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STATISTICAL INFERENCE BASED ON ALIGNED RANKS FOR TWO-WAY MANOVA WITH INTERACTION

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Abstract. Multiresponse experiments in two-way layouts with interactions, having equal number of observations per cell, are considered. Robust procedures based on aligned ranks for statistical inference of interactions, main effects and an overall mean response in the models are proposed. Large sample properties of the proposed tests, estimators and confidence regions as the cell size tends to infinity are investigated. For the univariate case, it is found that the asymptotic relative efficiencies (ARE's) of the proposed procedures relative to classical procedures agree with the ARE-results of the two-sample rank test relative to the t-test. In addition, robustness due to Huber (1981, *Robust Statistics*, Wiley, New York) can be drawn.

Key words and phrases: Asymptotically distribution-free procedure, multivariate analysis, test, point estimate, confidence region, asymptotic efficiency, robustness.

1. Introduction

For the present paper, we consider a two-way MANOVA model with interaction, having equal number of observations per cell. For the two-way model, the k-th observation $\boldsymbol{X}_{ijk} = (X_{ijk}^{(1)}, \dots, X_{ijk}^{(p)})'$ in the i-th level of the first factor and j-th level of the second factor is expressed as

$$\boldsymbol{X}_{ijk} = \boldsymbol{\mu} + \boldsymbol{\alpha}_i + \boldsymbol{\beta}_j + (\boldsymbol{\alpha\beta})_{ij} + \boldsymbol{e}_{ijk}, \quad (i = 1, \dots, I,\ j = 1, \dots, J,\ k = 1, \dots, n) \tag{1.1}$$

where $\sum_{i=1}^{I} \boldsymbol{\alpha}_i = \sum_{j=1}^{J} \boldsymbol{\beta}_j = \boldsymbol{0}$ and $\sum_{i=1}^{I} (\boldsymbol{\alpha\beta})_{ij} = \sum_{j=1}^{J} (\boldsymbol{\alpha\beta})_{ij} = \boldsymbol{0}$ for all i, j's. In (1.1), $\boldsymbol{\mu}$ is the overall mean response, $\boldsymbol{\alpha}_i$ is the effect of the i-th level of the first factor, $\boldsymbol{\beta}_j$ is the effect of the j-th level of the second factor, $(\boldsymbol{\alpha\beta})_{ij}$ is the interaction between the i-th level of the first factor and the j-th level of the second factor, and $\boldsymbol{e}_{ijk}$ is the error term with mean $\boldsymbol{0}$ and a positive-definite covariance matrix. The terms $\boldsymbol{\alpha}_i$ and $\boldsymbol{\beta}_j$ are also called main effects. It is assumed that $\boldsymbol{e}_{ijk}$'s are independent and identically distributed with continuous distribution function

$F(\boldsymbol{x})$. For the respective parameters, the null hypotheses of interest and the alternatives are respectively

$$(1.2)\qquad H;\ (\boldsymbol{\alpha\beta})_{ij}=\mathbf{0}\ \text{ for } i=1,\ldots,I \text{ and } j=1,\ldots,J \quad \text{v.s.}\quad A;\ (\boldsymbol{\alpha\beta})_{ij}\neq\mathbf{0} \text{ for some } (i,j),$$

$$(1.3)\qquad H^*;\ \boldsymbol{\alpha}_i=\mathbf{0}\ \text{ for } i=1,\ldots,I \quad \text{v.s.}\quad A^*;\ \boldsymbol{\alpha}_i\neq\mathbf{0} \text{ for some } i,$$

$$(1.4)\qquad H';\ \boldsymbol{\beta}_j=\mathbf{0}\ \text{ for } j=1,\ldots,J \quad \text{v.s.}\quad A';\ \boldsymbol{\beta}_j\neq\mathbf{0} \text{ for some } j,$$

and

$$(1.5)\qquad H^+;\ \boldsymbol{\mu}=\mathbf{0} \quad \text{v.s.}\quad A^+;\ \boldsymbol{\mu}\neq\mathbf{0}.$$

Sen and Puri (1977) proposed multivatiate aligned rank tests for the full rank linear models and investigated the asymptotic properties of the proposed tests. However, the linear models do not include our model (1.1) which is not a full rank model. For the two-way ANOVA (MANOVA) models without interaction, rank test procedures were proposed by Friedman (1937), Mehra and Sarangi (1967), Sen (1969), Mack and Skillings (1980) and others. Also, R-estimators for contrasts of treatment effects were proposed by Lehmann (1964), Puri and Sen (1967, 1968) and confidence regions based on the R-estimators were discussed by Puri and Sen (1967). Then, most of them investigated the asymptotic properties of these statistics as the number of blocks tends to infinity. On the other hand, Shiraishi (1989*a*) proposed the extended aligned rank tests, the Friedman-type tests (within-block rank tests) and the R-estimators of treatment effects for the two-way MANOVA models without interaction, and showed the asymptotic equivalence of the statistical inference based on aligned ranks and the one based on within-block ranks as the cell size tends to infinity. Furthermore, Shiraishi (1989*b*) derived the asymptotic properties of the R-estimators and confidence regions based on the R-estimators as the number of blocks tends to infinity.

Aligned rank test procedures for the hypotheses (1.2)–(1.5) in the model (1.1) are proposed and the asymptotic properties as the cell size n tends to infinity are derived. Next, the estimators of respective parameters based on aligned ranks are proposed, and the asymptotic properties are derived. Furthermore, the confidence regions are discussed.

2. Classical unbiased estimators

Because of the motivation in the proposed statistics and because of the comparison to robust procedures, unbiased least squares estimators are stated in Table 1.

Table 1. Classical unbiased estimators.

Parameter	$\boldsymbol{\mu}$	$\boldsymbol{\alpha}_i$	$\boldsymbol{\beta}_j$	$(\boldsymbol{\alpha\beta})_{ij}$
Estimator	$\bar{\boldsymbol{X}}_{\cdots}$	$\bar{\boldsymbol{X}}_{i\cdot\cdot}-\bar{\boldsymbol{X}}_{\cdots}$	$\bar{\boldsymbol{X}}_{\cdot j\cdot}-\bar{\boldsymbol{X}}_{\cdots}$	$\bar{\boldsymbol{X}}_{ij\cdot}-\bar{\boldsymbol{X}}_{i\cdot\cdot}-\bar{\boldsymbol{X}}_{\cdot j\cdot}+\bar{\boldsymbol{X}}_{\cdots}$

$\bar{\boldsymbol{X}}_{\cdots}=\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{n}\boldsymbol{X}_{ijk}/N$, $\bar{\boldsymbol{X}}_{i\cdot\cdot}=\sum_{j=1}^{J}\sum_{k=1}^{n}\boldsymbol{X}_{ijk}/(Jn)$, $\bar{\boldsymbol{X}}_{\cdot j\cdot}=\sum_{i=1}^{I}\sum_{k=1}^{n}\boldsymbol{X}_{ijk}/(In)$ and $\bar{\boldsymbol{X}}_{ij\cdot}=\sum_{k=1}^{n}\boldsymbol{X}_{ijk}/n$, where $N=IJn$. When $p=1$ and $F(x)$ is normal, it is simple to verify that these unbiased estimators are uniformly minimum variance unbiased estimators.

3. Linear rank statistics

For $p \times IJ$ matrix $\boldsymbol{t} = (\boldsymbol{t}_{11}, \dots, \boldsymbol{t}_{1J}, \boldsymbol{t}_{21}, \dots, \boldsymbol{t}_{IJ})$, $p \times I$ matrix $\boldsymbol{t}^* = (t_i^{*(l)})_{l=1,\dots,p,\, i=1,\dots,I} = (\boldsymbol{t}_1^*, \dots, \boldsymbol{t}_I^*)$ and p-dimensional column vector $\boldsymbol{t}^+ = (t^{+(1)}, \dots, t^{+(p)})'$, let us define aligned observations by $\boldsymbol{Y}_{ijk}(\boldsymbol{t}) = \boldsymbol{X}_{ijk} - (\bar{\boldsymbol{X}}_{i\cdot\cdot} + \bar{\boldsymbol{X}}_{\cdot j\cdot}) - \boldsymbol{t}_{ij}$, $\boldsymbol{Y}^*_{ijk}(\boldsymbol{t}^*) = \boldsymbol{X}_{ijk} - (\bar{\boldsymbol{X}}_{ij\cdot} - \bar{\boldsymbol{X}}_{i\cdot\cdot}) - \boldsymbol{t}_i^*$ and $\boldsymbol{Y}^+_{ijk}(\boldsymbol{t}^+) = \boldsymbol{X}_{ijk} - \bar{\boldsymbol{X}}_{ij\cdot} + \bar{\boldsymbol{X}}_{\cdots} - \boldsymbol{t}^+$, and let their l-th coordinates be respectively $Y^{(l)}_{ijk}(\boldsymbol{t})$, $Y^{*(l)}_{ijk}(\boldsymbol{t})$ and $Y^{+(l)}_{ijk}(\boldsymbol{t}^+)$, where $\boldsymbol{t}_{ij} = (t_{ij}^{(1)}, \dots, t_{ij}^{(p)})'$. Then let $R^{(l)}_{ijk}(\boldsymbol{t})$, $R^{*(l)}_{ijk}(\boldsymbol{t}^*)$ and $R^{+(l)}_{ijk}(\boldsymbol{t}^+)$ be the rank of $Y^{(l)}_{ijk}(\boldsymbol{t})$ among the N observations $Y^{(l)}_{111}(\boldsymbol{t}), \dots, Y^{(l)}_{IJn}(\boldsymbol{t})$, the rank of $Y^{*(l)}_{ijk}(\boldsymbol{t}^*)$ among the observations $Y^{*(l)}_{111}(\boldsymbol{t}^*), \dots, Y^{*(l)}_{IJn}(\boldsymbol{t}^*)$ and the rank of $|Y^{+(l)}_{ijk}(\boldsymbol{t}^+)|$ among the observations $|Y^{+(l)}_{111}(\boldsymbol{t}^+)|, \dots, |Y^{+(l)}_{IJn}(\boldsymbol{t}^+)|$ respectively for $l = 1, \dots, p$. Using these ranks and score functions $a^{(l)}_N(\cdot)$ and $a^{+(l)}_N(\cdot)$, which are maps from $\{1, \dots, N\}$ to real values ($N \geq 1$), for $\boldsymbol{t}$, $\boldsymbol{t}^\# = (\boldsymbol{t}^\#_{11}, \dots, \boldsymbol{t}^\#_{1J}, \boldsymbol{t}^\#_{21}, \dots, \boldsymbol{t}^\#_{IJ})$, $\boldsymbol{t}^*$ and $\boldsymbol{t}^+$, let us put;

$$(3.1)\quad S^{(l)}_{ij}(\boldsymbol{t}, \boldsymbol{t}^\#) = \sum_{k=1}^{n} \{a^{(l)}_N(R^{(l)}_{ijk}(\boldsymbol{t})) - \bar{a}^{(l)}_N(R^{(l)}_{\cdot jk}(\boldsymbol{t}^\#)) - \bar{a}^{(l)}_N(R^{(l)}_{i\cdot k}(\boldsymbol{t}^\#)) + \bar{a}^{(l)}_N\}/\sqrt{n},$$

$$(3.2)\quad S^{*(l)}_i(\boldsymbol{t}^*) = \sum_{j=1}^{J}\sum_{k=1}^{n} \{a^{(l)}_N(R^{*(l)}_{ijk}(\boldsymbol{t}^*)) - \bar{a}^{(l)}_N\}/(J\sqrt{n})$$

and

$$(3.3)\quad S^{+(l)}(\boldsymbol{t}^+) = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{n} \{\mathrm{sign}(Y^{+(l)}_{ijk}(\boldsymbol{t}^+))\} a^{+(l)}_N(R^{+(l)}_{ijk}(\boldsymbol{t}^+))/(IJ\sqrt{n}),$$

respectively, where $\bar{a}^{(l)}_N(R^{(l)}_{\cdot jk}(\boldsymbol{t}^\#)) = \sum_{i=1}^{I} a^{(l)}_N(R^{(l)}_{ijk}(\boldsymbol{t}^\#))/I$,

$$\bar{a}^{(l)}_N(R^{(l)}_{i\cdot k}(\boldsymbol{t}^\#)) = \sum_{j=1}^{J} a^{(l)}_N(R^{(l)}_{ijk}(\boldsymbol{t}^\#))/J \quad \text{and} \quad \bar{a}^{(l)}_N = \sum_{m=1}^{N} a^{(l)}_N(m)/N$$

and $\mathrm{sign}(x) = 1$ for $x > 0$; $= 0$ for $x = 0$; $= -1$ elsewhere. The values of $S^{(l)}_{ij}(\boldsymbol{t}, \boldsymbol{t}^\#)$, $S^{*(l)}_i(\boldsymbol{t}^*)$ and $S^{+(l)}(\boldsymbol{t}^+)$ depend on $\boldsymbol{t}^{(l)} = (t^{(l)}_{11}, \dots, t^{(l)}_{1J}, t^{(l)}_{21}, \dots, t^{(l)}_{IJ})$, $\boldsymbol{t}^{\#(l)}$, $\boldsymbol{t}^{*(l)} = (t_1^{*(l)}, \dots, t_I^{*(l)})$ and $t^{+(l)}$ respectively, but they are not independent of $\boldsymbol{t}^{(l')}$, $\boldsymbol{t}^{\#(l')}$, $\boldsymbol{t}^{*(l')}$ and $t^{+(l')}$ for $l' \neq l$.

4. Common assumptions and basic theorems

The following are some assumptions to discuss the asymptotic theory.

ASSUMPTION 1. Score function $a^{(l)}_N(\cdot)$ is generated by a function $\psi_l(u)$ $(0 < u < 1)$ in the following way $(l = 1, \dots, p)$:

$$a^{(l)}_N(m) = E\{\psi_l(U_N(m))\} \quad \text{or} \quad \psi_l(m/(N+1)) \quad \text{for} \quad m = 1, \dots, N,$$

where $U_N(m)$ is the m-th order statistic in a sample of size N from the rectangular $(0,1)$ distribution.

ASSUMPTION 1′. Score function $a_N^{+(l)}(\cdot)$ is defined in the same manner as in Assumption 1:

$$a_N^{+(l)}(m) = E\{\psi_l(1/2 + U_N(m)/2)\} \quad \text{or} \quad \psi_l(1/2 + m/\{2(N+1)\}) \qquad \text{for} \quad m = 1, \dots, N.$$

ASSUMPTION 2. The score generating function $\psi_l(u)$ is non-constant, non-decreasing and square integrable.

ASSUMPTION 3. Letting $F_l(x^{(l)})$ and $f_l(x^{(l)})$ be respectively the l-th marginal distribution function of $F(\boldsymbol{x})$ and its density function, for $l = 1, \dots, p$, $F_l(x^{(l)})$ possess finite Fisher's information, i.e.,

$$\int_{-\infty}^{\infty} \{-f_l'(x^{(l)})/f_l(x^{(l)})\}^2 f_l(x^{(l)}) dx^{(l)} < +\infty.$$

We derive asymptotic linearity for the rank statistics $S_{ij}^{(l)}(\boldsymbol{t}, \boldsymbol{t}^{\#})$, $S_i^{*(l)}(\boldsymbol{t}^*)$ and $S^{+(l)}(\boldsymbol{t}^+)$.

4.1 *Asymptotic linearity of* $S_{ij}^{(l)}(\boldsymbol{t}, \boldsymbol{t}^{\#})$

For $p \times IJ$ matrices $\boldsymbol{s}$ and $\boldsymbol{t}$, letting $Q_{ijk}^{(l)}(\boldsymbol{s}, \boldsymbol{t})$ be the rank of $X_{ijk}^{(l)} - s_{ij}^{(l)} - t_{ij}^{(l)}$ among the N observations $\{X_{ijk}^{(l)} - s_{ij}^{(l)} - t_{ij}^{(l)};\ i = 1, \dots, I,\ j = 1, \dots, J,\ k = 1, \dots, n\}$ for $l = 1, \dots, p$, we introduce the following statistic:

$$\tilde{S}_{ij}^{(l)}(\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{t}^{\#}) = \sum_{k=1}^{n} \{a_N^{(l)}(Q_{ijk}^{(l)}(\boldsymbol{s}, \boldsymbol{t})) - \bar{a}_N^{(l)}(Q_{\cdot jk}^{(l)}(\boldsymbol{s}, \boldsymbol{t}^{\#})) - \bar{a}_N^{(l)}(Q_{i \cdot k}^{(l)}(\boldsymbol{s}, \boldsymbol{t}^{\#})) + \bar{a}_N^{(l)}\}/\sqrt{n},$$

where $\bar{a}_N^{(l)}(Q_{\cdot jk}^{(l)}(\boldsymbol{s}, \boldsymbol{t}^{\#}))$ and $\bar{a}_N^{(l)}(Q_{i \cdot k}^{(l)}(\boldsymbol{s}, \boldsymbol{t}^{\#}))$ are respectively defined in a similar way to $\bar{a}_N^{(l)}(R_{\cdot jk}^{(l)}(\boldsymbol{t}^{\#}))$ and $\bar{a}_N^{(l)}(R_{i \cdot k}^{(l)}(\boldsymbol{t}^{\#}))$. Then, $\tilde{S}_{ij}^{(l)}(\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{t}^{\#})$ is a function of $(s_{11}^{(l)}, \dots, s_{IJ}^{(l)}, t_{11}^{(l)}, \dots, t_{IJ}^{(l)}, t_{11}^{\#(l)}, \dots, t_{IJ}^{\#(l)})$ and does not depend on $s_{ij}^{(l')}$, $t_{ij}^{(l')}$ and $t_{ij}^{\#(l')}$ for $l' \neq l$. Also, to reduce notational complexity, we set $\tilde{S}_{ij}^{(l)} = \tilde{S}_{ij}^{(l)}(\boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0})$.

LEMMA 4.1. *Let* $(X_{111}^{(l)}, \dots, X_{IJn}^{(l)})$ *have a joint density* $\prod_{i=1}^{I} \prod_{j=1}^{J} \prod_{k=1}^{n} f_l \cdot (x_{ijk}^{(l)})$ *and let* $\|\boldsymbol{z}\|_m = \sqrt{\boldsymbol{z} \cdot \boldsymbol{z}'}$ *for the* m*-dimensional row vector* $\boldsymbol{z}$*. Then under*

Assumptions 1–3, *for any positive* ε, C_1, C_2 *and* C_3;

$$\lim_{n\to\infty} P\left\{ \sup_{\substack{\|\boldsymbol{\rho}^{(l)}\|_{IJ}<C_1\\ \|\boldsymbol{\Delta}^{(l)}\|_{IJ}<C_2\\ \|\boldsymbol{\Delta}^{\#(l)}\|<C_3}} |\tilde{S}_{ij}^{(l)}(\boldsymbol{\rho}/\sqrt{n},\boldsymbol{\Delta}/\sqrt{n},\boldsymbol{\Delta}^{\#}/\sqrt{n}) - \tilde{S}_{ij}^{(l)} \right.$$
$$+ d_l\cdot(\rho_{ij}^{(l)} - \bar{\rho}_{i\cdot}^{(l)} - \bar{\rho}_{\cdot j}^{(l)} + \bar{\rho}_{\cdot\cdot}^{(l)}$$
$$\left. + \Delta_{ij}^{(l)} - \bar{\Delta}_{\cdot\cdot}^{(l)} - \bar{\Delta}_{i\cdot}^{\#(l)} - \bar{\Delta}_{\cdot j}^{\#(l)} + 2\bar{\Delta}_{\cdot\cdot}^{\#(l)})| > \varepsilon \right\} = 0,$$

where $\boldsymbol{\rho}^{(l)} = (\rho_{11}^{(l)},\ldots,\rho_{1J}^{(l)},\rho_{21}^{(l)},\ldots,\rho_{IJ}^{(l)})$, $\boldsymbol{\Delta}^{(l)} = (\Delta_{11}^{(l)},\ldots,\Delta_{1J}^{(l)},\Delta_{21}^{(l)},\ldots,\Delta_{IJ}^{(l)})$ *and* $\boldsymbol{\Delta}^{\#(l)}$ *are respectivery the* l*-th rows of* $p\times IJ$ *matrices* $\boldsymbol{\rho}$, $\boldsymbol{\Delta}$ *and* $\boldsymbol{\Delta}^{\#}$, *and*

$$d_l = -\int_0^1 \{\psi_l(u)\cdot f_l'(F_l^{-1}(u))/f_l(F_l^{-1}(u))\}du.$$

PROOF. Let us put

$$W_{ij}(\boldsymbol{\rho}/\sqrt{n},\boldsymbol{\Delta}/\sqrt{n}) = \sum_{k=1}^{n}\{a_N^{(l)}(Q_{ijk}^{(l)}(\boldsymbol{\rho}/\sqrt{n},\boldsymbol{\Delta}/\sqrt{n})) - a_N^{(l)}(Q_{ijk}^{(l)}(\mathbf{0},\mathbf{0}))\}/\sqrt{n}$$
$$+ d_l\cdot(\rho_{ij}^{(l)} - \bar{\rho}_{\cdot\cdot}^{(l)} + \Delta_{ij}^{(l)} - \bar{\Delta}_{\cdot\cdot}^{(l)}).$$

Then it suffices to show

$$\sup_{\substack{\|\boldsymbol{\rho}^{(l)}\|_{IJ}<C_1\\ \|\boldsymbol{\Delta}^{(l)}\|_{IJ}<C_2}} |W_{ij}(\boldsymbol{\rho}/\sqrt{n},\boldsymbol{\Delta}/\sqrt{n})| \xrightarrow{P} 0, \tag{4.1}$$

where $\xrightarrow{P}$ denotes convergence in probability. There exist K_i $(i=1,2)$ such that $|d_l\cdot C_i|/K_i < \varepsilon/8$. So we put the set

$$\begin{aligned} B = \{&(\rho_{11u_{11}}^{(l)},\rho_{12u_{12}}^{(l)},\ldots,\rho_{IJu_{IJ}}^{(l)},\Delta_{11v_{11}}^{(l)},\Delta_{12v_{12}}^{(l)},\ldots,\Delta_{IJv_{IJ}}^{(l)});\\ &\rho_{iju_{ij}}^{(l)} = -C_1 + u_{ij}C_1/K_1\\ &\qquad \text{for}\quad u_{ij} = 0,1,\ldots,2K_1;\ i=1,\ldots,I;\ j=1,\ldots,J \qquad \text{and}\\ &\Delta_{ijv_{ij}}^{(l)} = -C_2 + v_{ij}C_2/K_2\\ &\qquad \text{for}\quad v_{ij} = 0,1,\ldots,2K_2;\ i=1,\ldots,I;\ j=1,\ldots,J\}. \end{aligned}$$

Then, from Assumptions 1 and 2, $\sum_{k=1}^{n} a_N^{(l)}(Q_{ijk}^{(l)}(\boldsymbol{\rho}/\sqrt{n},\boldsymbol{\Delta}/\sqrt{n}))/\sqrt{n}$ is nonincreasing in $\rho_{ij}^{(l)}$ and $\Delta_{ij}^{(l)}$, while it is nondecreasing in $\rho_{i'j'}^{(l)}$ and $\Delta_{i'j'}^{(l)}$ for $(i',j')\neq(i,j)$. Here it follows that

$$\text{The l.h.s. of (4.1)} \leq \max_{(\boldsymbol{\rho}^{(l)},\boldsymbol{\Delta}^{(l)})\in B} |W_{ij}(\boldsymbol{\rho}/\sqrt{n},\boldsymbol{\Delta}/\sqrt{n})| + \varepsilon/2, \tag{4.2}$$

where the notation l.h.s. stands for a left-hand side. Using Assumption 3, which is a condition for the contiguity, from the proof similar to the proof on Lemma 3.8 of Jurečková (1969), we find that

$$W_{ij}(\boldsymbol{\rho}/\sqrt{n}, \boldsymbol{\Delta}/\sqrt{n}) \xrightarrow{P} 0. \tag{4.3}$$

Therefore (4.2) and (4.3) give (4.1). □

By using Lemma 4.1, it is simple to show

THEOREM 4.1. *Under the assumptions of Lemma* 4.1, *for any* $\varepsilon > 0$ *and any* $C_1, C_2 > 0$,

$$\lim_{n\to\infty} P\left\{ \sup_{\substack{\|\boldsymbol{\Delta}^{(l)}\|_{IJ}<C_1 \\ \|\boldsymbol{\Delta}^{\#(l)}\|_{IJ}<C_2}} |S_{ij}^{(l)}(\boldsymbol{\Delta}/\sqrt{n}, \boldsymbol{\Delta}^{\#}/\sqrt{n}) - \tilde{S}_{ij}^{(l)} + d_l \cdot (\Delta_{ij}^{(l)} - \bar{\Delta}_{\cdot\cdot}^{(l)} - \bar{\Delta}_{i\cdot}^{\#(l)} - \bar{\Delta}_{\cdot j}^{\#(l)} + 2\bar{\Delta}_{\cdot\cdot}^{\#(l)})| > \varepsilon \right\} = 0,$$

where $S_{ij}^{(l)}(\boldsymbol{t}, \boldsymbol{t}^{\#})$ *is defined by* (3.1).

We get two corollaries as a direct result of Theorem 4.1:

COROLLARY 4.1. *Let* $B(C) = \{\boldsymbol{\Delta}^{(l)}; \bar{\Delta}_{\cdot j}^{(l)} = \bar{\Delta}_{i\cdot}^{(l)} = \mathbf{0}$ *for all* i, j*'s*, $\|\boldsymbol{\Delta}^{(l)}\|_{IJ} < C\}$. *Then under the assumptions of Lemma* 4.1, *for any* $\varepsilon > 0$ *and any* $C_1, C_2 > 0$,

$$\lim_{n\to\infty} P\left\{ \sup_{\substack{\boldsymbol{\Delta}^{(l)}\in B(C_1) \\ \boldsymbol{\Delta}^{\#(l)}\in B(C_2)}} |S_{ij}^{(l)}(\boldsymbol{\Delta}/\sqrt{n}, \boldsymbol{\Delta}^{\#}/\sqrt{n}) - \tilde{S}_{ij}^{(l)} + d_l \cdot \Delta_{ij}^{(l)}| > \varepsilon \right\} = 0.$$

COROLLARY 4.2. *Under the assumptions of Lemma* 4.1, *for any* $\varepsilon > 0$ *and any* $C > 0$,

$$\lim_{n\to\infty} P\left\{ \sup_{\boldsymbol{\Delta}^{(l)}\in B(C)} ||S_{ij}^{(l)}(\boldsymbol{\Delta}/\sqrt{n}, (\widehat{\boldsymbol{\alpha\beta}}))| - |\tilde{S}_{ij}^{(l)} - d_l \cdot \Delta_{ij}^{(l)}|| > \varepsilon \right\} = 0,$$

where $(\widehat{\boldsymbol{\alpha\beta}}) = ((\widehat{\boldsymbol{\alpha\beta}})_{11}, \dots, (\widehat{\boldsymbol{\alpha\beta}})_{1J}, (\widehat{\boldsymbol{\alpha\beta}})_{21}, \dots, (\widehat{\boldsymbol{\alpha\beta}})_{IJ})$ *and* $(\widehat{\boldsymbol{\alpha\beta}})_{ij} = \bar{\boldsymbol{X}}_{ij\cdot} - \bar{\boldsymbol{X}}_{i\cdot\cdot} - \bar{\boldsymbol{X}}_{\cdot j\cdot} + \bar{\boldsymbol{X}}_{\cdot\cdot\cdot}$.

4.2 *Asymptotic linearity of* $S_i^{*(l)}(\boldsymbol{t}^*)$

Letting $Q_{ijk}^{*(l)}$ be the rank of $X_{ijk}^{(l)}$ among the N observations $\{X_{ijk}^{(l)}; i = 1, \ldots, I, j = 1, \ldots, J, k = 1, \ldots, n\}$, we introduce the following statistic:

$$\tilde{S}_i^{*(l)} = \sum_{j=1}^{J} \sum_{k=1}^{n} \{a_N^{(l)}(Q_{ijk}^{*(l)}) - \bar{a}_N^{(l)}\}/(J\sqrt{n}).$$

Proceeding as in the proof of Theorem 4.1, we get

THEOREM 4.2. *Under the assumptions of Lemma* 4.1, *for any* $\varepsilon > 0$ *and any* $C > 0$,

$$\lim_{n\to\infty} P\left\{ \sup_{\|\boldsymbol{\Delta}^{*(l)}\|<C} |S_i^{*(l)}(\boldsymbol{\Delta}^*/\sqrt{n}) - \tilde{S}_i^{*(l)} + d_l \cdot (\Delta_i^{*(l)} - \bar{\Delta}_{\cdot}^{*(l)})| > \varepsilon \right\} = 0,$$

where $S_i^{*(l)}(\boldsymbol{t}^*)$ *is defined by* (3.2) *and* $\bar{\Delta}_{\cdot}^{*(l)} = \sum_{i=1}^{I} \Delta_i^{*(l)}/I$.

4.3 *Asymptotic linearity of* $S^{+(l)}(\boldsymbol{t}^+)$

For $p \times IJ$ matrix $\boldsymbol{s}^+ = (s_{ij}^{+(l)})_{l,i,j}$ and p-dimensional column vector $\boldsymbol{t}^+$, we put $X_{ijk}^{+(l)}(\boldsymbol{s}^+, \boldsymbol{t}^+) = X_{ijk}^{(l)} - s_{ij}^{+(l)} - t^{+(l)}$. Letting $Q_{ijk}^{+(l)}(\boldsymbol{s}^+, \boldsymbol{t}^+)$ be the rank of $|X_{ijk}^{+(l)}(\boldsymbol{s}^+, \boldsymbol{t}^+)|$ among the N observations $\{|X_{ijk}^{+(l)}(\boldsymbol{s}^+, \boldsymbol{t}^+)|; i = 1, \ldots, I, j = 1, \ldots, J, k = 1, \ldots, n\}$, we introduce the following statistic:

$$\tilde{S}^{+(l)}(\boldsymbol{s}^+, \boldsymbol{t}^+) = \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{n} \{\mathrm{sign}(X_{ijk}^{+(l)}(\boldsymbol{s}^+, \boldsymbol{t}^+))\} a_N^{+(l)}(Q_{ijk}^{+(l)}(\boldsymbol{s}^+, \boldsymbol{t}^+))/(IJ\sqrt{n}).$$

Then $\tilde{S}^{+(l)}(\boldsymbol{s}^+, \boldsymbol{t}^+)$ is a function of $(s_{11}^{+(l)}, \ldots, s_{IJ}^{+(l)}, t^{+(l)})$ and does not depend on $s_{ij}^{+(l')}$ and $t^{+(l')}$ (all i's and all j's) for $l' \neq l$. Also, to reduce notational complexity, we set $\tilde{S}^{+(l)} = \tilde{S}^{+(l)}(\mathbf{0}, \mathbf{0})$.

LEMMA 4.2. *Suppose that* $(X_{111}^{(l)}, \ldots, X_{IJn}^{(l)})$ *has a joint density* $\prod_{i=1}^{I} \prod_{j=1}^{J} \prod_{k=1}^{n} f_l(x_{ijk}^{(l)})$ *and* $f_l(x)$ *is symmetric about* 0. *Then under Assumptions* 1′, 2 *and* 3, *for any positive* ε, C_1 *and* C_2,

$$\lim_{n\to\infty} P\left\{ \sup_{\substack{\|\boldsymbol{\rho}^{+(l)}\|_{IJ}<C_1 \\ |\Delta^{+(l)}|<C_2}} |\tilde{S}^{+(l)}(\boldsymbol{\rho}^+/\sqrt{n}, \boldsymbol{\Delta}^+/\sqrt{n}) - \tilde{S}^{+(l)} + d_l^+ \cdot (\bar{\rho}_{\cdot\cdot}^{+(l)} + \Delta^{+(l)})| > \varepsilon \right\} = 0,$$

where $\boldsymbol{\rho}^{+(l)} = (\rho_{11}^{+(l)}, \ldots, \rho_{IJ}^{+(l)})$ *and* $\Delta^{+(l)}$ *are respectively* l*-th rows of* $\boldsymbol{\rho}^{+}$ *and* $\boldsymbol{\Delta}^{+}$, $\bar{\rho}_{\cdot\cdot}^{+(l)} = \sum_{i=1}^{I} \sum_{j=1}^{J} \rho_{ij}^{+(l)}/(IJ)$ *and*

$$d_l^+ = -\int_0^1 \{\psi_l(1/2+u/2) \cdot f_l'(F_l^{-1}(1/2+u/2))/f_l(F_l^{-1}(1/2+u/2))\}du.$$

PROOF. Let us put

$$\begin{aligned} &W_{ij}^+(\boldsymbol{\rho}^+/\sqrt{n}, \boldsymbol{\Delta}^+/\sqrt{n}) \\ &= \sum_{k=1}^{n}\{\operatorname{sign}(X_{ijk}^{+(l)}(\boldsymbol{\rho}^+/\sqrt{n}, \boldsymbol{\Delta}^+/\sqrt{n}))a_N^{(l)}(Q_{ijk}^{+(l)}(\boldsymbol{\rho}^+/\sqrt{n}, \boldsymbol{\Delta}^+/\sqrt{n})) \\ &\qquad - \operatorname{sign}(X_{ijk}^{+(l)})a_N^{(l)}(Q_{ijk}^{+(l)}(\mathbf{0},\mathbf{0}))\}/\sqrt{n} + d_l^+ \cdot (\rho_{ij}^{+(l)} + \Delta^{+(l)}). \end{aligned}$$

Then it suffices to show

$$\sup_{\substack{\|\boldsymbol{\rho}^{+(l)}\|_{IJ} < C_1 \\ |\Delta^{+(l)}| < C_2}} |W_{ij}^+(\boldsymbol{\rho}^+/\sqrt{n}, \boldsymbol{\Delta}^+/\sqrt{n})| \xrightarrow{P} 0. \tag{4.4}$$

There exist K_i $(i = 1, 2)$ such that $|d_l^+ \cdot C_i|/K_i < \varepsilon/8$. So, we put the set

$$\begin{aligned} B^+ = \{(&\rho_{11u_{11}}^{+(l)}, \rho_{12u_{12}}^{+(l)}, \ldots, \rho_{IJu_{IJ}}^{+(l)}, \Delta^{+(l)}); \\ &\rho_{iju_{ij}}^{(l)} = -C_1 + u_{ij}C_1/K_1 \\ &\qquad \text{for } u_{ij} = 0, 1, \ldots, 2K_1;\ i = 1, \ldots, I;\ j = 1, \ldots, J \quad \text{and} \\ &\Delta_v^{(l)} = -C_2 + vC_2/K_2 \text{ for } v = 0, 1, \ldots, 2K_2\}. \end{aligned}$$

Then, from Assumptions 1′ and 2,

$$\sum_{k=1}^{n} \operatorname{sign}(X_{ijk}^{+(l)}(\boldsymbol{\rho}^+/\sqrt{n}, \boldsymbol{\Delta}^+/\sqrt{n}))a_N^{(l)}(Q_{ijk}^{+(l)}(\boldsymbol{\rho}^+/\sqrt{n}, \boldsymbol{\Delta}^+/\sqrt{n}))/\sqrt{n}$$

is nonincreasing in $\rho_{ij}^{+(l)}$ and $\Delta^{+(l)}$, while it is nondecreasing in $\rho_{i'j'}^{+(l)}$ for $(i', j') \neq (i, j)$. Here it follows that

$$\text{The l.h.s. of (4.4)} \leq \max_{(\boldsymbol{\rho}^{+(l)}, \Delta^{+(l)}) \in B^+} |W_{ij}^+(\boldsymbol{\rho}^+/\sqrt{n}, \boldsymbol{\Delta}^+/\sqrt{n})| + \varepsilon/2. \tag{4.5}$$

Using Assumption 3, which is a condition for the contiguity, from the proof similar to the proof on Lemma 3.8 of Jurečková (1969), we find that

$$W_{ij}^+(\boldsymbol{\rho}^+/\sqrt{n}, \boldsymbol{\Delta}^+/\sqrt{n}) \xrightarrow{P} 0. \tag{4.6}$$

Therefore, (4.5) and (4.6) give (4.4). □

Proceeding as in the proof for Theorem 4.1, we get

THEOREM 4.3. *Let* $B^+(C) = \{\Delta^{+(l)}; |\Delta^{+(l)}| < C\}$. *Then, under the assumptions of Lemma 4.2, for any* $\varepsilon > 0$ *and any* $C > 0$,

$$\lim_{n\to\infty} P\left\{ \sup_{\Delta^{+(l)}\in B^+(C)} |S^{+(l)}(\boldsymbol{\Delta}^+/\sqrt{n}) - \tilde{S}^{+(l)} + d_l^+ \cdot \Delta^{+(l)}| > \varepsilon \right\} = 0,$$

where $S^{+(l)}(\boldsymbol{t})$ *is defined by* (3.3).

5. Statistical inference for interactions

The distribution of $Y_{ijk}(\boldsymbol{t})$ is not dependent on $\boldsymbol{\alpha}_i$'s and $\boldsymbol{\beta}_j$'s. Neither do the ranks of $R_{ijk}^{(l)}(\boldsymbol{t})$'s depend on $\boldsymbol{\mu}$. Since the statistical inference is considered based on the ranks throughout this section, it is assumed without a loss of generality that

$$\boldsymbol{\mu} = \boldsymbol{\alpha}_1 = \cdots = \boldsymbol{\alpha}_I = \boldsymbol{\beta}_1 = \cdots = \boldsymbol{\beta}_J = \mathbf{0}. \tag{5.1}$$

Let us put pIJ column vectors $S(\boldsymbol{t},\boldsymbol{t}^\#) = (S_{11}(\boldsymbol{t},\boldsymbol{t}^\#)',\ldots,S_{1J}(\boldsymbol{t},\boldsymbol{t}^\#)', S_{21}(\boldsymbol{t},\boldsymbol{t}^\#)',\ldots,S_{IJ}(\boldsymbol{t},\boldsymbol{t}^\#)')'$ and $\tilde{S}(\boldsymbol{s},\boldsymbol{t},\boldsymbol{t}^\#) = (\tilde{S}_{11}(\boldsymbol{s},\boldsymbol{t},\boldsymbol{t}^\#)',\ldots,\tilde{S}_{1J}(\boldsymbol{s},\boldsymbol{t},\boldsymbol{t}^\#)', \tilde{S}_{21}(\boldsymbol{s},\boldsymbol{t},\boldsymbol{t}^\#)',\ldots,\tilde{S}_{IJ}(\boldsymbol{s},\boldsymbol{t},\boldsymbol{t}^\#)')'$, where the l-th coordinates of $S_{ij}(\boldsymbol{t},\boldsymbol{t}^\#)$ and $\tilde{S}_{ij}(\boldsymbol{s},\boldsymbol{t},\boldsymbol{t}^\#)$ are respectively $S_{ij}^{(l)}(\boldsymbol{t},\boldsymbol{t}^\#)$ and $\tilde{S}_{ij}^{(l)}(\boldsymbol{s},\boldsymbol{t},\boldsymbol{t}^\#)$. Also, to reduce notational complexity, we set $S = S(\mathbf{0},\mathbf{0})$, $\tilde{S} = \tilde{S}(\mathbf{0},\mathbf{0},\mathbf{0})$ and $R_{ijk}^{(l)} = R_{ijk}^{(l)}(\mathbf{0})$.

5.1 *Tests*

Based on the asymptotic distribution of $\boldsymbol{S}$ under H, we consider testing the null hypothesis H versus the alternative A.

LEMMA 5.1. *Suppose that Assumptions* 1–3 *are satisfied. Then under* H, *as* $n\to\infty$, $\boldsymbol{S}$ *has asymptotically a* pIJ*-variate normal distribution with mean* $\mathbf{0}$ *and a variance-covariance matrix* $\Lambda\otimes\Gamma$, *where* $\Lambda = (\lambda_{mm'})_{m,m'=1,\ldots,IJ}$,

$$\Gamma = (\gamma_{ll'})_{l,l'=1,\ldots,p}, \tag{5.2}$$

$\lambda_{mm'} = (1-1/I)(1-1/J)$ *if* $m = m'$; $= -1/J + 1/(IJ)$ *if* $m = (i-1)J + j$ *and* $m' = (i-1)J + j'$ *for* i *and* (j,j') *such that* $1 \le i \le I$ *and* $1 \le j \ne j' \le J$; $= -1/I + 1/(IJ)$ *if* $m = (i-1)J+j$ *and* $m' = (i'-1)J + j$ *for* (i,i') *and* j *such that* $1 \le i \ne i' \le I$ *and* $1 \le j \le J$; $= 1/(IJ)$ *elsewhere,*

$$\gamma_{ll'} = \begin{cases} \displaystyle\int_0^1 \{\psi_l(u) - \bar{\psi}_l\}^2 du & \text{if } l = l', \\ \displaystyle\int_{R^2} \{\psi_l(F_l(x)) - \bar{\psi}_l\}\{\psi_{l'}(F_{l'}(y)) - \bar{\psi}_{l'}\} dF_{ll'}(x,y) & \text{elsewhere,} \end{cases}$$

$\bar{\psi}_l = \int_0^1 \psi_l(u)du$, $F_{ll'}(x,y)$ *stands for the* (l,l')*-th marginal distribution of* $F(\boldsymbol{x})$ *and* $\otimes$ *denotes the Kronecker product.*

PROOF. Theorem 4.1 shows that $\boldsymbol{S} - \tilde{\boldsymbol{S}} \xrightarrow{P} \mathbf{0}$ under H. Furthermore, it is simple to verify that $\tilde{\boldsymbol{S}} \xrightarrow{\mathcal{L}} N_{pIJ}(\mathbf{0}, \Lambda \otimes \Gamma)$, where $\xrightarrow{\mathcal{L}} N_K(\boldsymbol{\mu}_0, \Sigma)$ denotes convergence in law to a K variate normal distribution with mean $\boldsymbol{\mu}_0$ and a variance-covariance matrix Σ. Hence, the conclusion is found. □

Next, we give a consistent estimator of Γ. Let us put $\Gamma(R) = (\hat{\gamma}_{ll'}(R))_{l,l'=1,\dots,p}$, where

$$\hat{\gamma}_{ll'}(R) = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{n}\{a_N^{(l)}(R_{ijk}^{(l)}) - \bar{a}_N^{(l)}\}\{a_N^{(l')}(R_{ijk}^{(l')}) - \bar{a}_N^{(l')}\}/(N-1). \tag{5.3}$$

ASSUMPTION 4. $\psi_l(u)$ is absolutely continuous for $l = 1, \dots, p$.

LEMMA 5.2. *Suppose that Assumptions* 1–4 *are satisfied. Then, under* H, $\Gamma(R)$ *converges in probability to* Γ.

PROOF. The proof is similar to the proof of Lemma 4.2 of Shiraishi (1989*a*) and is therefore omitted. □

We reject H when the following statistic is too large: $AL = \boldsymbol{S}'\{\Lambda \otimes \Gamma(R)\}^{-}\boldsymbol{S}$. Since the generalized inverse of $\Lambda \otimes \Gamma(R)$ is not unique, we take $E_{IJ} \otimes \Gamma(R)^{-1}$ as the generalized inverse, where E_m is the identity matrix of order m. Then, we have $AL = \boldsymbol{S}'\{E_{IJ} \otimes \Gamma(R)^{-1}\}\boldsymbol{S}$.

ASSUMPTION 5. Γ is positive definite.

Then combining Lemma 5.1 with Lemma 5.2, we get

THEOREM 5.1. *Suppose that Assumptions* 1 *through* 5 *are satisfied. Then under* H, *as* $n \to \infty$, AL *has asymptotically a* χ^2*-distribution with* $p(I-1)(J-1)$ *degrees of freedom.*

Next, we consider the sequence of local alternatives A_n; $(\boldsymbol{\alpha\beta})_{ij} = \boldsymbol{\Delta}_{ij}/\sqrt{n}$, $\boldsymbol{\Delta}_{ij} \neq \boldsymbol{\Delta}_{i'j'}$ for some $(i,j) \neq (i',j')$ and $\sum_{i=1}^{I}\boldsymbol{\Delta}_{ij} = \sum_{j=1}^{J}\boldsymbol{\Delta}_{ij} = \mathbf{0}$ for all i, j's, where $\boldsymbol{\Delta}_{ij} = (\Delta_{ij}^{(1)}, \dots, \Delta_{ij}^{(p)})'$.

If we suppose the following Assumption 6, proceeding as in the proof of Theorem VI.2.1 of Hájek and Šidák (1967), we find that A_n is contiguous to H as $n \to \infty$.

ASSUMPTION 6. $\partial f(\boldsymbol{x})/\partial x^{(l)}$'s are continuous and

$$\int_{R^p}\{-\partial f(\boldsymbol{x})/\partial x^{(l)}/f(\boldsymbol{x})\}^2 f(\boldsymbol{x})d\boldsymbol{x} < \infty \quad \text{for} \quad l = 1, \dots, p.$$

THEOREM 5.2. *Suppose that Assumptions* 1–6 *are satisfied. Then, under* A_n, *as* $n \to \infty$, AL *has asymptotically a noncentral* χ^2*-distribution with*

$p(I-1)(J-1)$ *degrees* of freedom and noncentrality ***parameter*** δ^2, ***where*** $\delta^2 = \sum_{i=1}^{I}\sum_{j=1}^{J}\boldsymbol{\nu}_{ij}'\Gamma^{-1}\boldsymbol{\nu}_{ij}, \boldsymbol{\nu}_{ij} = (\nu_{ij}^{(1)}, \dots, \nu_{ij}^{(p)})'$ *and* $\nu_{ij}^{(l)} = d_l \cdot \Delta_{ij}^{(l)}$.

PROOF. From Theorem 4.1, we get under H

$$|S_{ij}^{(l)}(-\boldsymbol{\Delta}/\sqrt{n}, -\boldsymbol{\Delta}/\sqrt{n}) - \tilde{S}_{ij}^{(l)} - \nu_{ij}^{(l)}| \xrightarrow{P} 0.$$

Here it follows that $\boldsymbol{S}(-\boldsymbol{\Delta}/\sqrt{n}, -\boldsymbol{\Delta}/\sqrt{n}) \xrightarrow{\mathcal{L}} N(\boldsymbol{\nu}, \Lambda \otimes \Gamma)$ under H, which is equivalent to the relation that

$$\boldsymbol{S} \xrightarrow{\mathcal{L}} N(\boldsymbol{\nu}, \Lambda \otimes \Gamma) \quad \text{under } A_n. \tag{5.4}$$

The contiguity of A_n with respect to H and Lemma 5.2 implies that $\Lambda \otimes \Gamma(R) \xrightarrow{P}$ $\Lambda \otimes \Gamma$ under A_n. Combining this with (5.4), we get the conclusion. ❑

5.2 Point estimates

Using a similar method to that of Shiraishi (1989*a*), we propose the R-estimators of matrix $(\boldsymbol{\alpha\beta}) = ((\boldsymbol{\alpha\beta})_{11}, \dots, (\boldsymbol{\alpha\beta})_{1J}, (\boldsymbol{\alpha\beta})_{21}, \dots, (\boldsymbol{\alpha\beta})_{IJ})$ on the model (1.1), based on the aligned ranks. Let $\|\boldsymbol{s}\| = \sum_{i=1}^{I}\sum_{j=1}^{J}|s_{ij}|$ for IJ-dimensional row vector $\boldsymbol{s}$. Then, we put

$$\begin{aligned}\Omega_n(R) &= \Big\{\boldsymbol{\theta} : \sum_{l=1}^{p} \|S^{(l)}(\boldsymbol{\theta}, \widehat{(\boldsymbol{\alpha\beta})})\| = \text{minimum} \\ &\qquad \text{under} \sum_{i=1}^{I}\boldsymbol{\theta}_{ij} = \sum_{j=1}^{J}\boldsymbol{\theta}_{ij} = 0 \text{ (all } i, j\text{'s)}\Big\} \\ &= \Big\{\boldsymbol{\theta} : \|S^{(l)}(\boldsymbol{\theta}, \widehat{(\boldsymbol{\alpha\beta})})\| = \text{minimum} \\ &\qquad \text{under} \sum_{i=1}^{I}\theta_{ij}^{(l)} = \sum_{j=1}^{J}\theta_{ij}^{(l)} = 0 \text{ (all } i, j\text{'s) for } l = 1, \dots, p\Big\},\end{aligned}$$

where $\boldsymbol{\theta} = (\boldsymbol{\theta}_{11}, \dots, \boldsymbol{\theta}_{1J}, \boldsymbol{\theta}_{21}, \dots, \boldsymbol{\theta}_{IJ})$, $\boldsymbol{\theta}_{ij} = (\theta_{ij}^{(1)}, \dots, \theta_{ij}^{(p)})'$, $S^{(l)}(\boldsymbol{t}, \boldsymbol{t}^{\#}) = (S_{11}^{(l)}(\boldsymbol{t}, \boldsymbol{t}^{\#}), \dots, S_{IJ}^{(l)}(\boldsymbol{t}, \boldsymbol{t}^{\#}))$ is the l-th row vector of $S(\boldsymbol{t}, \boldsymbol{t}^{\#})$ and $\widehat{(\boldsymbol{\alpha\beta})}$ is defined in Corollary 4.2. Sine $S_{ij}^{(l)}(\boldsymbol{t}, \boldsymbol{t}^{\#})$ takes finite values in $(t_{11}^{(l)}, \dots, t_{IJ}^{(l)})$, $\Omega_n(R)$ is not empty. We propose some point $\hat{\boldsymbol{\theta}}_n$ in $\Omega_n(R)$ as an aligned rank estimator of $(\boldsymbol{\alpha\beta})$. It is simple to verify;

$$\begin{aligned}&\Big\{\boldsymbol{\theta} : \sum_{l=1}^{p} \|S^{(l)}(\boldsymbol{\theta} + (\boldsymbol{\alpha\beta}), \widehat{(\boldsymbol{\alpha\beta})})\| = \text{minimum} \\ &\qquad \text{under} \sum_{i=1}^{I}\boldsymbol{\theta}_{ij} = \sum_{j=1}^{J}\boldsymbol{\theta}_{ij} = 0 \text{ (all } i, j\text{'s)}\Big\} \\ &= \{\boldsymbol{\theta} - (\boldsymbol{\alpha\beta}); \boldsymbol{\theta} \in \Omega_n(R)\}.\end{aligned} \tag{5.5}$$

If $\Omega_n(R)$ is a convex set, a natural choice of $\hat{\boldsymbol{\theta}}_n$ is the center of gravity of $\Omega_n(R)$. We add

ASSUMPTION 7. $d_l > 0$ for $l = 1, \ldots, p$.

THEOREM 5.3. *Suppose that Assumptions* 1–3 *and* 7 *are satisfied. Then* $\sqrt{n} \cdot \mathrm{vec}(\hat{\boldsymbol{\theta}}_n - (\boldsymbol{\alpha\beta}))$ *has a* pIJ*-variate normal distribution with mean* $\mathbf{0}$ *and a variance-covariance matrix* $\Lambda \otimes \Xi$, *where* $\mathrm{vec}(A)$ *denotes* $(\boldsymbol{a}_1', \ldots, \boldsymbol{a}_m')'$ *for* $p \times m$ *matrix* $A = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m)$, $\Xi = (\xi_{ll'})_{l,l'=1,\ldots,p}$ *and* $\xi_{ll'} = \gamma_{ll'}/(d_l \cdot d_{l'})$. *Furthermore,*

$$\lim_{n\to\infty} P\left\{ \sup_{\boldsymbol{\theta}\in\Omega_n(R)} \sqrt{n}\, \|\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}\|_{pIJ} > \varepsilon \right\} = 0 \quad \textit{for} \quad \varepsilon > 0.$$

where $\|A\|_{pm} = \sqrt{\{\mathrm{vec}(A)\}' \cdot \{\mathrm{vec}(A)\}}$.

PROOF. From (5.5), we may assume without a loss of generality that $(\boldsymbol{\alpha\beta}) = \mathbf{0}$. Let us define the solution for system of the following equations by $\hat{\boldsymbol{\theta}} = (\hat{\theta}^{(1)'}, \ldots, \hat{\theta}^{(p)'})'$ where $\hat{\boldsymbol{\theta}}^{(l)} = (\hat{\theta}_{11}^{(l)}, \ldots, \hat{\theta}_{1J}^{(l)}, \hat{\theta}_{21}^{(l)}, \ldots, \hat{\theta}_{IJ}^{(l)})$.

$$\tilde{S}_{ij}^{(l)} = \sqrt{n} \cdot d_l \theta_{ij}^{(l)} \quad \text{for} \quad j = 1, \ldots, J \quad \text{and} \quad l = 1, \ldots, p.$$

$\hat{\boldsymbol{\theta}}^{(l)}$ is given by $\hat{\boldsymbol{\theta}}^{(l)} = \tilde{\boldsymbol{S}}^{(l)}/(\sqrt{n} \cdot d_l)$, where $\tilde{\boldsymbol{S}}^{(l)} = (\tilde{S}_{11}^{(l)}, \ldots, \tilde{S}_{IJ}^{(l)})$. Hence, the asymptotic normality of $\mathrm{vec}(\tilde{\boldsymbol{S}}^{(1)'}, \ldots, \tilde{\boldsymbol{S}}^{(p)'})'$ implies that $\sqrt{n} \cdot \mathrm{vec}(\hat{\boldsymbol{\theta}})$ has asymptotically a multivariate normal distribution with mean $\mathbf{0}$ and a variance-covariance matrix $\Lambda \otimes \Xi$. Also, using Corollary 4.1, the convergence of $\sqrt{n} \cdot \mathrm{vec}(\hat{\boldsymbol{\theta}})$ and Assumption 7, along the lines, on the proof of Appendix of Shiraishi (1989*a*), we can show

$$\sup_{\boldsymbol{\theta}\in\Omega_n(R)} \sqrt{n}\, \|\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_{pIJ} \xrightarrow{P} 0.$$

Therefore, all the conclusions are found. □

5.3 *Confidence regions*

If Assumptions 1–3, 5 and 7 are satisfied, from Theorem 5.3, we can find that

$$n\{\mathrm{vec}(\hat{\boldsymbol{\theta}}_n - (\boldsymbol{\alpha\beta}))\}'(E_{IJ} \otimes \Xi^{-1})\{\mathrm{vec}(\hat{\boldsymbol{\theta}}_n - (\boldsymbol{\alpha\beta}))\} \xrightarrow{\mathcal{L}} \chi^2_{p(I-1)(J-1)}.$$

Letting $\hat{\Sigma}$ be a consistent estimator of the unknown Ξ^{-1}, if we put

$$CR(\tau) = \{\boldsymbol{\theta}; n\{\mathrm{vec}(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_n)\}'(E_{IJ} \otimes \hat{\Sigma})\{\mathrm{vec}(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_n)\} \le \chi^2_{p(I-1)(J-1)}(\tau)\},$$

$CR(\tau)$ is an asymptotically $100(1-\tau)$ percent distribution-free confidence region for $(\boldsymbol{\alpha\beta})$, where $\chi^2_m(\tau)$ is the upper 100τ percent point of the χ^2-distribution with m degrees of freedom. So we construct the consistent estimator.

From Theorem 4.1, we can get

LEMMA 5.3. *For some positive c, let us define, for* $i = 1, \ldots, I$, $j = 1, \ldots, J$ *and* $l = 1, \ldots, p$,

$$\hat{d}^{(l)}_{n[i,j]} = \{S^{(l)}_{ij}(\hat{\boldsymbol{\theta}}_n - c\mathcal{E}_{ij}/\sqrt{n}, \widehat{(\boldsymbol{\alpha\beta})}) - S^{(l)}_{ij}(\hat{\boldsymbol{\theta}}_n + c\mathcal{E}_{ij}/\sqrt{n}, \widehat{(\boldsymbol{\alpha\beta})})\}/[2c(1 - 1/IJ)],$$

where $\mathcal{E}_{ij}$ *is a* $p \times (IJ)$ *matrix with* $\mathbf{1}_p$ *at the* $\{(i-1)J + j\}$*-th column and zero vector elsewhere for each* (i, j) *and* $\mathbf{1}_p = (1, \ldots, 1)'$. *Suppose that Assumptions* 1–3 *are satisfied. Then* $\hat{d}^{(l)}_{n[i,j]}$ *converges in probability to* d_l.

As an estimator of d_l, we choose $\hat{d}_l = \sum_{i=1}^{I}\sum_{j=1}^{J}\hat{d}^{(l)}_{n[i,j]}/(IJ)$ and put $\hat{D} = \mathrm{diag}(\hat{d}_1, \ldots, \hat{d}_p)$. Replacing $R^{(l)}_{ijk}$ by $R^{(l)}_{ijk}(\hat{\boldsymbol{\theta}}_n)$ for all i, j, k and l in (5.3), we denote the corresponding random variable by $\hat{\gamma}_{ll'}(R(\hat{\boldsymbol{\theta}}_n))$ and set

$$\Gamma(R(\hat{\boldsymbol{\theta}}_n)) = (\hat{\gamma}_{ll'}(R(\hat{\boldsymbol{\theta}}_n)))_{l,l'=1,\ldots,p}.$$

Proceeding as in the proof of Lemma 5.2, we get

LEMMA 5.4. *Suppose that Assumptions* 1–4 *and* 7 *are satisfied. Then,* $\Gamma(R(\hat{\boldsymbol{\theta}}_n))$ *converges in probability to* Γ.

Hence, combining Lemma 5.3 with Lemma 5.4, we get

THEOREM 5.4. *Suppose that Assumptions* 1–5 *and* 7 *are satisfied. Let* $\hat{\Sigma} = \hat{D}\{\Gamma(R(\hat{\boldsymbol{\theta}}_n))\}^{-}\hat{D}$. *Then,* $CR(\tau)$ *is an asymptotically* $100(1-\tau)$ *percent distribution-free confidence region for* $(\boldsymbol{\alpha\beta})$.

6. Statistical inference for main effects

We consider statistical inference for $\boldsymbol{\alpha}_i$'s and $\boldsymbol{\beta}_j$'s in this section, but we will concentrate our effects on the statistical inference for $\boldsymbol{\alpha}_i$'s as we recognize that the statistical inference for $\boldsymbol{\beta}_j$'s can be obtained simply by reversing the first and second factor. We consider the statistical inference based on $R^{*(l)}_{ijk}(\boldsymbol{t})$'s in this section. The distribution of statistics under the model (1.1) does not depend on $\boldsymbol{\mu}$, $\boldsymbol{\beta}_j$'s and $(\boldsymbol{\alpha\beta})_{ij}$'s.

Let us put pI column vectors $\boldsymbol{S}^*(\boldsymbol{t}) = (\boldsymbol{S}^*_1(\boldsymbol{t})', \ldots, \boldsymbol{S}^*_I(\boldsymbol{t})')'$, where the l-th coordinate of $\boldsymbol{S}^*_i(\boldsymbol{t})$ is $S^{*(l)}_i(\boldsymbol{t})$. Also, to reduce notational complexity, we set $\boldsymbol{S}^* = \boldsymbol{S}^*(\boldsymbol{0})$, $\tilde{\boldsymbol{S}}^* = \tilde{\boldsymbol{S}}^*(\boldsymbol{0}, \boldsymbol{0})$ and $R^{*(l)}_{ijk} = R^{*(l)}_{ijk}(\boldsymbol{0})$.

6.1 *Tests*

Based on the asymptotic distribution of $\boldsymbol{S}^*$ under H^*, we consider to test the null hypothesis H^* versus the alternative A^*. Using Theorem 4.2, as in the proof of Lemma 5.1, we get

LEMMA 6.1. *Suppose that Assumptions* 1–3 *are satisfied. Then under* H^*, *as* $n \to \infty$, $\boldsymbol{S}^*$ *has asymptotically a* pI*-variate normal distribution with mean*

$\mathbf{0}$ *and a variance-covariance matrix* $\Lambda^* \otimes \Gamma^*$, *where* $\Lambda^* = E_I - \mathbf{1}_I \cdot \mathbf{1}_I'/I$, $\Gamma^* = (\gamma_{ll'}/J)_{l,l'=1,\ldots,p}$ *and* $\gamma_{ll'}$ *is defined by* (5.2).

Next, we draw a consistent estimator of Γ^*, based on $R_{ijk}^{*(l)}$'s. Let us put $\Gamma^*(R^*) = (\hat{\gamma}_{ll'}(R^*)/J)_{l,l'=1,\ldots,p}$, where

$$\hat{\gamma}_{ll'}(R^*) = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{n}\{a_N^{(l)}(R_{ijk}^{*(l)}) - \bar{a}_N^{(l)}\}\{a_N^{(l')}(R_{ijk}^{*(l')}) - \bar{a}_N^{(l')}\}/(N-1). \tag{6.1}$$

Proceeding as in the proof of Lemma 5.2, we get

LEMMA 6.2. *Suppose that Assumptions* 1–4 *are satisfied. Then under* H^*, $\Gamma^*(R^*)$ *converges in probability to* Γ^*.

We reject H^* when the following statistic is too large: $AL^* = \boldsymbol{S}^{*\prime}\{\Lambda^* \otimes \Gamma^*(R^*)\}^- \boldsymbol{S}^*$. Then we get

PROPOSITION 6.1. *Suppose that* $\Gamma^*(R^*)$ *is positive definite. Then* AL^* *does not depend on the choice of generalized inverse* $\Lambda^* \otimes \Gamma(R^*)$ *and is expressed as* $AL^* = \boldsymbol{S}^{*\prime}\{E_I \otimes \Gamma^*(R^*)^{-1}\}\boldsymbol{S}^*$.

Then, combining Lemma 6.1 with Lemma 6.2, we get

THEOREM 6.1. *Suppose that Assumptions* 1–5 *are satisfied. Then under* H^*, *as* $n \to \infty$, AL^* *has asymptotically a* χ^2*-distribution with* $p(I-1)$ *degrees of freedom.*

Next, we consider the sequence of local alternatives $A_n^*; \boldsymbol{\alpha}_i = \boldsymbol{\Delta}_i^*/\sqrt{n}$, $\boldsymbol{\Delta}_i^* \neq \boldsymbol{\Delta}_{i'}^*$ for some $i \neq i'$ and $\sum_{i=1}^I \boldsymbol{\Delta}_i^* = \mathbf{0}$, where $\boldsymbol{\Delta}_i^* = (\Delta_i^{*(l)}, \ldots, \Delta_i^{*(p)})'$.
Using Theorem 4.2, as in the proof of Theorem 5.3, we get

THEOREM 6.2. *Suppose that Assumptions* 1–6 *are satisfied. Then under* A_n^*, *as* $n \to \infty$, AL^* *has asymptotically a noncentral* χ^2*-distribution with* $p(I-1)$ *degrees of freedom and noncentrality parameter* δ^2, *where* $\delta^2 = \sum_{i=1}^I \boldsymbol{\nu}_i^{*\prime}\Gamma^{*-1}\boldsymbol{\nu}_i^*$, $\boldsymbol{\nu}_i^* = (\nu_i^{*(1)}, \ldots, \nu_i^{*(p)})'$ *and* $\nu_i^{*(l)} = d_l \cdot \Delta_i^{*(l)}$.

6.2 *Point estimates*

We propose the R-estimators of matrix $\boldsymbol{\alpha}^* = (\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_I)$ on the model (1.1). Let $\|\boldsymbol{s}\|^* = \sum_{i=1}^I |s_i|$ for I-dimensional row vector $\boldsymbol{s}$. Then we put

$$\begin{aligned}\Omega_n^*(R) &= \left\{\boldsymbol{\theta}^* : \sum_{l=1}^{p} \|\boldsymbol{S}^{*(l)}(\boldsymbol{\theta}^*)\|^* = \text{minimum under } \sum_{i=1}^{I}\boldsymbol{\theta}_i^* = \mathbf{0}\right\}\\ &= \left\{\boldsymbol{\theta}^* : \|\boldsymbol{S}^{*(l)}(\boldsymbol{\theta}^*)\|^* = \text{minimum under } \sum_{i=1}^{I}\theta_i^{*(l)} = 0 \text{ for } l = 1, \ldots, p\right\},\end{aligned}$$

where $\boldsymbol{\theta}^* = (\boldsymbol{\theta}_1^*, \ldots, \boldsymbol{\theta}_I^*)$, $\boldsymbol{\theta}_i^* = (\theta_i^{*(1)}, \ldots, \theta_1^{*(p)})'$ and $\boldsymbol{S}^{*(l)}(\boldsymbol{t}^*) = (S_1^{*(l)}(\boldsymbol{t}^*), \ldots, S_I^{*(l)}(\boldsymbol{t}^*))$ is the l-th row vector of $\boldsymbol{S}^*(\boldsymbol{t})$. We propose some point $\hat{\boldsymbol{\theta}}_n^*$ in $\Omega_n^*(R)$ as an aligned rank estimator of $\boldsymbol{\alpha}^*$. It is simple to verify;

$$\left\{\boldsymbol{\theta}^* : \sum_{l=1}^{p} \|S^{*(l)}(\boldsymbol{\theta}^* + \boldsymbol{\alpha}^*)\|^* = \text{minimum under } \sum_{i=1}^{I} \boldsymbol{\theta}_i^* = 0\right\}$$
$$= \{\boldsymbol{\theta}^* - \boldsymbol{\alpha}^*; \boldsymbol{\theta}^* \in \Omega_n^*(R)\}.$$

If $\Omega_n^*(R)$ is a convex set, a natural choice of $\hat{\boldsymbol{\theta}}_n^*$ is the center of gravity of $\Omega_n^*(R)$. Even if $\Omega_n^*(R)$ is not convex, we can show

THEOREM 6.3. *Suppose that Assumptions* 1–3 *and* 7 *are satisfied. Then,* $\sqrt{n}\cdot\mathrm{vec}(\hat{\boldsymbol{\theta}}_n^* - \boldsymbol{\alpha}^*)$ *has a* pI*-variate normal distribution with mean* $\mathbf{0}$ *and a variance-covariance matrix* $\Lambda^* \otimes \Xi/J$*, where* Ξ *is defined in Theorem* 5.3. *Furthermore,*

$$\lim_{n\to\infty} P\left\{\sup_{\boldsymbol{\theta}^*\in\Omega_n^*(R)} \sqrt{n}\,\|\hat{\boldsymbol{\theta}}_n^* - \boldsymbol{\theta}^*\|_{pI} > \varepsilon\right\} = 0 \quad \textit{for} \quad \varepsilon > 0.$$

PROOF. The proof is similar to that of Theorem 5.3 and is therefore omitted. □

6.3 *Confidence regions*

If Assumptions 1–3, 5 and 7 are satisfied, from Theorem 6.3, we find

$$nJ\{\mathrm{vec}(\hat{\boldsymbol{\theta}}_n^* - \boldsymbol{\alpha}^*)\}'(E_I \otimes \Xi^{-1})\{\mathrm{vec}(\hat{\boldsymbol{\theta}}_n^* - \alpha^*)\} \xrightarrow{\mathcal{L}} \chi^2_{p(I-1)}.$$

Letting $\hat{\Sigma}^*$ be a consistent estimator of the unknown Ξ^{-1}, if we put

$$CR^*(\tau) = \{\boldsymbol{\theta}^*; n\{\mathrm{vec}(\boldsymbol{\theta}^* - \hat{\boldsymbol{\theta}}_n^*)\}'(E_I \otimes \hat{\Sigma}^*)\{\mathrm{vec}(\boldsymbol{\theta}^* - \hat{\boldsymbol{\theta}}_n^*)\} \le \chi^2_{p(I-1)}(\tau)\},$$

$CR^*(\tau)$ is an asymptotically $100(1-\tau)$ percent distribution-free confidence region for $\boldsymbol{\alpha}^*$. So, we construct the consistent estimator. Using Theorem 4.1, we get

LEMMA 6.3. *For some positive c, let us define, for* $i = 1, \ldots, I$,

$$\hat{d}_{n(i)}^{*(l)} = \{S_i^{*(l)}(\hat{\boldsymbol{\theta}}_n^* - c\mathcal{E}_i/\sqrt{n}) - S_i^{*(l)}(\hat{\boldsymbol{\theta}}_n^* + c\mathcal{E}_i/\sqrt{n})\}/\{(2c)(1-1/I)\},$$

where $\mathcal{E}_i$ *is a* $p \times I$ *matrix with* $\mathbf{1}_p$ *at the i-th column and zero vector elsewhere for each i. Suppose that Assumptions* 1–3 *are satisfied. Then,* $\hat{d}_{n(i)}^{*(l)}$ *converges in probability to* d_l.

As an estimator of d_l, we choose $\hat{d}_l^* = \sum_{i=1}^{I} \hat{d}_{n(i)}^{*(l)}/I$ and put $\hat{D}^* = \mathrm{diag}(\hat{d}_1^*, \ldots, \hat{d}_p^*)$. Replacing $R_{ijk}^{*(l)}$ by $R_{ijk}^{*(l)}(\hat{\boldsymbol{\theta}}_n^*)$ for all i, j, k and l in (6.1), we denote the corresponding random variable by $\hat{\gamma}_{ll'}(R^*(\hat{\boldsymbol{\theta}}_n^*))$ and set

$$\Gamma(R^*(\hat{\boldsymbol{\theta}}_n^*)) = (\hat{\gamma}_{ll'}(R^*(\hat{\boldsymbol{\theta}}_n^*)))_{l,l'=1,\ldots,p}.$$

Proceeding as in the proof of Lemma 5.2, we get

LEMMA 6.4. *Suppose that Assumptions* 1–4 *and* 7 *are satisfied. Then,* $\Gamma(R^*(\hat{\boldsymbol{\theta}}_n^*))$ *converges in probability to* Γ.

Hence, combining Lemma 6.3 with Lemma 6.4, we get

THEOREM 6.4. *Suppose that Assumptions* 1–5 *and* 7 *are satisfied. Let* $\hat{\Sigma}^* = \hat{D}^*\{\Gamma(R^*(\hat{\boldsymbol{\theta}}_n^*))\}^-\hat{D}^*$. *Then* $CR^*(\tau)$ *is an asymptotically* $100(1-\tau)$ *percent distribution-free confidence region for* $\boldsymbol{\alpha}^*$.

7. Statistical inference for overall mean response

Throughout this section, we add

ASSUMPTION 8. $f(\boldsymbol{x}) = \partial^p F(\boldsymbol{x})/\partial x_1 \cdots \partial x_p$ is diagonally symmetric about $\mathbf{0}$, that is, $f(-\boldsymbol{x}) = f(\boldsymbol{x})$.

Since we consider the statistical inference based on $R_{ijk}^{+(l)}(\boldsymbol{t}^+)$'s in this section and since the distributions of statistics under the model (1.1) do not depend on $\boldsymbol{\alpha}_i$'s, $\boldsymbol{\beta}_j$'s, or $(\boldsymbol{\alpha\beta})_{ij}$'s, throughout this section it is assumed without a loss of generality that

$$(7.1) \qquad \boldsymbol{\alpha}_1 = \cdots = \boldsymbol{\alpha}_I = \boldsymbol{\beta}_1 = \cdots = \boldsymbol{\beta}_J = (\boldsymbol{\alpha\beta})_{11} = \cdots = (\boldsymbol{\alpha\beta})_{IJ} = \mathbf{0}.$$

Let us put p column vectors $\boldsymbol{S}^+(\boldsymbol{t}^+) = (S^{+(1)}(\boldsymbol{t}^+), \ldots, S^{+(p)}(\boldsymbol{t}^+))'$ and $\tilde{\boldsymbol{S}}^+(\boldsymbol{s}^+, \boldsymbol{t}^+) = (\tilde{S}^{+(1)}(\boldsymbol{s}^+, \boldsymbol{t}^+), \ldots, \tilde{S}^{+(p)}(\boldsymbol{s}^+, \boldsymbol{t}^+))'$. Also, to reduce notational complexity, we set $\boldsymbol{S}^+ = \boldsymbol{S}^+(\mathbf{0})$, $\tilde{\boldsymbol{S}}^+ = \tilde{\boldsymbol{S}}^+(\mathbf{0}, \mathbf{0})$, $Y_{ijk}^{+(l)} = Y_{ijk}^{+(l)}(\mathbf{0})$, $R_{ijk}^{+(l)} = R_{ijk}^{+(l)}(\mathbf{0})$ and $Q_{ijk}^{+(l)} = Q_{ijk}^{+(l)}(\mathbf{0}, \mathbf{0})$.

7.1 *Tests*

Based on the asymptotic distribution of $\boldsymbol{S}^+$ under H^+, we test the null hypothesis H^+ versus the alternative A^+.

LEMMA 7.1. *Suppose that Assumptions* $1'$, 2, 3 *and* 8 *are satisfied. Then under* H^+, *as* $n \to \infty$, $\boldsymbol{S}^+$ *has asymptotically a p-variate normal distribution with mean* $\mathbf{0}$ *and a variance-covariance matrix* Γ^+, *where* $\Gamma^+ = (\gamma_{ll'}^+/(IJ))_{l,l'=1,\ldots,p}$ *and*

$$\gamma_{ll'}^+ = \begin{cases} \displaystyle\int_0^1 \{\psi_l(u)\}^2 du & \text{if } l = l', \\ \displaystyle\int_{R^2} \operatorname{sign}(x) \cdot \operatorname{sign}(y) \cdot \psi_l(F_l(|x|))\psi_{l'}(F_{l'}(|y|)) dF_{ll'}(x, y) & \text{elsewhere.} \end{cases}$$

PROOF. Theorem 4.3 shows that $\boldsymbol{S}^+ - \tilde{\boldsymbol{S}}^+ \xrightarrow{P} \mathbf{0}$ under H^+. Furthermore, it is simple to verify that $\tilde{\boldsymbol{S}}^+ \xrightarrow{\mathcal{L}} N_p(\mathbf{0}, \Gamma^+)$. Hence, the conclusion is found. □

Next, we draw a consistent estimator of Γ^+. Let us put $\Gamma^+(R^+) = (\hat{\gamma}^+_{ll'}(R^+)/(IJ))_{l,l'=1,\dots,p}$ and $\Gamma^+(Q^+) = (\hat{\gamma}^+_{ll'}(Q^+)/(IJ))_{l,l'=1,\dots,p}$, where

$$(7.2)\quad \hat{\gamma}^+_{ll'}(R^+) = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{n} \operatorname{sign}(Y^{+(l)}_{ijk}) \cdot \operatorname{sign}(Y^{+(l')}_{ijk}) \cdot a^{+(l)}_N(R^{+(l)}_{ijk}) a^{+(l')}_N(R^{+(l')}_{ijk})/N$$

and

$$\hat{\gamma}^+_{ll'}(Q^+) = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{n} \operatorname{sign}(X^{(l)}_{ijk}) \cdot \operatorname{sign}(X^{(l')}_{ijk}) \cdot a^{+(l)}_N(Q^{+(l)}_{ijk}) a^{+(l')}_N(Q^{+(l')}_{ijk})/N.$$

LEMMA 7.2. *Suppose that Assumptions* 1′, 2–4 *and* 8 *are satisfied. Then under* H^+, $\Gamma^+(R^+)$ *converges in probability to* Γ^+.

PROOF. Theorem 5.4.4 of Puri and Sen (1985) shows $\Gamma^+(Q^+) \xrightarrow{P} \Gamma^+$. By the proof similar to that of Lemma 5.2, we can verify that $\hat{\gamma}^+_{ll'}(R^+) - \hat{\gamma}^+_{ll'}(Q^+) \xrightarrow{P} 0$ for all l and l'. Thus, the conclusion is found. □

We reject H^+ when the following statistic is too large: $AL^+ = \boldsymbol{S}^{+\prime} \cdot \{\Gamma^+(R^+)\}^- \boldsymbol{S}^+$.

ASSUMPTION 9. Γ^+ is positive definite.

Then, combining Lemma 7.1 with Lemma 7.2, we get

THEOREM 7.1. *Suppose that Assumptions* 1′, 2–4, 8 *and* 9 *are satisfied. Then under* H^+, *as* $n \to \infty$, AL^+ *has asymptotically a* χ^2*-distribution with* p *degrees of freedom.*

Next, we consider the sequence of local alternatives A^+_n; $\boldsymbol{\mu} = \boldsymbol{\Delta}^+/\sqrt{n}$, $\boldsymbol{\Delta}^+ \neq \mathbf{0}$, where $\boldsymbol{\Delta}^+ = (\Delta^{+(1)}, \dots, \Delta^{+(p)})'$.

Using Theorem 4.3, as in the proof of Theorem 7.1, we get

THEOREM 7.2. *Suppose that Assumptions* 1′, 2–4, 6, 8 *and* 9 *are satisfied. Then under* A^+_n, *as* $n \to \infty$, AL^+ *has asymptotically a noncentral* χ^2*-distribution with* p *degrees of freedom and noncentrality parameter* δ^2, *where* $\delta^2 = \boldsymbol{\nu}^{+\prime}(\Gamma^+)^{-1}\boldsymbol{\nu}^+$, $\boldsymbol{\nu}^+ = (\nu^{+(1)}, \dots, \nu^{+(p)})'$ *and* $\nu^{+(l)} = d^+_l \cdot \Delta^{+(l)}$.

7.2 *Point estimates*

Using a method similar to that of Hodges and Lehmann (1963), we propose $\hat{\boldsymbol{\theta}}_n^+ = (\hat{\theta}_n^{+(1)}, \ldots, \hat{\theta}_n^{+(p)})'$ as an estimator of $\boldsymbol{\mu} = (\mu^{(1)}, \ldots, \mu^{(p)})'$, based on aligned ranks, where

$$\hat{\theta}_n^{+(l)} = [\inf\{t^{+(l)}; S^{+(l)}(\boldsymbol{t}^+) < 0\} + \sup\{t^{+(l)}; S^{+(l)}(\boldsymbol{t}^+) > 0\}]/2 \quad \text{for} \quad l = 1, \ldots, p.$$

We find that $\hat{\boldsymbol{\theta}}_n^+$ is the center of gravity of the set of admissible solutions of $\boldsymbol{t}^+$ for which $\sum_{l=1}^p |S^{+(l)}(\boldsymbol{t}^+)| = \text{minimum}$.

By using Theorem 4.3, we get

THEOREM 7.3. *Suppose that Assumptions* $1'$, 2, 3 *and* 8 *are satisfied. Then* $\sqrt{n} \cdot \mathrm{vec}(\hat{\boldsymbol{\theta}}_n^+ - \boldsymbol{\mu})$ *has a p-variate normal distribution with mean* $\mathbf{0}$ *and a variance-covariance matrix* Ξ^+, *where* $\Xi^+ = (\xi_{ll'}^+/(IJ))_{l,l'=1,\ldots,p}$ *and* $\xi_{ll'}^+ = \gamma_{ll'}^+/(d_l^+ \cdot d_{l'}^+)$.

7.3 *Confidence regions*

If Assumptions $1'$, 2, 3, 8 and 9 are satisfied, from Theorem 7.3, we find that

$$n(\hat{\boldsymbol{\theta}}_n^+ - \boldsymbol{\mu})'(\Xi^+)^{-1}(\hat{\boldsymbol{\theta}}_n^+ - \boldsymbol{\mu}) \xrightarrow{\mathcal{L}} \chi_p^2.$$

Letting $\hat{\Sigma}^+$ be a consistent estimator of the unknown $(\Xi^+)^{-1}$, if we put

$$CR^+(\tau) = \{\boldsymbol{\theta}^+; n(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_n^+)'\hat{\Sigma}^+(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_n^+) \leq \chi_p^2(\tau)\},$$

$CR^+(\tau)$ is an asymptotically $100(1-\tau)$ percent distribution-free confidence region for $\boldsymbol{\mu}$. So, we construst the consistent estimator.

LEMMA 7.3. *For some positive c, let us define, for* $l = 1, \ldots, p$,

$$\hat{d}^{+(l)} = \{S^{+(l)}(\hat{\boldsymbol{\theta}}_n^+ - c\mathbf{1}_p/\sqrt{n}) - S^{+(l)}(\hat{\boldsymbol{\theta}}_n^+ + c\mathbf{1}_p/\sqrt{n})\}/(2c).$$

Then under Assumptions $1'$–3 *and* 8, $\hat{d}^{+(l)}$ *converges in probability to* d_l^+.

PROOF. From Theorem 4.3, we can find

$$S^{+(l)}(\hat{\boldsymbol{\theta}}_n - c\mathbf{1}_p/\sqrt{n}) - S^{+(l)}(\hat{\boldsymbol{\theta}}_n + c\mathbf{1}_p/\sqrt{n}) \xrightarrow{P} 2cd_l^+,$$

which implies the conclusion. □

We put $\hat{D}^+ = \mathrm{diag}(\hat{d}_1^+, \ldots, \hat{d}_p^+)$. Replacing $R_{ijk}^{+(l)}$ by $R_{ijk}^{+(l)}(\hat{\boldsymbol{\theta}}_n^+)$ for all i, j, k and l in (7.2), we denote the corresponding random variable by $\hat{\gamma}_{ll'}^+(R^+(\hat{\boldsymbol{\theta}}_n^+))$ and put

$$\Gamma(R^+(\hat{\boldsymbol{\theta}}_n^+)) = (\hat{\gamma}_{ll'}^+(R^+(\hat{\boldsymbol{\theta}}_n^+)))_{l,l'=1,\ldots,p}.$$

Proceeding as in the proof of Lemma 7.2, we get

LEMMA 7.4. *Suppose that Assumptions* 1′, 2, 3, 8 *and* 9 *are satisfied. Then,* $\Gamma(R^+(\hat{\boldsymbol{\theta}}_n^+))$ *converges in probability to* Γ^+.

Hence, combining Lemma 7.3 with Lemma 7.4, we get

THEOREM 7.4. *Let* $\hat{\Sigma}^+ = \hat{D}^+\{\Gamma^+(R^+(\hat{\boldsymbol{\theta}}_n^+))\}^-\hat{D}^+$. *Then under the assumptions of Lemma* 7.4, $CR^+(\tau)$ *is an asymptotically* $100(1-\tau)$ *percent distribution-free confidence region for* $\boldsymbol{\mu}$.

8. ARE and robustness

We discuss the asymptotic relative efficiencies (ARE's) of the proposed tests and estimators with respect to the normal theory parametric tests and classical unbiased estimators. For $p \geq 2$, the ARE's are complicated, especially in the case of the tests, the ARE under A_n (A_n^*, A_n^+) depends on parameter $\boldsymbol{\Delta}$ ($\boldsymbol{\Delta}^*$, $\boldsymbol{\Delta}^+$) and we can discuss the ARE as in Section 4 of Sen (1971). So we give the ARE's for $p = 1$. Normal theory F-tests were reviewed by Dunn and Clark ((1987), Chapter 7). It is simple to verify that (normalized likelihood ratio F-test) $\xrightarrow{\mathcal{L}} \chi^2_{(I-1)(J-1)}$ under H and $\xrightarrow{\mathcal{L}} \chi^2_{(I-1)(J-1)}(\eta^2)$ under A_n, where $\eta^2 = \boldsymbol{\Delta}\cdot\boldsymbol{\Delta}'/\operatorname{Var}(e_{111})$. Also, we find $\sqrt{n}((\widehat{\boldsymbol{\alpha\beta}}) - (\boldsymbol{\alpha\beta}))' \xrightarrow{\mathcal{L}} N(0, \operatorname{Var}(e_{111})\cdot\Lambda)$, where $(\widehat{\boldsymbol{\alpha\beta}})$ is a least squares estimator and is defined in Corollary 4.2. Combining these facts with Theorems 5.1, 5.2 and 5.3, we get

$$
\begin{aligned}
&\operatorname{ARE}(AL, \text{the F-test}) \\
&\quad = \operatorname{ARE}(\hat{\boldsymbol{\theta}}_n, (\widehat{\boldsymbol{\alpha\beta}})) \\
&\quad = \operatorname{Var}(e_{111}) \\
&\quad \cdot \left[\int_0^1 \psi_1(u)\cdot\{-f'(F^{-1}(u))/f(F^{-1}(u))\}du\right]^2 \Big/ \int_0^1 \{\psi_1(u) - \bar{\psi}_1\}^2 du,
\end{aligned}
$$

which is equivalent to the classical ARE-results of the two-sample rank test with respect to the t-test, where $\operatorname{ARE}(C, D)$ stands for the asymptotic relative efficiency of C with respect to D. The ARE-results of the proposed R-confidence regions for $(\boldsymbol{\alpha\beta})$ and the other proposed statistical inferences relative to F-tests and classical inferences based on unbiased estimators likewise, remain identical in the case.

Moreover, as in Section 6 of Shiraishi (1989*a*), the choice of an asymptotic optimal score generating function can be given, and asymptotically maximin power tests and minimax variance estimators due to Huber (1981) can be drawn.

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