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ON SOME JOINT LAWS IN FLUCTUATIONS OF SUMS OF RANDOM VARIABLES

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Abstract. This paper deals with the joint and marginal distributions of certain random variables concerning the fluctuations of partial sums $N_r = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_r$, $r = 1, 2, \dots, n$; $N_0 = 0$ of independent Pascal random variables $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$, thus generalizing and extending the previous work due to Saran (1977, *Z. Angew. Math. Mech.*, **57**, 610–613) and Saran and Sen (1979, *Mathematische Operationsforschung und Statistik, Series Statistics*, **10**, 469–478). The random variables considered are $\Lambda_n^{(c)}$, $\phi_n^{(c)}$, $\phi_n^{(-c)}$, Z_n and $\max_{1\le r\le n}(N_r - r)$ where $c = 0, 1, 2, \dots$ and $\Lambda_n^{(c)}$, $\phi_n^{(\pm c)}$ and Z_n denote, respectively, the number of subscripts $r = 1, 2, \dots, n$ for which $N_r = r + c$, $N_{r-1} = N_r = r \pm c$ and $N_{r-1} = N_r$.

Key words and phrases: Pascal random variables, partial sums, lattice path, rotation procedure, random walk, composed path, ballot problems.

1. Introduction

This paper is a continuation of two papers (Saran (1977), Saran and Sen (1979)) and deals with the derivation of joint distributions of certain random variables concerning the fluctuations of partial sums of independent and identically distributed (i.i.d.) random variables.

As in Saran (1977) and Saran and Sen (1979), here also we consider the sequence $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ of i.i.d. random variables where ε_r denotes the number of failures between the $(r-1)$-th and r-th successes (before the first success when $r = 1$) in a Bernoulli sequence with p $(0 < p < 1)$ the probability of success and $q = 1 - p$ the probability of failure in a single trial so that

$$p\{\varepsilon_r = j\} = pq^j, \qquad j = 0, 1, 2, \dots. \tag{1.1}$$

Let us denote by

$\Lambda_n^{(c)}$: the number of subscripts $r = 1, 2, \dots, n$ for which $N_r = r + c$,
$\phi_n^{(c)}$: the number of subscripts $r = 1, 2, \dots, n$ for which $N_{r-1} = N_r = r + c$,
$\Delta_n^{*(c)}$: the number of subscripts $r = 1, 2, \dots, n$ for which $N_r \le r + c$,

Z_n: the number of subscripts $r = 1, 2, \ldots, n$ for which $N_{r-1} = N_r$, where $N_r = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_r$, $r = 1, 2, \ldots, n$ and $N_0 = 0$. In this paper we propose to investigate, for a non-negative integer c, the probability distributions of the vectors $\{\phi_n^{(c)}, Z_n\}$, $\{\phi_n^{(-c)}, Z_n\}$ and $\{\Lambda_n^{(c)}, Z_n, \max_{1 \le r \le n}(N_r - r)\}$, and the marginal distributions derived therefrom under the condition that $N_n = k$ is fixed, thus generalizing and extending the earlier work in Saran (1977) and Saran and Sen (1979). These distributions were derived by employing the Gnedenko's technique (Gnedenko and Korolyuk (1951)) of path methods as used by Csáki and Vincze (1961), Sen (1968, 1969), Saran (1977) and Saran and Sen (1979) and the method of composed paths introduced by Srivastava (1973) and Vellore (1972). Finally, we give some applications of these results in deriving the generalized ballot problems (Takács (1967, 1970)).

2. Lattice path, random walk and rotation procedure

Let $N_r = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_r$, $r = 1, 2, \ldots, n$, $N_0 = 0$ and $N_n = k$. Then

$$P\{N_n = j\} = \binom{n+j-1}{n-1} p^n q^j, \qquad j = 0, 1, 2, \ldots. \tag{2.1}$$

Let us represent the sequence $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ of non-negative integers by a minimal lattice path in the following manner: (i) the path starts from the origin; (ii) for every r, ε_r represents one horizontal unit followed by ε_r vertical units. The section of the path contributed by ε_r starts where the section of the path contributed by ε_{r-1} ended (see Fig. 1). Such a path from $(0,0)$ to (n, N_n) is called a minimal lattice path (see Mohanty (1966)).

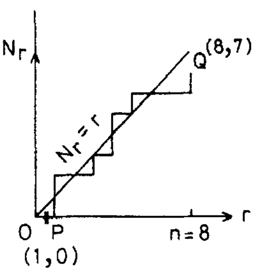

Fig. 1. Lattice path for the sequence $\varepsilon_1 = 2$, $\varepsilon_2 = 0$, $\varepsilon_3 = 1$, $\varepsilon_4 = 2$, $\varepsilon_5 = 1$, $\varepsilon_6 = 0$, $\varepsilon_7 = 0$, $\varepsilon_8 = 1$.

One can observe from (1.1) and (2.1) that the sequence $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ of Pascal random variables possesses the property of all possible lattice paths from $(0,0)$ to (n, k) to be equally probable each with a probability

$$p^n q^k = \binom{n+k-1}{n-1}^{-1} P\{N_n = k\}.$$

Let $\theta_1, \theta_2, \ldots$ be independent random variables associated with the outcomes of a Bernoulli sequence as follows:

$$\theta_i = \begin{cases} -1 & \text{if the } i\text{-th trial gives success,} \\ +1 & \text{if the } i\text{-th trial gives failure,} \quad i = 1, 2, \ldots, \end{cases}$$

with $P\{\theta_i = -1\} = p$, $P\{\theta_i = +1\} = q = 1 - p$. Let $S_0 = 0$, $S_i = \theta_1 + \theta_2 + \cdots + \theta_i$, $i = 1, 2, \ldots$. If the points (i, S_i) are represented in a plane and each of them is connected with the next one, we get a "simple random walk" path generated by the sequence $\{\theta_i\}$.

Further, in the following the random walk path defined above is said to have R runs if the total number of changes from positive direction to negative direction and vice versa is $R - 1$. We shall use in the sequel the notion of a "composed path", introduced by Srivastava (1973) and Vellore (1972), different from the ordinary path as defined above. A composed path is made up of runs $(\theta_{i_j+1} = \theta_{i_j+2} = \cdots = \theta_{i_{j+1}}$, $j = 0, 1, \ldots, R-1$, $i_0 = 0)$ where two consecutive runs are not necessarily of different kind (i.e., $\theta_{i_j} \neq \theta_{i_j+1}$ need not hold). To specify a composed path, its runs are to be previously specified (cf. Srivastava (1973), Vellore (1972)).

It is important to note that among the n non-negative variables $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$, if exactly m $(\leq n)$ assumes the value zero at each, then the corresponding lattice path (defined in Fig. 1) will have either $2n - 2m$ or $2n - 2m - 1$ changes from a horizontal to a vertical direction and vice versa according to whether ε_n equals zero or not.

The "rotation procedure" used in the sequel is defined as follows: On the rotating of the lattice path from $(0,0)$ to (n,k) (Fig. 1) about the origin through $45°$ in a clockwise direction and referring to the line $N_r = r$ as the x-axis, we observe it is equivalent to a simple random walk from $(0,0)$ to $(n+k, k-n)$ starting with a negative step, i.e., with $S_1 = -1$ (see Figs. 1 and 2).

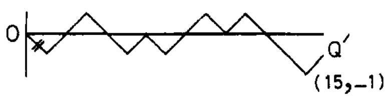

Fig. 2.

We shall also use in the sequel some of the path operations defined by Saran and Sen ((1979), Section 5).

3. Notations

The following symbols will be used.

$E_{m,n}$: a random walk path $(S_0, S_1, \ldots, S_m)$ from $(0,0)$ to (m,n), i.e., with $S_m = n$.

$V^{(t)}$ point: a point (i, S_i) of an $E_{m,n}$ path for which $S_i = t$. This is called a return to the line $y = t$.

$W^{(t)}$ = t-wave: the segment of a path included between two consecutive $V^{(t)}$ points.

$W_+^{(t)}$ ($W_-^{(t)}$): a t-wave with $S_i > t$ ($S_i < t$) at the intervening positions called a positive (negative) t-wave.

$T^{(t)}$ point: a point (i, S_i) of an $E_{m,n}$ path for which $S_i = t$ and $S_{i-1} \cdot S_{i+1} = t^2 - 1$. This is called a crossing or intersection of the line $y = t$.

$S_+^{(t)}$ ($S_-^{(t)}$): the segment of a path included between two consecutive $T^{(t)}$ points with $S_i \geq t$ ($S_i \leq t$) at the intervening positions.

$S_+^{(\cdot)}$: an $S_+^{(t)}$ for some t.

$E_{m,n}^R$: an $E_{m,n}$ path having R runs.

$E_{m,n}^{R+}$ ($E_{m,n}^{R-}$): an $E_{m,n}^R$ path starting with a positive (negative) step.

$E_{m,n,t}^{R+,l}$ ($E_{m,n,t}^{R-,l}$): an $E_{m,n}^{R+}$ ($E_{m,n}^{R-}$) path having l $T^{(t)}$ points.

$E_{m,n,t}^{R+,l,p}$ ($E_{m,n,t}^{R-,l,p}$): an $E_{m,n,t}^{R+,l}$ ($E_{m,n,t}^{R-,l}$) path having p $V^{(t)}$ points.

$F_{m,n}^{2r+,\cdot,\cdot,j}$: a composed path from $(0,0)$ to (m,n) starting with a positive step, having r positive runs and $j+r$ negative runs where the last j runs are negative and the remainder of the runs alternate, and the $(j+1)$-th run from the end being negative.

$F_{m,n}^{2r-,\cdot,\cdot,j}$: a composed path from $(0,0)$ to (m,n) starting with a negative step, having r positive runs and $j+r$ negative runs where the first j runs are negative and the remainder of the runs alternate, and the $(j+1)$-th run from the beginning being negative.

$F_{m,n}^{(2r-1)-,\cdot,\cdot,j}$: a composed path from $(0,0)$ to (m,n) starting with a negative step, having $r-1$ positive runs and $j+r$ negative runs, where the first j runs are negative and the remainder of the runs alternate, and the $(j+1)$-th run from the beginning being negative.

$F_{m,n,t}^{2r+,l,\cdot,j}$: an $F_{m,n}^{2r+,\cdot,\cdot,j}$ path having l $T^{(t)}$ points.

$N(A)$: the number of all possible random walk paths of type A, e.g.,

$$N(E_{m,n}^{2r+}) = \binom{\frac{m+n}{2} - 1}{r-1}\binom{\frac{m-n}{2} - 1}{r-1}.$$

$N[\alpha, \beta, \ldots, N_n]$: the number of lattice paths from $(0,0)$ to (n, N_n) having characteristics $\alpha, \beta, \ldots$, e.g. $N[\phi_n^{(c)} = j,\ Z_n = m,\ N_n = k]$ = the number of lattice paths from $(0,0)$ to (n,k) with $\phi_n^{(c)} = j$ and $Z_n = m$.

4. Some auxiliary results

Srivastava ((1973), (3.1), (3.2)), for short hereafter denoted by Srivastava ((3.1), (3.2)), proved that for $k \geq 0$,

$$(4.1) \quad N(F_{2n,2k,0}^{2r+,0,\cdot,j}) = \binom{n+k-1}{r-1}\binom{n-k-1}{r+j-1} - \binom{n+k-1}{r-2}\binom{n-k-1}{r+j}.$$

We can easily show that for $k \geq 0$,

$$(4.2) \quad N(F_{2n+1,2k+1,0}^{2r+,0,\cdot,j}) = \binom{n+k}{r-1}\binom{n-k-1}{r+j-1} - \binom{n+k}{r-2}\binom{n-k-1}{r+j}.$$

To prove (4.2), we consider the reversed path (i.e. apply the γ-operation (Saran and Sen (1979), Section 5) on an $F^{2r+,0,\cdot,j}_{2n+1,2k+1,0}$ path) and observe that

$$N(F^{2r+,0,\cdot,j}_{2n+1,2k+1,0}) = N(F^{2r-,\cdot,\cdot,j}_{2n+1,2k+1}) - N(F^{2r-,*,\cdot,j}_{2n+1,2k+1,2k+1})$$

where $F^{2r-,*,\cdot,j}_{2n+1,2k+1,2k+1}$ denotes an $F^{2r-,\cdot,\cdot,j}_{2n+1,2k+1}$ path crossing the line $y = 2k+1$ at least once. Let P be the point of the $F^{2r-,*,\cdot,j}_{2n+1,2k+1,2k+1}$ path where its first $(j+1)$ negative runs end. To determine the number of such paths we apply a transformation at the last point of intersection, say Q, of the path with $y = 2k+1$. Change the order of the θ's of the segment from P to Q (i.e., apply γ-operation) and then reflect the remaining portion of the path beyond Q about $y = 2k+1$ (i.e., apply β-operation (Saran and Sen (1979), Section 5)). The result is an $F^{(2r-1)-,\cdot,\cdot,j+1}_{2n+1,2k+1}$ path and the number of such paths is $\binom{n+k}{r-2}\binom{n-k-1}{r+j}$. The number of $F^{2r-,\cdot,\cdot,j}_{2n+1,2k+1}$ paths is obviously $\binom{n+k}{r-1}\binom{n-k-1}{r+j-1}$. This leads to the required result (4.2).

Combining (4.1) and (4.2), we have the following

LEMMA 4.1. *For $n \geq 0$,*

$$N(F^{2r+,0,\cdot,j}_{m,n,0}) = \binom{\frac{m+n}{2}-1}{r-1}\binom{\frac{m-n}{2}-1}{r+j-1} - \binom{\frac{m+n}{2}-1}{r-2}\binom{\frac{m-n}{2}-1}{r+j}. \tag{4.3}$$

Using (4.3) we now prove the following

LEMMA 4.2. *For $t > n$,*

$$N(E^{2r-,0,p}_{m,n,t}) = \begin{cases} \binom{\frac{m+n}{2}-t-1}{r-1}\binom{\frac{m-n}{2}+t-p-2}{r-p-1} - \binom{\frac{m+n}{2}-t-1}{r}\binom{\frac{m-n}{2}+t-p-2}{r-p-2}, & p \geq 1 \quad (4.4) \\ \binom{\frac{m+n}{2}-1}{r-1}\binom{\frac{m-n}{2}-1}{r-1} - \binom{\frac{m+n}{2}-t}{r}\binom{\frac{m-n}{2}+t-2}{r-2}, & p = 0, \quad (4.5) \end{cases}$$

$$N(E_{m,n,t}^{(2r+1)-,0,p})$$

$$= \begin{cases} \binom{\frac{m+n}{2}-t-1}{r-1}\binom{\frac{m-n}{2}+t-p-2}{r-p} & \quad (4.6) \\ \qquad -\binom{\frac{m+n}{2}-t-1}{r}\binom{\frac{m-n}{2}+t-p-2}{r-p-1}, & p \geq 1 \\ \binom{\frac{m+n}{2}-1}{r-1}\binom{\frac{m-n}{2}-1}{r} & \quad (4.7) \\ \qquad -\binom{\frac{m+n}{2}-t}{r}\binom{\frac{m-n}{2}+t-2}{r-1}, & p = 0, \end{cases}$$

and for $t = n > 0$,

$$N(E_{m,n,n}^{2r-,0,p}) = \begin{cases} \binom{\frac{m+n}{2}-p-1}{r-p}\binom{\frac{m-n}{2}-1}{r-1} & \\ \qquad -\binom{\frac{m+n}{2}-p-1}{r-p-1}\binom{\frac{m-n}{2}-1}{r}, & p \geq 1 \\ 0, & p = 0. \end{cases} \tag{4.8}$$

PROOF. To prove (4.4) for $p \geq 1$, let $OO_1O_2O_3P_1Q_1O_4P_2Q_2\cdots R$ (Fig. 3) be an $E_{m,n,t}^{2r-,0,p}$ path $(t > n)$ with O_1 as the point where its first negative run ends and O_2 and O_3 be the first $T^{(t-1)}$ and the first $V^{(t)}$ points, respectively, thus dividing the path into four segments, viz., OO_1, O_1O_2, O_2O_3 and O_3R. Now we apply the following transformation to the segment O_1R, shifting its starting point O_1 to the origin, and then attach the segment OO_1 at the end of the transformed segment O_1R. The last segment so attached is counted as a separate run (as done in a composed path).

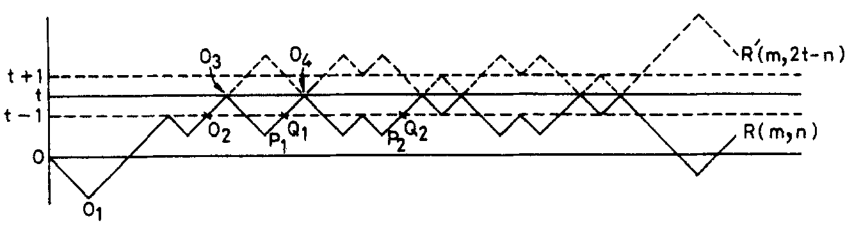

Fig. 3.

By reversing the order of θ's of the segment O_1O_2 (i.e., applying γ-operation), we get the segment O_2O_1 (see Fig. 4) which does not cross the x-axis. Remove

O_2O_3 and reflect about $y = t$ the segment O_3R (i.e. apply β-operation) as shown by the dotted lines in Fig. 3. Now the segment O_3R thus obtained contains $(p-1)$ $W_+^{(t)}$. Let h $(0 \le h \le p-1)$ out of $(p-1)$ $W_+^{(t)}$ be of length two each, i.e., each having two runs as well. Draw a line $y = t+1$ and remove the portions of the path between $y = t$ and $y = t+1$. Then, joining the remaining segments end-to-end in order, at the end of the segment O_2O_1, we get an $F_{m-2p,2t-n-2,0}^{2(r-h-1)+,0,\cdot,1}$ path having $(p-h-1)$ $S_+^{(\cdot)}$ with terminal points at $Q_1, Q_2, \ldots, Q_{p-h-1}$ (see Fig. 4).

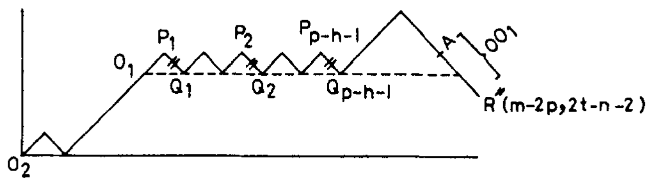

Fig. 4.

We further transform the path so obtained (Fig. 4) as follows. Keep the segment $O_2O_1P_1$ unaltered. Then after removing the portions $P_1Q_1, P_2Q_2, \ldots,$ $P_{p-h-1}Q_{p-h-1}$, attach in order, the remaining segments $Q_1P_2, Q_2P_3, \ldots,$ $Q_{p-h-2}P_{p-h-1}$, and $Q_{p-h-1}A$ to the end P_1. Then attach to it in order, the segments $P_{p-h-1}Q_{p-h-1}, \ldots, P_2Q_2, P_1Q_1$ and AR'' (see Fig. 5). Part of the transformed path between the origin and its last turning point is considered to be an ordinary path (where a "turning point" is a point where the path changes its direction from positive to negative or vice versa). In the remaining part, the run end points are kept as they are, moreover each straight segment thus attached forms a separate run. The result is an $F_{m-2p,2t-n-2,0}^{2(r-p)+,0,\cdot,p-h}$ path (see Fig. 5). A one-to-one correspondence can be easily established (cf. Srivastava ((1973), p. 215)). Hence, since any h out of $(p-1)$ $W_+^{(t)}$ can be of length two each,

$$N(E_{m,n,t}^{2r-,0,p}) = \sum_{h=0}^{p-1} \binom{p-1}{h} N(F_{m-2p,2t-n-2,0}^{2(r-p)+,0,\cdot,p-h}), \qquad p \ge 1$$

leading to (4.4) using (4.3). Likewise (4.6) and (4.8) can be established. To derive (4.5) and (4.7), let $E_{m,n,t}^{R-,*}$ be an $E_{m,n}^{R-}$ path crossing the line $y = t$ at least once. Then from results (2.3) and (2.6) of Vellore (1972), we have

$$N(E_{m,n,t}^{(2r+1)-,*}) = \binom{\frac{m-n}{2}+t-1}{r-1}\binom{\frac{m+n}{2}-t-1}{r} \tag{4.9}$$

and

$$N(E_{m,n,t}^{2r-,*}) = \binom{\frac{m-n}{2}+t-1}{r-2}\binom{\frac{m+n}{2}-t-1}{r}, \tag{4.10}$$

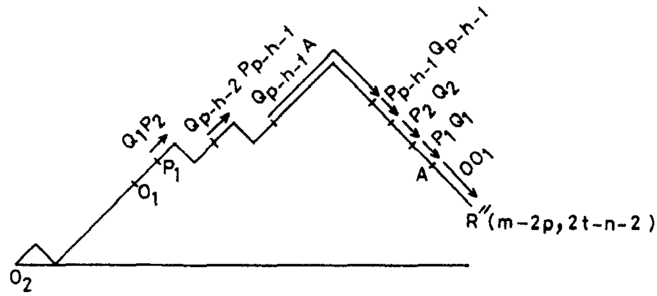

Fig. 5.

respectively. Thus, for $p = 0$

$$N(E^{2r-,0,0}_{m,n,t}) = N(E^{2r-}_{m,n}) - N(E^{2r-,*}_{m,n,t-1}),$$

leading to (4.5) using (4.10). Similarly (4.7) follows immediately from (4.9). This completes the proof of Lemma 4.2.

In like manner, one can easily prove the following

LEMMA 4.3. *For $n > 0$, $p \geq 0$*

$$N(E^{2r+,0,p}_{m,n,0}) = \binom{\frac{m+n}{2} - p - 2}{r-p-1}\binom{\frac{m-n}{2} - 1}{r-1} - \binom{\frac{m+n}{2} - p - 2}{r-p-2}\binom{\frac{m-n}{2} - 1}{r}, \tag{4.11}$$

$$N(E^{(2r+1)+,0,p}_{m,n,0}) = \binom{\frac{m+n}{2} - p - 2}{r-p}\binom{\frac{m-n}{2} - 1}{r-1} - \binom{\frac{m+n}{2} - p - 2}{r-p-1}\binom{\frac{m-n}{2} - 1}{r} \tag{4.12}$$

and

$$N(E^{2r-,0,p}_{2n,0,0}) = \frac{p}{r}\binom{n-p-1}{r-p}\binom{n-1}{r-1}. \tag{4.13}$$

5. Some joint distributions

THEOREM 5.1. *For $c = 0, 1, 2, \ldots$ and $0 \leq j \leq m \leq n - j$,*

$$\binom{n+k-1}{n-1} P\{\phi^{(c)}_n = j, Z_n = m \mid N_n = k\} = \binom{n+c}{n-m-j}\binom{k-c-1}{n-m+j-1} \tag{5.1}$$

$$- \binom{n+c}{n-m-j-1}\binom{k-c-1}{n-m+j}, \qquad 0 \le k < n+c$$

and

$$\binom{n+k-1}{n-1} P\{\phi_n^{(c)} = j, Z_n = m \mid N_n = k\} \tag{5.2}$$
$$= \binom{k-1}{n-m-j-1}\binom{n}{m-j}$$
$$- \binom{k-1}{n-m-j-2}\binom{n}{m-j-1}, \qquad k \ge n+c,$$

provided the left-hand sides are defined.

PROOF. To prove (5.1), we have for $0 \le k < n+c$

$$\begin{aligned} N[\phi_n^{(c)} = j, Z_n = m, N_n = k] &= N[\phi_n^{(c)} = j, Z_n = m, \varepsilon_n > 0, N_n = k] \\ &\quad + N[\phi_n^{(c)} = j, Z_n = m, \varepsilon_n = 0, N_n = k]. \end{aligned} \tag{5.3}$$

The first factor on the right-hand side of (5.3) involves the enumeration of lattice paths from $(0,0)$ to (n,k), $0 \le k < n+c$, having exactly j horizontal crossings of the line $N_r = r + c$ and $2n - 2m - 1$ changes from horizontal to vertical direction and vice versa (see Fig. 6). Applying the "rotation procedure" (see Section 2) to the path in Fig. 6, the result is an $E_{n+k,k-n,c}^{2(n-m)-,2j}$ path (see Fig. 7). In a similar manner it can be shown that the second factor on the right-hand side of (5.3) equals $N(E_{n+k,k-n,c}^{(2n-2m+1)-,2j})$. Hence for $0 \le k < n+c$

$$N[\phi_n^{(c)} = j, Z_n = m, N_n = k] = N(E_{n+k,k-n,c}^{2(n-m)-,2j}) + N(E_{n+k,k-n,c}^{(2n-2m+1)-,2j}),$$

leading to (5.1) using Srivastava ((4.12), (4.13)) which are given for even $n + k$ and $k - n$ but can easily be proved to be true for any $n + k$ and $k - n$.

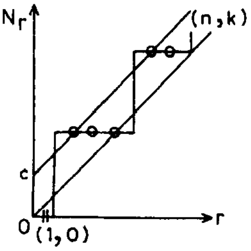

Fig. 6.

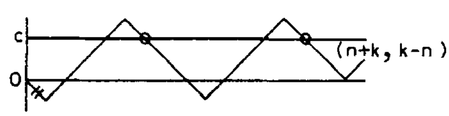

Fig. 7.

To prove (5.2), we have for $k > n + c$

$$
\begin{aligned}
N[\phi_n^{(c)} = j, Z_n = m, N_n = k] &= N[\phi_n^{(c)} = j, Z_n = m, \varepsilon_n > 0, N_n = k] \\
&\quad + N[\phi_n^{(c)} = j, Z_n = m, \varepsilon_n = 0, N_n = k] \\
&= N(E_{n+k,k-n,c}^{2(n-m)-,2j+1}) + N(E_{n+k,k-n,c}^{(2n-2m+1)-,2j+1}),
\end{aligned}
$$

leading to (5.2), for $k > n + c$, using Srivastava (4.2) and Srivastava ((4.4) for $j = 0$). And for $k = n + c$

$$
\begin{aligned}
N[\phi_n^{(c)} = j, Z_n = m, N_n = n + c] &= N[\phi_n^{(c)} = j, Z_n = m, \varepsilon_n > 0, N_n = n + c] \\
&\quad + N[\phi_n^{(c)} = j, Z_n = m, \varepsilon_n = 0, N_n = n + c] \\
&= N(E_{2n+c,c,c}^{2(n-m)-,2j}) + N(E_{2n+c,c,c}^{(2n-2m+1)-,2j+1}),
\end{aligned}
$$

leading to (5.2), for $k = n + c$, using Srivastava ((4.7), (4.9)). This completes the proof of Theorem 5.1.

Deductions. (i) For $c = 0$, (5.1) and (5.2) verify result (6) in Saran (1977).

(ii) Summing (5.1) and (5.2) each over m and using the summation formula in Feller ((1968), Chapter II(12.9)), for short hereafter denoted by Feller (12.9), we get for $c = 0, 1, 2, \ldots$

$$
\begin{aligned}
(5.4) \qquad & \binom{n+k-1}{n-1} P\{\phi_n^{(c)} = j \mid N_n = k\} \\
& = \frac{n-k+2c+4j+1}{n+k+1} \binom{n+k+1}{k-c-2j}, \qquad 0 \le k < n + c
\end{aligned}
$$

and

$$
\begin{aligned}
(5.5) \qquad & \binom{n+k-1}{n-1} P\{\phi_n^{(c)} = j \mid N_n = k\} \\
& = \frac{k-n+4j+3}{n+k+1} \binom{n+k+1}{n-2j-1}, \qquad k \ge n + c,
\end{aligned}
$$

which respectively verify results (6) and (7) in Saran and Sen (1979) for $c = 0$.

THEOREM 5.2. *For* $c = 1, 2, \ldots$

$$
\begin{aligned}
(5.6) \qquad & \binom{n+k-1}{n-1} P\{\phi_n^{(-c)} = j, Z_n = m \mid N_n = k\} \\
& = \binom{k-1}{n-m+j-2}\binom{n}{m+j-1} \\
& \quad - \binom{k-1}{n-m+j-1}\binom{n}{m+j}, \qquad 0 \le k < n - c
\end{aligned}
$$

and

$$(5.7)\quad \binom{n+k-1}{n-1}P\{\phi_n^{(-c)}=j, Z_n=m \mid N_n=k\}$$
$$=\binom{n-c}{n-m+j-1}\binom{k+c-1}{n-m-j}$$
$$-\binom{n-c}{n-m+j}\binom{k+c-1}{n-m-j-1},\qquad k\geq n-c$$

provided the left-hand sides are defined.

PROOF. For $0\leq k<n-c$, we have

$$(5.8)\quad N[\phi_n^{(-c)}=j, Z_n=m, N_n=k]$$
$$=N[\phi_n^{(-c)}=j, Z_n=m, \varepsilon_n>0, N_n=k]$$
$$+N[\phi_n^{(-c)}=j, Z_n=m, \varepsilon_n=0, N_n=k].$$

The first (second) factor on the right-hand side of (5.8) involves the enumeration of lattice paths from $(0,0)$ to (n,k), $0\leq k<n-c$, having exactly j horizontal crossings of the line $N_r=r-c$, and $2n-2m-1$ $(2n-2m)$ changes from horizontal to vertical direction and vice versa (see Figs. 8 and 9). Applying the "rotation procedure", the right-hand side of (5.8) equals

$$N(E_{n+k,-(n-k),-c}^{2(n-m)-,2j-1})+N(E_{n+k,-(n-k),-c}^{(2n-2m+1)-,2j-1}),$$

which applying the β-operation reduces to

$$N(E_{n+k,n-k,c}^{2(n-m)+,2j-1})+N(E_{n+k,n-k,c}^{(2n-2m+1)-,2j-1}),$$

leading to (5.6) using Srivastava ((4.2), (4.3)).

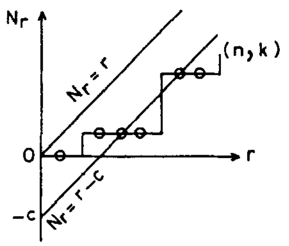

Fig. 8.

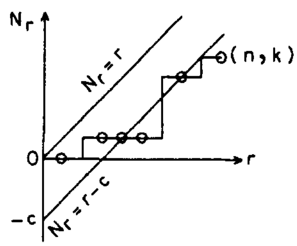

Fig. 9.

To derive (5.7), there arise the following three contingencies corresponding to different values of k, viz. (i) $k=n-c$, (ii) $n-c<k\leq n$, (iii) $k>n$. Using similar arguments as in the proof for (5.6), we have for $k=n-c$,

$$N[\phi_n^{(-c)}=j, Z_n=m, N_n=n-c]$$
$$=N(E_{2n-c,c,c}^{2(n-m)+,2j-1})+N(E_{2n-c,c,c}^{(2n-2m+1)+,2j})$$

and, for $n - c < k \le n$,

$$N[\phi_n^{(-c)} = j, Z_n = m, N_n = k]$$
$$= N(E_{n+k,n-k,c}^{2(n-m)+,2j}) + N(E_{n+k,n-k,c}^{(2n-2m+1)+,2j}),$$

leading to (5.7) for $k = n - c$ and $n - c < k \le n$, respectively, using Srivastava ((4.6), (4.8), (4.11), (4.13)).

For $k > n$, we have

$$N[\phi_n^{(-c)} = j, Z_n = m, N_n = k]$$
$$= N(E_{n+k,k-n,-c}^{2(n-m)-,2j}) + N(E_{n+k,k-n,-c}^{(2n-2m+1)-,2j}). \tag{5.9}$$

To evaluate the first factor on the right-hand side of (5.9), let $OQ_1P_1Q_2P_2Q_3 \cdots P_{2j-1}Q_{2j}R$ (Fig. 10) denote an $E_{n+k,k-n,-c}^{2(n-m)-,2j}$ path where $Q_1, Q_2, \ldots, Q_{2j}$ are the points of intersection of the path with the line $y = -c$ and $P_1, P_2, \ldots, P_{2j-1}$ as the last turning points of the path before intersecting the line $y = -c$ at $Q_2, \ldots, Q_{2j}$. Let the coordinates of Q_1 be $(q, -c)$. Now we apply the following transformation. Concerning the section of the path between O to $Q_1(q, -c)$, let us first alter the signs and then the direction, i.e., we replace $\theta_1, \theta_2, \ldots, \theta_q$ by $-\theta_q, -\theta_{q-1}, \ldots, -\theta_1$. Then, to the end of the transformed segment OQ_1 attach in order the segments $Q_1P_1, Q_2P_2, \ldots, Q_{2j-1}P_{2j-1}, Q_{2j}R$ with the signs of the θ's changed for all such segments lying below the line $y = -c$. Then attach to this path, in order, the remaining segments $P_{2j-1}Q_{2j}, P_{2j-2}Q_{2j-1}, \ldots, P_1Q_2$, again, such that the direction of all those segments lying below the line $y = -c$ is changed. As in the proof of Lemma 4.2, part of the transformed path (Fig. 11) between the origin and its last turning point is considered to be an ordinary path. In the remaining part the run end points are kept as they are, moreover, each section attached forms a separate run. The result is an $F_{n+k,k-n+2c,0}^{2(n-m-j+1)+,0,\cdot,2j-2}$ path (see Fig. 11). By reversing the procedure a one-to-one correspondence can be easily verified.

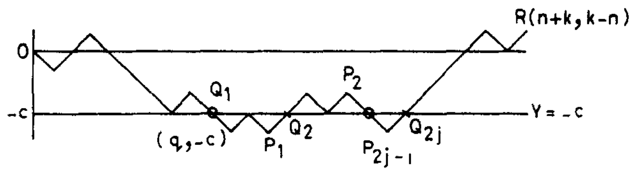

Fig. 10.

In a similar manner a one-to-one correspondence between $E_{n+k,k-n,-c}^{(2n-2m+1)-,2j}$ and $F_{n+k,k-n+2c,0}^{2(n-m-j+1)+,0,\cdot,2j-1}$ paths can be established. Thus the right-hand side of (5.9) equals

$$N(F_{n+k,k-n+2c,0}^{2(n-m-j+1)+,0,\cdot,2j-2}) + N(F_{n+k,k-n+2c,0}^{2(n-m-j+1)+,0,\cdot,2j-1}),$$

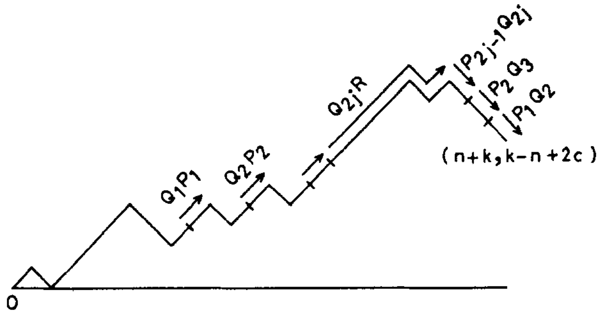

Fig. 11.

which, using (4.3) leads to (5.7) for $k > n$. This completes the proof of Theorem 5.2.

Deduction. Summing (5.6) and (5.7) each over m and using Feller (12.9), we get for $c = 1, 2, \ldots$

$$\binom{n+k-1}{n-1} P\{\phi_n^{(-c)} = j \mid N_n = k\} = \frac{n-k+4j-3}{n+k+1}\binom{n+k+1}{n+2j-1}, \quad 0 \le k < n-c \tag{5.10}$$

and

$$\binom{n+k-1}{n-1} P\{\phi_n^{(-c)} = j \mid N_n = k\} = \frac{k-n+2c+4j-1}{n+k+1}\binom{n+k+1}{k+c+2j}, \quad k \ge n-c. \tag{5.11}$$

THEOREM 5.3. *For* $0 \le k < n + c$,

$$\binom{n+k-1}{n-1} P\{\Lambda_n^{(c)} = j, \Delta_n^{*(c)} = n, Z_n = m \mid N_n = k\} = \begin{cases} \dbinom{n+c-j-1}{m+c-1}\dbinom{k-c-1}{n-m-1} \\ \qquad -\dbinom{n+c-j-1}{m+c}\dbinom{k-c-1}{n-m}, & j \ge 1 \\ \dbinom{k-1}{n-m-1}\dbinom{n}{n-m} \\ \qquad -\dbinom{n+c-1}{n-m-1}\dbinom{k-c}{n-m}, & j = 0 \end{cases} \tag{5.12}$$

and for $k = n + c$,

$$\binom{2n+c-1}{n-1} P\{\Lambda_n^{(c)} = j, \Delta_n^{*(c)} = n, Z_n = m \mid N_n = n+c\} \tag{5.13}$$

$$= \begin{cases} \binom{n+c-j-1}{m+c-1}\binom{n-1}{m} \\ \qquad - \binom{n+c-j-1}{m+c}\binom{n-1}{m-1}, & j \geq 1 \\ 0, & j = 0 \end{cases}$$

where $c = 1, 2, \ldots$ *and* $0 < m \leq n - j$.

PROOF. For $0 \leq k < n + c$, we have

$$\begin{aligned} (5.14) \qquad & N[\Lambda_n^{(c)} = j, \Delta_n^{*(c)} = n, Z_n = m, N_n = k] \\ &= N[\Lambda_n^{(c)} = j, \Delta_n^{*(c)} = n, Z_n = m, \varepsilon_n > 0, N_n = k] \\ &\quad + N[\Lambda_n^{(c)} = j, \Delta_n^{*(c)} = n, Z_n = m, \varepsilon_n = 0, N_n = k]. \end{aligned}$$

The first factor on the right-hand side of (5.14) involves the enumeration of lattice paths from $(0,0)$ to (n,k), $0 \leq k < n+c$, having exactly j vertical contacts with the line $N_r = r + c$, $(2n - 2m - 1)$ changes from horizontal to vertical direction and vice versa, and never rising above the line $N_r = r + c$ (see Fig. 12). Applying the "rotation procedure" to the path in Fig. 12, the result is an $E_{n+k,k-n,c}^{2(n-m)-,0,j}$ path (Fig. 13). Similarly, the second factor on the right-hand side of (5.14) can be shown to be equal to $N(E_{n+k,k-n,c}^{(2n-2m+1)-,0,j})$. Thus for $0 \leq k < n + c$

$$\begin{aligned} & N[\Lambda_n^{(c)} = j, \Delta_n^{*(c)} = n, Z_n = m, N_n = k] \\ &= N(E_{n+k,k-n,c}^{2(n-m)-,0,j}) + N(E_{n+k,k-n,c}^{(2n-2m+1)-,0,j}), \end{aligned}$$

leading to (5.12) using (4.4) to (4.7).

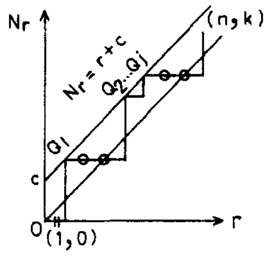

Fig. 12.

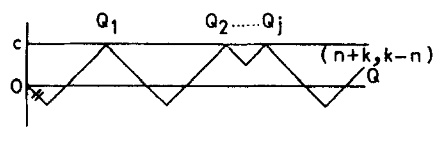

Fig. 13.

Now $N_n = n + c$ and $\Delta_n^{*(c)} = n$ imply that $\varepsilon_n > 0$. Thus by the "rotation procedure"

$$\begin{aligned} & N[\Lambda_n^{(c)} = j, \Delta_n^{*(c)} = n, Z_n = m, N_n = n + c] \\ &= N[\Lambda_n^{(c)} = j, \Delta_n^{*(c)} = n, Z_n = m, \varepsilon_n > 0, N_n = n + c] \\ &= N(E_{2n+c,c,c}^{2(n-m)-,0,j}), \end{aligned}$$

leading to (5.13) using (4.8).

Deductions. (i) Summing (5.12) and (5.13) each over j from 0 to $n-m$ and using Feller (12.6), we get for $c = 1, 2, \dots$

$$\binom{n+k-1}{n-1} P\left\{\max_{1\le r\le n}(N_r - r) \le c, Z_n = m \mid N_n = k\right\} \tag{5.15}$$
$$= \begin{cases} \binom{k-1}{n-m-1}\binom{n}{n-m} \\ \qquad -\binom{n+c}{n-m-1}\binom{k-c-1}{n-m}, & 0 \le k < n+c \\ \binom{n+c-1}{n-m-1}\binom{n-1}{n-m-1} \\ \qquad -\binom{n+c-1}{n-m-2}\binom{n-1}{n-m}, & k = n+c, \end{cases}$$

which is in agreement with (12) in Saran (1977).

(ii) Summing (5.12) and (5.13) each over m and using Feller (12.9), we get, respectively, for $c = 1, 2, \dots$ and $0 \le k < n+c$,

$$\binom{n+k-1}{n-1} P\{\Lambda_n^{(c)} = j, \Delta_n^{*(c)} = n \mid N_n = k\} \tag{5.16}$$
$$= \begin{cases} \dfrac{n-k+j+2c}{n+k-j}\dbinom{n+k-j}{n+c}, & j \ge 1 \\ \dbinom{n+k-1}{n-1} - \dbinom{n+k-1}{n+c}, & j = 0 \end{cases}$$

and

$$\binom{2n+c-1}{n-1} P\{\Lambda_n^{(c)} = j, \Delta_n^{*(c)} = n \mid N_n = n+c\} \tag{5.17}$$
$$= \begin{cases} \dfrac{j+c}{2n+c-j}\dbinom{2n+c-j}{n+c}, & j \ge 1 \\ 0, & j = 0. \end{cases}$$

(iii) Setting $j = 0$ in (5.12) and (5.13), we get for $c = 1, 2, \dots$

$$\binom{n+k-1}{n-1} P\left\{\max_{1\le r\le n}(N_r - r) < c, Z_n = m \mid N_n = k\right\} \tag{5.18}$$
$$= \binom{k-1}{n-m-1}\binom{n}{n-m}$$
$$- \binom{n+c-1}{n-m-1}\binom{k-c}{n-m}, \quad 0 \le k \le n+c.$$

(iv) Summing (5.15) over m and using Feller (12.9) or summing (5.16) and

(5.17) each over j from 0 to n and using Feller (12.6), we get for $c = 1, 2, \ldots$

$$(5.19) \qquad P\{N_r \le r + c \text{ for } r = 1, 2, \ldots, n \mid N_n = k\} = 1 - \frac{\binom{n+k-1}{n+c+1}}{\binom{n+k-1}{n-1}},$$
$$0 \le k \le n + c.$$

(v) Summing (5.18) over m and using Feller (12.9) or setting $j = 0$ in (5.16) and (5.17), we get for $c = 1, 2, \ldots$

$$(5.20) \qquad P\{N_r < r + c \text{ for } r = 1, 2, \ldots, n \mid N_n = k\} = 1 - \frac{\binom{n+k-1}{n+c}}{\binom{n+k-1}{n-1}},$$
$$0 \le k \le n + c.$$

THEOREM 5.4. *For $0 < m \le n - j$ and $0 \le k < n$,*

$$(5.21) \qquad \begin{aligned} &\binom{n+k-1}{n-1} P\{\Lambda_n^{(0)} = j, \Delta_n^{*(0)} = n, Z_n = m \mid N_n = k\} \\ &\quad = \binom{n-j-1}{n-m-j}\binom{k-1}{n-m-1} \\ &\quad - \binom{n-j-1}{n-m-j-1}\binom{k-1}{n-m}, \qquad j \ge 0 \end{aligned}$$

and

$$(5.22) \qquad \begin{aligned} &\binom{2n-1}{n-1} P\{\Lambda_n^{(0)} = j, \Delta_n^{*(0)} = n, Z_n = m \mid N_n = n\} \\ &\quad = \frac{j}{n-m}\binom{n-j-1}{n-m-j}\binom{n-1}{n-m-1}, \qquad j \ge 0. \end{aligned}$$

PROOF. Similarly as above, we have by the "rotation procedure", for $0 \le k < n$,

$$\begin{aligned} N[\Lambda_n^{(0)} = j, \Delta_n^{*(0)} = n, Z_n = m, N_n = k] \\ = N(E_{n+k,-(n-k),0}^{2(n-m)-,0,j}) + N(E_{n+k,-(n-k),0}^{(2n-2m+1)-,0,j}) \\ = N(E_{n+k,n-k,0}^{2(n-m)+,0,j}) + N(E_{n+k,n-k,0}^{(2n-2m+1)+,0,j}), \end{aligned}$$

by β-operation, leading to (5.21) using (4.11) and (4.12).

Further, it is obvious that

$$\begin{aligned} N[\Lambda_n^{(0)} = j, \Delta_n^{*(0)} = n, Z_n = m, N_n = n] \\ = N[\Lambda_n^{(0)} = j, \Delta_n^{*(0)} = n, Z_n = m, \varepsilon_n > 0, N_n = n] \\ = N(E_{2n,0,0}^{2(n-m)-,0,j}), \end{aligned}$$

leading to (5.22), using (4.13).

Deductions. (i) Summing (5.21) and (5.22) each over j from 0 to $n-m$ and using Feller ((12.6), (12.16)), we get

$$\binom{n+k-1}{n-1} P\left\{\max_{1\le r\le n}(N_r - r) \le 0, Z_n = m \mid N_n = k\right\} = \binom{k-1}{n-m-1}\binom{n}{n-m} - \binom{k-1}{n-m}\binom{n}{n-m-1}, \quad 0 \le k \le n. \tag{5.23}$$

Comparing (5.15) and (5.23) we observe that (5.15) holds good for $c = 0$ too.

(ii) Summing (5.21) and (5.22) each over m and using Feller (12.9), we get, for $0 \le k \le n$,

$$\binom{n+k-1}{n-1} P\{\Lambda_n^{(0)} = j, \Delta_n^{*(0)} = n \mid N_n = k\} = \frac{n-k+j}{n+k-j}\binom{n+k-j}{n}, \quad j \ge 0. \tag{5.24}$$

(iii) Summing (5.23) over m and using Feller (12.9), we get

$$P\{N_r \le r \text{ for } r = 1, 2, \ldots, n \mid N_n = k\} = 1 - \frac{k(k-1)}{n(n+1)} \quad \text{for} \quad 0 \le k \le n. \tag{5.25}$$

(iv) Setting $j = 0$ in (5.21) and (5.22), we get for $0 < m \le n$,

$$\binom{n+k-1}{n-1} P\left\{\max_{1\le r\le n}(N_r - r) < 0, Z_n = m \mid N_n = k\right\} = \binom{n-1}{n-m}\binom{k-1}{n-m-1} - \binom{n-1}{n-m-1}\binom{k-1}{n-m}, \quad 0 \le k \le n. \tag{5.26}$$

6. Application of the results in deriving ballot problems

Summing (5.26) over m and using Feller (12.9) or setting $j = 0$ in (5.24), we get

$$P\{N_r < r \text{ for } r = 1, \ldots, n \mid N_n = k\} = 1 - \frac{k}{n}, \quad \text{for} \quad k = 0, 1, \ldots, n, \tag{6.1}$$

verifying Takács' lemma ((1970), (1), p. 360) only for i.i.d. random variables, which is a generalization of the classical ballot theorem formulated below.

Suppose that in a ballot candidate A scores a votes and candidate B scores b votes and that all possible $\binom{a+b}{a}$ voting records are equally probable. Denote by α_r and β_r the number of votes registered for A and B, respectively, among the first r votes recorded. Let μ be a non-negative integer. Define the random variables ν_r, $r = 1, 2, \ldots, a+b$, as follows: $\nu_r = 0$ if the r-th vote is the cast for A and $\nu_r = (\mu+1)$ if the r-th vote is the cast for B. Then $\nu_1, \nu_2, \ldots, \nu_{a+b}$ are i.i.d. random variables, taking non-negative integers for which $\nu_1 + \nu_2 + \cdots + \nu_{a+b} = b(\mu+1)$. Set $N_r = \nu_1 + \nu_2 + \cdots + \nu_r$ for $r = 1, 2, \ldots, a+b$ and $N_0 = 0$. Since $N_r = (\mu+1)\beta_r$ and $r = \alpha_r + \beta_r$ for $r = 1, 2, \ldots, a+b$, the inequality $\alpha_r > \mu\beta_r$ holds if and only if $N_r < r$. Thus putting $n = a+b$ and $k = b(\mu+1)$ in (6.1), we get

$$P\{\alpha_r > \mu\beta_r \text{ for } r = 1, 2, \ldots, a+b \mid N_{a+b} = b(\mu+1)\} = \frac{a-\mu b}{a+b}, \tag{6.2}$$

for $a \geq \mu b$, thus verifying the classical ballot theorem (Takács (1967), (1), p. 2).

Similarly other results may also be applied in deriving the generalized ballot problems, The random variables $\Lambda_n^{(c)}$, $\phi_n^{(c)}$, $\phi_n^{(-c)}$, $\Delta_n^{*(c)}$ and Z_n are equivalent to certain characteristics of the ballot problem as follows:

$\Lambda_{a+b}^{(c)}$: the number of subscripts $r = 1, 2, \ldots, a+b$ for which $\alpha_r = \mu\beta_r - c$,

$\phi_{a+b}^{(\pm c)}$: the number of subscripts $r = 1, 2, \ldots, a+b$ for which $\alpha_r = \mu\beta_r \mp c$ but $\alpha_{r-1} = \mu\beta_{r-1} \mp c - 1$ and $\beta_{r-1} = \beta_r$,

$\Delta_{a+b}^{*(c)}$: the number of subscripts $r = 1, 2, \ldots, a+b$ for which $\alpha_r \geq \mu\beta_r - c$,

Z_{a+b}: the number of subscripts $r = 1, 2, \ldots, a+b$ for which $\beta_{r-1} = \beta_r$.

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