b_n \leq n$ and $na_n^2 \geq b_n(b_n + 1)$. It can be shown by (2.4) that admissibility requires $a_n = (n + 1)^{1/2}$ and $b_n = n$.

Let a and b be any pair of real numbers. Suppose that $f_n(x; a, b)$ is a lower bound of the Mills' ratio and it is uniformly no smaller than one of the lower bound $f_n(x; a_n, b_n)$ with a_n and b_n satisfying (2.1). Since it is bounded from above by $f_n\{x; (n + 1)^{1/2}, n\}$, by (2.4) we have that $(b - n)x + (n + 1)^{1/2}b - na \geq 0$ for each $x > 0$. Therefore, $b \geq n$. By (2.4) again, for fixed b it suffices to consider $a \geq 0$. On the other hand, suppose that $f_n(x; a, b)$ is an upper bound of the Mills' ratio and it is uniformly no larger than $f_n\{x; (n + 1)^{1/2}, n\}$. Since it is bounded from below by $f_n(x; 0, n)$, by (2.4) we have that $(n - b)x + na \geq 0$ for each $x > 0$. Therefore $a \geq 0$. By (2.4) again, for fixed a it suffices to consider $b \geq 0$.

Let $P(x)$ and $Q(x)$ be any pair of polynomials with the degree of $P(x)$ no more than n . Suppose that $P(x)/Q(x)$ is a lower bound of the Mills' ratio and $f_n(x; a, b) \leq P(x)/Q(x)$ for a pair of nonnegative real numbers a and b . It suffices to consider $P(x) = x^n + p_{n-1}x^{n-1} + \dots + p_0$ and $Q(x) = q_mx^m + \dots + q_0$ for some integer m . Then $P(x)/Q(x) \leq f_n(x; A, B)$ with $A = (n + 1)^{1/2}$ and $B = n$. By the fact that a continued fraction

$$(2.5) \quad \frac{b_0}{x+} \frac{b_1}{x+} \frac{b_2}{x+} \dots \frac{b_n}{x+} a_n$$

can be expressed by $b_0(x^n + a_n x^{n-1} + \dots)/(x^{n+1} + a_n x^n + \dots)$,

$$q_m x^{m+n} + (aq_m + q_{m-1})x^{m+n-1} + \dots \leq x^{2n+1} + (a + p_{n-1})x^{2n} + \dots$$

and

$$x^{2n+1} + (A + p_{n-1})x^{2n} + \dots \leq q_m x^{m+n} + (Aq_m + q_{m-1})x^{m+n-1} + \dots$$

for each $x \geq 0$. It follows that $m = n + 1$, $q_m = 1$ and $q_{m-1} = p_{n-1}$. Taking the reciprocal and then subtracting x , we have

$$\frac{1}{x+} \frac{2}{x+} \dots \frac{n-1}{x+} \frac{b}{x+} a \geq \{Q(x) - xP(x)\}/P(x) \geq \frac{1}{x+} \frac{2}{x+} \dots \frac{n-1}{x+} \frac{B}{x+} A$$

where $Q(x) - xP(x)$ is a polynomial with degree no more than $n - 1$. Successive reduction will establish that $P(x)/Q(x)$ is the continued fraction f_n in (1.2). The results for the case that $P(x)/Q(x)$ is an upper bound of the Mills' ratio follow similarly. \square

3. Convergence and properties of f_n

The sequence $\{L_n\}$ of Laplace continued fractions in (1.1) is by Theorem 2.1 a sequence of admissible bounds $\{f_n\}$ in (1.2) when $a_n = 0$ and $b_n = n$. It is known that

$$(3.1) \quad L_1 < L_3 < \cdots < \bar{\Phi}(x)/\phi(x) < \cdots < L_4 < L_2$$

if $x > 0$. By the fact that $L_n - L_{n-1} = (-1)^n/C_n(x)C_{n+1}(x)$ (see Abramowitz and Stegun ((1972), p. 19)), the difference converges to zero for each fixed $x > 0$. It follows that (1.3) holds for each $x > 0$. However, $L_1 = L_3 = \cdots = 0$ and $L_2 = L_4 = \cdots = \infty$ at the origin. Consequently, its convergence rate is very slow for small x .

The Laplace continued fraction may be written in another form $\{f_n(x; n + 1, n)\}$. However, this sequence also fails to converge at the origin. Furthermore, the bound $f_n(x; n + 1, n)$ is not admissible. By Theorem 2.1, the superior admissible bound is

$$f_n(x; (n + 1)^{1/2}, n) = L_n^r = \frac{1}{x+} \frac{1}{x+} \frac{2}{x+} \cdots \frac{n}{x+} (n + 1)^{1/2}.$$

It can be shown that

$$L_2^r < L_4^r < \cdots < \bar{\Phi}(x)/\phi(x) < \cdots < L_3^r < L_1^r.$$

It can also be shown that L_n^r satisfies (1.3) for each $x \geq 0$ including the origin. It is trivial that $L_{2n-1} < L_{2n}^r$ and $L_{2n+1} < L_{2n}^r$ if $x < (2n + 1)^{1/2}$, and $L_{2n+1}^r < L_{2n}$, $L_{2n+1}^r < L_{2n+2}$ if $x < (2n + 2)^{1/2}$. One would prefer L_n^r to L_n .

In the remainder of this article, the continued fraction $f_n(x; a_n, b_n)$ will be reserved for the case

$$a_n = 2\{(b_n + 1)(b_n - n)/b_n\}^{1/2} \quad \text{and} \quad n \leq b_n \leq (n^2 + n + 1)^{1/2} + (n - 1).$$

It can be shown that

$$(3.2) \quad L_{2r-1} \leq f_{2r+1} \leq \bar{\Phi}(x)/\phi(x) \leq f_{2r} \leq L_{2r-2}$$

for each $x \geq 0$. By Stirling's formula, f_n satisfies (1.3) including the origin if and only if

$$(3.3) \quad b_n = 2n + O(n^r)$$

with $r < 1$. The coefficient b_n , denoted by $b_n(0)$, which minimizes the absolute error $|\bar{\Phi}(x) - f_n\phi(x)|$ at the origin is of the form

$$(3.4) \quad b_n(0) = (n^2 + n + 1)^{1/2} + (n - 1)$$

and $\{f_n\}$, with $b_n = b_n(0)$, satisfies (3.1) and (1.3) including the origin. The corresponding a_n can be expressed by

$$2\{2(n^2 + n + 1)^{1/2} + (n - 1)\}/3^{1/2}.$$

The continued fraction f_n has an absolute error uniformly no more than L_{n-1}^r if and only if $b_n^3 \leq 4n(b_n + 1)(b_n - n)$. The coefficient $b_n(0)$ satisfies this constraint. The solution to the cubic polynomial is of the form

$$(3.5) \quad b_n = 4\{n + (n^2 + 3n)^{1/2} \sin(\theta/3 - \pi/6)\}/3$$

where $\theta = \arctan[\{27(8n^2 + 13n + 16)/n\}^{1/2}/(9 - 4n)]$. The absolute error of f_n with b_n in (3.5) is less than that of f_n with b_n in (3.4) if $x \geq 1$. One would prefer f_n with b_n in (3.4) or (3.5) to L_n and L_n^r .

4. Bounds of Mills' ratio

The first three orders of L_n , L_n^r or f_n are respectively

$$\begin{aligned} &(x + a_1)/(x^2 + a_1x + b_1), \\ &(x^2 + a_2x + b_2)/\{x^3 + a_2x^2 + (b_2 + 1)x + a_2\} \quad \text{and} \\ &\{x^3 + a_3x^2 + (b_3 + 2)x + 2a_3\}/\{x^4 + a_3x^3 + (b_3 + 3)x^2 + 3a_3x + b_3\}. \end{aligned}$$

It will be shown that these bounds are superior to bounds of the Mills' ratio in the literature where the Laplace continued fraction L_n has seldom been recognized as a bound. The upper bound and the lower bound of the Mills' ratio in Gordon (1941) are $L_0 = 1/x$ and $L_1 = x/(x^2 + 1)$ respectively. The lower bound and the upper bound of the Mills' ratio in Gross and Hosmer (1978) are L_1 and $L_2 = (x^2 + 2)/(x^3 + 3x)$ respectively. For lower bounds of the Mills' ratio, L_1 is superior to the Laplace (1785) asymptotic expansion $1/x - 1/x^3$ (see Feller ((1968), p. 175 and p. 193)), L_2^r is superior to $\{(x^2 + 4)^{1/2} - x\}/2$ of Birnbaum (1942), and $f_3\{x; a_3, b_3(0)\}$ is better than $\pi/\{(\pi - 1)x + (x^2 + 2\pi)^{1/2}\}$ of Boyd (1959) if $x \geq 0.1$. For upper bounds of the Mills' ratio, L_3^r is superior to $4/\{3x + (x^2 + 8)^{1/2}\}$ of Sampford (1953), $f_2\{x; a_2, b_2(0)\}$ is better than $2/\{x + (x^2 + 2)^{1/2}\}$ of Komatu (1955), $2/\{x + (x^2 + 8/\pi)^{1/2}\}$ of Pollak (1956) if $x \geq 0.1$ and $\pi/[2x + \{2\pi + (\pi - 2)^2x^2\}^{1/2}]$ of Boyd (1959) if $x \geq 0.7$. Continued fractions f_n and L_n^r provide the sharpest tail bounds of the Mills' ratio in the literature if $x \geq 0.7$.

Other complicated bounds of the Mills' ratio are also investigated. Laplace asymptotic expansions, the six rational bounds in Shenton ((1954), p. 188) and the three rational lower bounds in Ruben ((1963), p. 362) can be expressed by the continued fraction (2.5). While the leading coefficients $b_0, b_1, b_2, b_3, \dots$ have the pattern 1, 1, 2, 3, \dots , there is always a first negative coefficient, say b_{n+1} , with two exceptions. These bounds can be expressed as ratios of two polynomials with the degree of the numerator no less than $n + 1$. Laplace asymptotic expansions are inferior to $L_0, L_1, L_2, L_3, \dots$, except that the first one $1/x$ is L_0 . The six rational bounds of Shenton (1954) are inferior to L_3, L_5, L_4, L_6, L_5 and L_6 respectively. The three rational lower bounds of Ruben (1963) are inferior to L_1, L_3 and L_5

respectively except that the first one is L_1 . The sequence of lower bounds of the Mills' ratio of the form $(x^2 + 2r + 2)^{-1/2}$ in (2.10) of Ruben (1964) are inferior to $L_2^r, L_4^r, L_6^r, \dots$. Even though the numbers of terms in L_n^r are larger, expressions and computations are, however, much simpler.

When x is small, say $x \leq 1$, the Laplace (1785) series

$$(4.1) \quad \Phi(x) = \Phi(0) + \phi(0) \sum_{t=0}^n (-1)^t x^{2t+1} / 2^t (2t+1)t! + \varepsilon_n$$

and the Laplace (1812) series, rediscovered by Pólya (1949),

$$(4.2) \quad \Phi(x) = \Phi(0) + \phi(x) \sum_{t=0}^n x^{2t+1} / 1 \cdot 3 \cdot 5 \cdots (2t+1) + \varepsilon_n$$

are very good bounds of $\Phi(x)$. If the same number of terms is used, the absolute error of (4.1) is less than that of (4.2) in the range $x \leq 2$. While the Laplace series is alternatively an upper bound and a lower bound, the Laplace-Pólya series is always a lower bound. These bounds will be compared to other approximations to $\Phi(x)$ in the last section.

5. Asymptotic optimality

Consider admissible bounds $f_n(x; a, b)$ with the coefficients a and b satisfying (2.1). The coefficient b is said to be optimal at a fixed x_0 if it has the smallest absolute error $|f_n(x_0; a, b) - R(x_0)|$. Since a/b is unimodal with the unique mode $b_n(0)$, by (2.4) we have that every coefficient b , $n < b \leq b_n(0)$, is optimal at a unique finite x , $x \geq 0$, with the exception of $b = n$ when $f_n = L_n$. By (2.4), $f_n\{x; a_n, b_n(0)\}$ has smaller absolute error than L_n if $x < 2\{[2(n^2 + n + 1)^{1/2} + (n + 2)]/3\}^{1/2}$. Suppose that one has a specific x_0 in mind. The optimal b_n is the solution to the equation

$$x_0 = \{3n + 2(n-1)b_n - b_n^2\} / \{b_n(b_n + 1)(b_n - n)\}^{1/2}$$

in the range of $n < b_n \leq b_n(0)$. The expression of this solution is rather complicated. However, it can be expressed asymptotically by

$$(5.1) \quad b_n(x_0) = 2n - x_0 n^{1/2} + (x_0^2 - 1)/2 + \varepsilon_n$$

with $\varepsilon_n = O(n^{-1/2})$. The solution $b_n(x_0)$ is of the form (3.3). Some selected values of $b_n(x_0)$ are provided in Table 1. When $x_0 = 0$, $b_n(0)$ is given in (3.4) and the corresponding ε_n lies between 0 and $3/(8n + 4)$. It is interesting to note that b_n in (3.5) can be expressed by $b_n(2^{1/2}) + \varepsilon_n$. The value $b_n(x_0)$ is monotone decreasing in x_0 . However, the three-term expression in (5.1) is not. To remedy this deficiency, one may use

$$(5.2) \quad b_n(x_0) = (3n - 1)/2 + \varepsilon_n \quad \text{if } x_0 > n^{1/2}$$

or expand ε_n in (5.1) to several extra terms. In view of (3.2) and the fact that L_n is a very good approximation for large x , the use of the three terms in (5.1) is adequate.

Table 1. Optimal values of $b_n(x_0)/n$.

n	x_0						
	0	1	2	3	4	5	6
1	1.732	1.376	1.215	1.135	1.090	1.064	1.047
2	1.823	1.465	1.283	1.184	1.126	1.091	1.068
3	1.869	1.526	1.335	1.224	1.157	1.115	1.087
4	1.896	1.571	1.377	1.259	1.185	1.137	1.105
5	1.914	1.605	1.412	1.289	1.210	1.157	1.121
10	1.954	1.703	1.524	1.396	1.305	1.239	1.190
20	1.976	1.783	1.631	1.512	1.418	1.344	1.286
50	1.990	1.860	1.748	1.652	1.569	1.498	1.437
100	1.995	1.901	1.815	1.739	1.670	1.608	1.552

Table 2. Absolute errors of L_n and f_n as approximations to $\Phi(x)^\dagger$.

x	degree n							
	1	2	3	4	5	6	7	8
0.5 L_n	$\bar{0}.17$	$\bar{0}.18$	$\bar{0}.11$	$\bar{0}.10$	$\bar{1}.72$	$\bar{1}.69$	$\bar{1}.51$	$\bar{1}.48$
f_n	$\bar{1}.30$	$\bar{2}.14$	$\bar{3}.33$	$\bar{3}.13$	$\bar{4}.62$	$\bar{4}.34$	$\bar{4}.20$	$\bar{4}.13$
1.0 L_n	$\bar{1}.38$	$\bar{1}.23$	$\bar{1}.13$	$\bar{2}.89$	$\bar{2}.58$	$\bar{2}.40$	$\bar{2}.28$	$\bar{2}.20$
f_n	$\bar{1}.38$	$\bar{3}.52$	$\bar{4}.82$	$\bar{4}.22$	$\bar{5}.78$	$\bar{5}.33$	$\bar{5}.16$	$\bar{6}.81$
1.5 L_n	$\bar{2}.70$	$\bar{2}.31$	$\bar{2}.15$	$\bar{3}.77$	$\bar{3}.42$	$\bar{3}.24$	$\bar{3}.14$	$\bar{4}.88$
f_n	$\bar{3}.56$	$\bar{4}.53$	$\bar{5}.85$	$\bar{5}.20$	$\bar{6}.62$	$\bar{6}.22$	$\bar{7}.89$	$\bar{7}.39$
2.0 L_n	$\bar{2}.12$	$\bar{3}.39$	$\bar{3}.15$	$\bar{4}.63$	$\bar{4}.28$	$\bar{4}.14$	$\bar{5}.68$	$\bar{5}.35$
f_n	$\bar{3}.18$	$\bar{5}.25$	$\bar{6}.19$	$\bar{7}.51$	$\bar{7}.17$	$\bar{8}.64$	$\bar{8}.25$	$\bar{8}.10$
2.5 L_n	$\bar{3}.17$	$\bar{4}.44$	$\bar{4}.13$	$\bar{5}.47$	$\bar{5}.18$	$\bar{6}.70$	$\bar{6}.30$	$\bar{6}.13$
f_n	$\bar{3}.20$	$\bar{5}.72$	$\bar{6}.50$	$\bar{7}.54$	$\bar{8}.79$	$\bar{8}.15$	$\bar{9}.35$	$\bar{10}.95$
3.0 L_n	$\bar{4}.20$	$\bar{5}.43$	$\bar{5}.10$	$\bar{6}.31$	$\bar{7}.96$	$\bar{7}.33$	$\bar{7}.12$	$\bar{8}.45$
f_n	$\bar{4}.76$	$\bar{5}.31$	$\bar{6}.23$	$\bar{7}.25$	$\bar{8}.34$	$\bar{9}.58$	$\bar{9}.12$	$\bar{10}.26$
3.5 L_n	$\bar{5}.21$	$\bar{6}.36$	$\bar{7}.74$	$\bar{7}.18$	$\bar{8}.47$	$\bar{8}.14$	$\bar{9}.42$	$\bar{9}.14$
f_n	$\bar{4}.17$	$\bar{6}.68$	$\bar{7}.49$	$\bar{8}.50$	$\bar{9}.64$	$\bar{10}.99$	$\bar{10}.18$	$\bar{11}.36$
4.0 L_n	$\bar{6}.18$	$\bar{7}.25$	$\bar{8}.44$	$\bar{9}.88$	$\bar{9}.20$	$\bar{10}.49$	$\bar{10}.13$	$\bar{11}.38$
f_n	$\bar{5}.28$	$\bar{6}.10$	$\bar{8}.66$	$\bar{9}.61$	$\bar{10}.72$	$\bar{10}.10$	$\bar{11}.16$	$\bar{12}.30$

$^\dagger \bar{0}.17$ stands for 0.17 and $\bar{1}.30$ for 0.030.

Let $a_n = 2\{(b_n + 1)(b_n - n)/b_n\}^{1/2}$ and $b_n = 2n - xn^{1/2} + (x^2 - 1)/2$. The normal integral $\Phi(x)$ is approximated by

$$(5.3) \quad \Phi(x) = 1 - \phi(x) \left\{ \frac{1}{x+} \frac{1}{x+} \frac{2}{x+} \frac{3}{x+} \cdots \frac{(n-1)}{x+} \frac{b_n}{x+} a_n \right\} + \varepsilon_n.$$

Absolute errors of (5.3) and those corresponding to L_n for $x = 0.5, 1.0, 1.5, \dots, 4.0$ and $n = 1, 2, \dots, 8$ are provided in Table 2. The notation $\bar{a}.bc$, used by Ruben (1964), is to be interpreted as $0.bc \times 10^{-a}$, e.g. $\bar{0}.17 = 0.17 \times 10^{-0} = 0.17$ and $\bar{1}.30 = 0.30 \times 10^{-1} = 0.030$. The values of these two approximations are upper bounds of $\Phi(x)$ if n is odd and lower bound if n is even. The approximation (5.3) is superior to its predecessor L_n . If (5.2) is also used, then absolute errors of f_n will remain the same if $x \leq n^{1/2}$ and they are identical to those of L_n if $n = 1$ and $x \geq 1$. However, it is superior for almost all other combination of x and n in Table 2. For instance, its absolute errors at $n = 2$ and $x \geq 1.5$ are $\bar{4}.54, \bar{5}.16, \bar{6}.14, \bar{7}.86, \bar{7}.21, \bar{8}.31$; at $n = 8$ and $x \geq 3.0$ are $\bar{10}.25, \bar{11}.25, \bar{12}.15$. While L_n^r is a good bound of the Mills' ratio, it is not as efficient as f_n in approximation $\Phi(x)$. Its absolute errors when $n = 8$ are $\bar{2}.27, \bar{3}.23, \bar{4}.15, \bar{6}.86, \bar{7}.42, \bar{8}.18, \bar{10}.68$ and $\bar{11}.22$ respectively at $x = 0.5, 1.0, \dots, 4.0$.

6. Approximations to $\Phi(x)$

There is a vast literature in approximating the normal integral. For example, the result in Hastings ((1955), p. 185)

$$\bar{\Phi}(x) = \frac{1}{2} \left(\sum_{t=0}^4 c_t x^t \right)^{-4} + \varepsilon$$

not only has a fixed maximum absolute error 0.00025 but also requires a table of six significant digit coefficients. The rational approximation in Gray and Schucany (1968) is inferior to f_7 of the same degree in (5.3) except in a very small neighbourhood where the error changes sign. Convergent approximations having simple expressions are L_n, f_n in (5.3), Shenton's (1954) continued fraction, the Laplace-Pólya series in (4.2), the Laplace series in (4.1) and Kerridge and Cook's (1976) expansion. Shenton (1954) proposed a continued fraction

$$\Phi(x) = \Phi(0) + \phi(x) \left\{ \frac{x}{1-} \frac{x^2}{3+} \frac{2x^2}{5-} \frac{3x^2}{7+} \frac{4x^2}{9-} \frac{5x^2}{11+} \cdots \frac{nx^2}{2n+1} \right\} + \varepsilon_n$$

and Kerridge and Cook (1976) derived an expansion

$$\Phi(x) = \Phi(0) + x\phi(x/2) \sum_{t=0}^n \frac{\theta_{2t}(x/2)}{2t+1} + \varepsilon_n$$

where $\theta_0(x) = 1, \theta_1(x) = x^2$ and $\theta_n(x) = x^2\{\theta_{n-1}(x) - \theta_{n-2}(x)\}/n$ for $n \geq 2$. While L_n, f_n , the Laplace-Pólya series and the Laplace series are bounds of the normal integral, the Kerridge and Cook expansion is not.

Table 3. Degree n or number of terms required for accuracy 10^{-d} .

x	d	degree n		terms			
		L_n	f_n	Shenton	Laplace -Pólya	Laplace	Kerridge and Cook
0.5	4	84	5	3	3	3	2
	5	132	9	4	4	3	3
	6	190	17	4	4	4	3
	7	259	28	5	5	5	7
1.0	4	20	3	5	5	5	3
	5	32	5	5	6	5	3
	6	47	8	6	7	6	4
	7	64	12	7	8	7	4
1.5	4	8	2	6	7	7	2
	5	14	3	7	8	8	4
	6	20	5	8	9	9	5
	7	28	7	9	10	10	6
2.0	4	4	2	8	9	9	5
	5	7	2	9	10	10	6
	6	11	3	10	12	12	6
	7	15	4	11*	13	13	7
2.5	4	2	2	10	12	12	5
	5	4	2	11	13	14	6
	6	6	3	12	15	15	7
	7	9	4	13*	16	17	7
3.0	4	1	1	13	14	16	5
	5	2	2	14	16	17	8
	6	4	3	15*	18	19	9
	7	5	4	16*	19	21	10
3.5	4	1	1	15	17	20	8
	5	1	2	17	19	22	9
	6	2	2	18	21	24	10
	7	3	3	19	23	25	10
4.0	4	1	1	18	20	25	8
	5	1	1	20*	22	27	10
	6	1	2	21*	25	29*	12
	7	2	2	22*	26	30*	13

*Specified accuracy not attainable in single precision programming.

For desk computation and single precision programming, we are interested in absolute error no more than 10^{-4} , 10^{-5} , 10^{-6} and 10^{-7} . Degree n or number of terms required to achieve the desired accuracy for the six convergent approximations in the order of simplicity in computation are provided in Table 3 for $x = 0.5, 1.0, \dots, 4.0$. It is found that $f_n(x)\phi(x)$ is the best approximation for the normal tail probability $\bar{\Phi}(x)$ in the range $x \geq 1.5$. For $x \leq 1$, the last four approximations in Table 3 are very efficient. It is remarkable to observe that the simple expression (5.3) has a maximum absolute error no more than 10^{-4} for the entire range of $x \geq 0$ in $n = 12$.

The last four approximations in Table 3 are also computed at $x = 0.5, 1.0, \dots, 4.0$ and $n = 1, 2, \dots, 8$. Shenton's continued fraction is better than f_n in Table 2 if $x = 0.5$ or $x = 1.0$ and $n \geq 5$. The same result holds true for both the Laplace-Pólya series and the Laplace series except $n \geq 6$ at $x = 1.0$. These three approximations are not recommended for $x \geq 2$ because of large errors. For example, at $x = 4$ and $n = 8$, the Shenton continued fraction has as absolute error 0.49, the Laplace-Pólya series 0.24 and the Laplace series 21.33. The Kerridge and Cook expansion is better than f_n in Table 2 if $x = 0.5$, $x = 1.0$ or $x = 1.5$ and $n \geq 5$. However, it is not so easy to compute. For the remaining x and n in Table 2, f_n with b_n in (5.1) and (5.2) has the smallest absolute error. The continued fraction f_n in (5.3) is recommended for desk computation in the range of $x \geq 1$.

Table 4. Degree n or number of terms required for accuracy 10^{-d} .

x	d	degree n		terms			
		L_n	f_n	Shenton	Laplace -Pólya	Laplace	Kerridge and Cook
1	10	131	35	9	10	10	6
	15	297	110	12	14	13	9
2	10	32	11	14	16	16	9
	15	73	34	18	21	21	13
3	10	13	8	19	24	25	13
	15	31	18	25	30	31	17
4	10	6	5	26	32	35	15
	15	16	12	*	39	*	21
5	10	2	3	34	41	48	20
	15	9	8	*	50	*	24
6	10	1	2	*	51	63	23
	15	4	5	*	61	*	30

*Specified accuracy not attainable in double precision programming.

The six approximations are also compared at accuracy of 10^{-10} and 10^{-15} when double precision programming is used. The results for $x = 1, 2, \dots, 6$ are provided in Table 4. It is found that f_n is the best if $x \geq 3$. If (5.2) is also used, the eight degrees for $x = 3, 4, 5$ and 6 are 7, 18, 4, 12, 2, 8, 1, and 4 respectively, and they are 6, 16, 4, 10, 2, 7, 1, and 3 respectively if ε_n in (5.1) is approximated by $x_0(6 - x_0^2)/8n^{1/2} + (x_0^3 - 4x_0^2 + 2x_0 - 4)/8n$. For $x \leq 1$, the last four approximations in Table 4 are very efficient. Double precision computation carries 16 significant digits during the course of execution. If one subtracts two numbers with the matching leading digits then the accuracy of the difference will suffer by the same number of digits. That is why both the Shenton continued fraction and the Laplace series fail to attain the specified accuracies in Table 4. On the other hand, the Laplace continued fraction L_n is tested at $x = 0.001$. It is found that its absolute error will eventually be reduced to less than 10^{-15} once $n \geq 2.934 \times 10^8$. The accuracy of f_n and Laplace continued fraction L_n are superior because the effects of rounding errors are not cumulative. While the values of f_n in (5.3) and the Laplace-Pólya series lie between 0 and 1, the values of Shenton's (1954) continued fraction, the Laplace series and Kerridge and Cook's (1976) expansion may be less than 0 or greater than 1.

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