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ADMISSIBILITY OF UNBIASED TESTS FOR A COMPOSITE HYPOTHESIS WITH A RESTRICTED ALTERNATIVE

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Abstract. This paper discusses α-admissibility and d-admissibility which are important concepts in studying the performance of statistical tests for composite hypotheses. A sufficient condition for α-admissibility is presented. When $\alpha = 1/m$, the Nomakuchi-Sakata test, which is uniformly more powerful than the likelihood ratio test for hypotheses $\min(\theta_1, \theta_2) = 0$ versus $\min(\theta_1, \theta_2) > 0$, is generalized for a class of distributions in an exponential family, and its unbiasedness and α-admissibility are shown. Finally, the case of $\alpha \neq 1/m$ is discussed in brief.

Key words and phrases: Nomakuchi-Sakata test, α-admissibility, d-admissibility, unbiasedness, exponential family, completeness.

1. Introduction

Let $X = (X_1, X_2)'$ be a bivariate normal random vector with unknown mean $\theta = (\theta_1, \theta_2)'$ and the identity covariance matrix I. We consider testing H_0 : $\min(\theta_1, \theta_2) = 0$ versus $H_1 : \min(\theta_1, \theta_2) > 0$. The likelihood ratio test of size α for this hypothesis has been given by

$$\varphi_{LR}(x_1, x_2) = \begin{cases} 1 & \text{if } \ x_1, x_2 > z(\alpha), \\ 0 & \text{otherwise,} \end{cases}$$

where $z(\alpha)$ is the upper $100\alpha\%$ point of the standard normal distribution (cf. Inada (1978), Sasabuchi (1980, 1988*a*, 1988*b*)). The admissibility of this test was shown by Cohen *et al.* (1983) and also Nomakuchi and Sakata (1987).

This might sound as if there exist no uniformly more powerful tests than φ_{LR}. In fact it is well known that likelihood ratio tests are optimum in many testing problems. Interestingly, however, Nomakuchi and Sakata (1987) showed that

$$\varphi_U(x_1, x_2) = \begin{cases} 1 & \text{if } \ z(i/m) < x_1, x_2 < z((i-1)/m), \ i = 1, \dots, m, \\ 0 & \text{otherwise,} \end{cases}$$

is a level $1/m$ unbiased test and uniformly most powerful than the likelihood ratio test. This test is essentially proposed by Lehmann (1952). Lehmann (1952) considered testing null hypothesis H_1 versus $H_2 : \min(\theta_1, \theta_2) \leq 0$ and gave a class of similar tests including φ_U. Gutmann (1987) and Berger (1989) also gave tests which are uniformly more powerful than φ_{LR}.

In this paper, we generalize the Nomakuchi-Sakata test for the distributions which belong to an exponential family and show its unbiasedness and α-admissibility. Since the Nomakuchi-Sakata test is not practical, our discussion might only satisfy theoretical interest. But it is an interesting fact that this test is optimal in the sense that there exists no test which is uniformly more powerful than φ_U.

2. Preliminary

2.1 *The d-admissibility and α-admissibility*

To begin with, we consider the α-admissibility and d-admissibility which are important concepts for the problem of testing composite hypotheses. Let $\{P_\theta; \theta \in \Theta\}$ be a parametric family of probability measures. The expectation of a function f with respect to P_θ is denoted by $E_\theta[f]$. The following two definitions are due to Lehmann (1986).

DEFINITION 2.1. A test function φ is said to be d-admissible for $H_0 : \theta \in \Theta_0$ versus $H_1 : \theta \in \Theta_1$ if for any test function ψ the inequalities

$$\begin{aligned} &E_\theta[\varphi] \geq E_\theta[\psi], \qquad \text{for all } \theta \in \Theta_0 \\ &E_\theta[\varphi] \leq E_\theta[\psi], \qquad \text{for all } \theta \in \Theta_1 \end{aligned} \tag{2.1}$$

imply $E_\theta[\varphi] = E_\theta[\psi]$ for all $\theta \in \Theta_0 \cup \Theta_1$.

DEFINITION 2.2. A test function φ is said to be α-admissible for $H_0 : \theta \in \Theta_0$ versus $H_1 : \theta \in \Theta_1$ if for any test ψ of level α, the inequalities (2.1) imply $E_\theta[\varphi] = E_\theta[\psi]$ for all $\theta \in \Theta_1$.

Characteristics of the d-admissibility and α-admissibility were discussed in Lehmann (1986). The α-admissibility guarantees the nonexistence of uniformly more powerful test of level α, but the d-admissibility does not. The likelihood ratio test φ_{LR} is d-admissible but not α-admissible. Thus it is not unusual that the Nomakuchi-Sakata test is uniformly more powerful than φ_{LR}.

The d-admissibility is easier to check than α-admissibility. We introduce the following proposition which is obtained from Theorem 9 of Chapter 6 in Lehmann (1986).

For a level α test φ, we put

$$\Theta_0^* = \{\theta \in \Theta_0; E_\theta[\varphi] = \alpha\}.$$

PROPOSITION 2.1. *We assume that Θ_0^* is not empty. If a level α test φ is d-admissible for $H_0^* : \theta \in \Theta_0^*$ versus $H_1 : \theta \in \Theta_1$, then it is α-admissible for $H_0 : \theta \in \Theta_0$ versus H_1.*

PROOF. Let ψ be any level α test for H_0 which satisfies

$$E_\theta[\psi] \geq E_\theta[\varphi] \qquad \text{for all } \theta \in \Theta_1.$$

Since ψ is a level α test, we have

$$E_\theta[\psi] \leq \alpha = E_\theta[\varphi] \qquad \text{for all } \theta \in \Theta_0^*.$$

Then the d-admissibility of φ for H_0^* versus H_1 implies

$$E_\theta[\psi] = E_\theta[\varphi] \qquad \text{for all } \theta \in \Theta_1.$$

□

2.2 *The lemmas*

We develop two lemmas which will be used in the next section. We assume for a finite continuous measure μ that

$$dP_\theta(x) = c(\theta)\exp(\theta x)d\mu(x) \qquad \text{and} \qquad \int \exp(\theta x)d\mu(x) < \infty$$

for all $\theta \in \boldsymbol{R}$.

LEMMA 2.1. *Let f be any measurable function on $\boldsymbol{R}$ such that for some $a \in \boldsymbol{R}$,*

$$f(x) \begin{cases} \geq 0 & \text{for } \; x > a, \\ \leq 0 & \text{for } \; x < a, \end{cases}$$

and $\mu\{x; f(x) > 0\} > 0$. Then if $E_{\theta_0}[f] = 0$ for some $\theta_0 \in \boldsymbol{R}$, we have

$$E_{\theta_0+t}[f] \begin{cases} > 0 & \text{for all } \; t > 0, \\ < 0 & \text{for all } \; t < 0. \end{cases}$$

PROOF.

$$\begin{aligned} E_{\theta_0+t}[f] &= c(\theta_0+t)\int f(x)\exp\{(\theta_0+t)x\}d\mu \\ &= c(\theta_0+t)\exp(ta)\int f(x)\exp\{(\theta_0+t)x - ta\}d\mu \\ &= c(\theta_0+t)\exp(ta)\left[\int_{\{x-a\geq 0\}} f(x)\exp(\theta_0 x)\exp\{t(x-a)\}d\mu \right. \\ &\qquad \left. + \int_{\{x-a<0\}} f(x)\exp(\theta_0 x)\exp\{t(x-a)\}d\mu\right]. \end{aligned} \tag{2.2}$$

In the case that t is a positive constant, we have

$$\begin{aligned} \exp\{t(x-a)\} > 1 \quad &\text{if} \quad x > a, \\ 0 < \exp\{t(x-a)\} < 1 \quad &\text{if} \quad x < a. \end{aligned}$$

According to the condition on f, we have

$$E_{\theta_0+t}[f] > c(\theta_0+t)\exp(ta)\int f(x)\exp(\theta_0 x)d\mu$$
$$= c(\theta_0+t)\exp(ta)E_{\theta_0}[f]/c(\theta_0) = 0.$$

The proof of the other case is similar. □

Lemma 2.1 is a special case of Theorem 3.1 of Chapter 5 in Karlin (1968). The following lemma is due to Birnbaum (1955) and Stein (1956).

LAMMA 2.2. *Let f be any bounded measurable function on $\boldsymbol{R}$ such that for some $a \in \boldsymbol{R}$,*

$$f(x) \geq 0 \quad \text{for} \quad x > a,$$

and $\mu\{x; f(x) > 0 \text{ and } x > a\} > 0$. Then for any θ_0 and some sufficiently large t, it follows that $E_{\theta_0+t}[f] > 0$.

PROOF. The first integral in (2.2) approaches $+\infty$ as $t \to +\infty$ and the second integral is bounded. This completes the proof. □

3. Main results

Suppose that X_1 and X_2 are independent random variables such that each variable is distributed as one parameter exponential family $\{P_{\theta_i}; \theta_i \in \boldsymbol{R}\}$, where

$$dP_{\theta_i} = c(\theta_i)\exp(\theta_i x_i)d\mu, \qquad i = 1, 2,$$

for a finite continuous measure μ.

We consider testing $H_0 : \min(\theta_1, \theta_2) = 0$ versus $H_1 : \min(\theta_1, \theta_2) > 0$ at level $1/m$ $(m = 2, 3, \ldots)$.

We employ the test φ_U^* defined by

$$\varphi_U^*(x_1, x_2) = \begin{cases} 1 & \text{if } (x_1, x_2) \in A_i,\ i = 1, \ldots, m, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$A_i = \{(x_1, x_2); q(i/m) < x_1, x_2 < q((i-1)/m)\}, \qquad i = 1, \ldots, m,$$

and $q(\alpha)$ is the upper $100\alpha\%$ point of P_0. In the normal case φ_U^* is nothing but φ_U discussed in the introduction.

Now let ψ be any test such that

$$E_{(\theta_1,\theta_2)}[\varphi_U^*] \begin{cases} \geq E_{(\theta_1,\theta_2)}[\psi] & \text{under } H_0, \\ \leq E_{(\theta_1,\theta_2)}[\psi] & \text{under } H_1. \end{cases}$$

Since $E_{(\theta_1,\theta_2)}[\ ]$ is continuous in (θ_1, θ_2), we have

$$E_{(\theta_1,\theta_2)}[\varphi_U^* - \psi] = \int (\varphi_U^* - \psi) dP_{\theta_1}(x_1) dP_{\theta_2}(x_2) = 0 \quad \text{under } H_0. \tag{3.1}$$

Because of the completeness of $\{P_{\theta_i}; \theta_i \geq 0\}$, (3.1) is equivalent to the following two conditions,

$$\int (\varphi_U^* - \psi) dP_{\theta_i}(x_i)|_{\theta_i=0} = 0 \quad \text{for almost all } x_j,\ j \neq i,\ i = 1, 2;$$

that is,

$$\int (\varphi_U^* - \psi) d\mu(x_i) = 0 \quad \text{for almost all } x_j,\ j \neq i,\ i = 1, 2. \tag{3.2}$$

In order to examine the d-admissibility of φ_U^*, it is sufficient to consider a class of the tests which satisfy (3.1) or (3.2) as competitors.

In the sequel $\mu \otimes \mu$ represents the direct product of measures.

LEMMA 3.1. *Let f be any bounded measurable function such that for some real numbers a and b,*

$$f(x_1, x_2) \begin{cases} \geq 0 & \textit{for } \ x_1 > a, x_2 > b \ \textit{ or } \ x_1 < a, x_2 < b, \\ \leq 0 & \textit{for } \ x_1 < a, x_2 > b \ \textit{ or } \ x_1 > a, x_2 < b, \end{cases}$$

and that $\mu \otimes \mu\{(x_1, x_2); f(x_1, x_2) > 0\} > 0$. If it holds that $E_{(\theta_1,\theta_2)}\ [f(X_1, X_2)] = 0$ for all (θ_1, θ_2) such that $\min(\theta_1, \theta_2) = 0$, then we have $E_{(\theta_1,\theta_2)}[f(X_1, X_2)] > 0$ for all (θ_1, θ_2) such that $\min(\theta_1, \theta_2) > 0$.

PROOF. Put

$$g(x_1, \theta_2) = \int f(x_1, x_2) c(\theta_2) \exp(\theta_2 x_2) d\mu(x_2), \tag{3.3}$$

then

$$E_{(\theta_1,\theta_2)}[f(X_1, X_2)] = \int g(x_1, \theta_2) c(\theta_1) \exp(\theta_1 x_1) d\mu(x_1).$$

Fix $\theta_2 > 0$. Since $g(x_1, 0) = 0$, it follows from Lemma 2.1 that $g(x_1, \theta_2) \geq 0$ for $x_1 > a$. We also have $g(x_1, \theta_2) \leq 0$ for $x_1 < a$ by reversed version of Lemma 2.1. Since $E_{(0,\theta_2)}[f(X_1, X_2)] = 0$ and $\mu\{x_1; g(x_1, \theta_2) > 0\} > 0$, from Lemma 2.1 we also have $E_{(\theta_1,\theta_2)}[f(X_1, X_2)] > 0$ for any $\theta_1 > 0$. □

LEMMA 3.2. *Let f be any bounded measurable function such that for some real numbers a, b,*

$$f(x_1, x_2) \begin{cases} \geq 0 & \textit{for } \ x_1 > a, x_2 > b, \\ \leq 0 & \textit{for } \ x_1 > a, x_2 < b, \end{cases}$$

and that $\mu \otimes \mu\{(x_1, x_2); x_1 > a, x_2 > b, f(x_1, x_2) > 0\} > 0$. *If it holds that* $E_{(\theta_1,\theta_2)}[f(X_1, X_2)] = 0$ *for all* (θ_1, θ_2) *such that* $\min(\theta_1, \theta_2) = 0$, *then there exist* $\theta_1 > 0$ *and* $\theta_2 > 0$ *which satisfy* $E_{(\theta_1,\theta_2)}[f(X_1, X_2)] > 0$.

PROOF. Let $g(x_1, \theta_2)$ be that defined by (3.3). Since we have

$$g(x_1, \theta_2) \geq 0 \qquad \text{for all } \theta_2 > 0 \text{ and } x_1 > a,$$

and $\mu\{x_1 > a, g(x_1, \theta_2) > 0\} > 0$, the proof of the lemma is obvious from Lemma 2.2. □

Nomakuchi and Sakata (1987) showed that the φ_U discussed in the introduction is an unbiased test under the assumption of the Schur-concavity of the joint density function. In the following theorem the Schur-concavity is not assumed; therefore, the test and distribution are generalized to the exponential family.

THEOREM 3.1. φ_U^* *is an unbiased test.*

PROOF. We define for $i, j = 1, \ldots, m$,

$$A_{ij} = \{(x_1, x_2); q(i/m) < x_1 < q((i-1)/m), q(j/m) < x_2 < q((j-1)/m)\},$$

$$f_{ij}(x_1, x_2) = \begin{cases} 1/m & \text{if } (x_1, x_2) \in A_{ii} \cup A_{jj}, \\ -1/m & \text{if } (x_1, x_2) \in A_{ij} \cup A_{ji}, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\varphi_U^*(x_1, x_2) = 1/m + \sum_{i<j} f_{ij}(x_1, x_2).$$

Since each f_{ij} $(i < j)$ satisfies the condition of Lemma 3.1 for $a = b = q(j/m)$, it holds that

$$E_{(\theta_1,\theta_2)}[\varphi_U^*] \begin{cases} > 1/m & \text{if } \min(\theta_1, \theta_2) > 0, \\ = 1/m & \text{if } \min(\theta_1, \theta_2) = 0. \end{cases}$$

Thus φ_U^* is an unbiased test. □

THEOREM 3.2. φ_U^* *is* $\alpha(= 1/m)$*-admissible.*

PROOF. From Proposition 2.1, it is enough to show that φ_U^* is d-admissible, that is, there exist $\theta_1 > 0$, $\theta_2 > 0$ such that $E_{(\theta_1,\theta_2)}[\varphi_U^*] > E_{(\theta_1,\theta_2)}[\psi]$ for any ψ satisfying (3.2) and $\mu\otimes\mu\{(x_1, x_2); \varphi_U^*(x_1, x_2) - \psi(x_1, x_2) \neq 0\} > 0$. Let $f(x_1, x_2) = \varphi_U^*(x_1, x_2) - \psi(x_1, x_2)$. Note that we have

$$f(x_1, x_2) \begin{cases} \geq 0 & \text{if } (x_1, x_2) \in A_i, i = 1, \ldots, m, \\ \leq 0 & \text{otherwise.} \end{cases}$$

$\mu\otimes\mu\{(x_1, x_2); f(x_1, x_2) > 0\} = 0$ implies $\mu\otimes\mu\{(x_1, x_2); f(x_1, x_2) \neq 0\} = 0$. Thus it is enough to consider the case where $\mu \otimes \mu\{(x_1, x_2); f(x_1, x_2) > 0\} > 0$. Let i_0 be the maximum number among i's satisfying

$$\mu \otimes \mu\{(x_1, x_2) \in A_i; f(x_1, x_2) > 0\} > 0.$$

Since it holds from (3.2) that $\mu \otimes \mu\{(x_1, x_2); [x_1 > q((i_0 - 1)/m)$ or $x_2 > q((i_0 - 1)/m)]$ and $f(x_1, x_2) < 0\} = 0$, we have

$$\mu \otimes \mu\{(x_1, x_2); x_1 > q(i_0/m), x_2 > q(i_0/m), f(x_1, x_2) < 0\} = 0,$$
$$\mu \otimes \mu\{(x_1, x_2); x_1 > q(i_0/m), x_2 < q(i_0/m), f(x_1, x_2) > 0\} = 0.$$

Since we can ignore the set of measure zero, it holds from Lemma 3.2 that

$$E_{(\theta_1,\theta_2)}[\varphi_U^*] > E_{(\theta_1,\theta_2)}[\psi] \quad \text{for some} \quad \theta_1 > 0, \theta_2 > 0.$$

This implies the d-admissibility of φ_U^*. □

Berger (1989) gave a class of tests which are uniformly more powerful than the likelihood ratio test in the normal case. His null hypothesis is slightly different from ours and represented by $H_2 : \min(\theta_1, \theta_2) \leq 0$. The constant tests are only unbiased tests and all tests are d-admissible for this hypothesis (cf. Lehmann (1952)). For the distributions in the exponential family, the Berger test may be generalized to

$$\varphi(x_1, x_2) = \begin{cases} 1 & \text{if} \quad (x_1, x_2) \in A_i,\ i = k, \ldots, m, \\ 0 & \text{otherwise,} \end{cases}$$

for some integer $k = 2, \ldots, m$. We apply the Berger test to our null hypothesis. It is easy to show that the Berger test is uniformly less powerful than the Nomakuchi-Sakata test, but the d-admissibility of the Berger test is shown in the same way as Theorem 3.2.

Finally, we consider the case of $\alpha \neq 1/m$ $(m = 2, 3, \ldots)$.

When $\alpha < 1/m$ $(m = 1, 2, \ldots)$, it was mentioned by Nomakuchi and Sakata (1987) that a randomized test

$$\varphi_1(x_1, x_2) = \begin{cases} \alpha m & \text{if} \quad (x_1, x_2) \in A_i,\ i = 1, \ldots, m, \\ 0 & \text{otherwise} \end{cases}$$

is unbiased. But it is shown as follows that φ_1 is not *admissible.* To begin with, we divide A_1 into four regions

$$B_{ij} = \{(x_1, x_2); q(i/2m) < x_1 < q((i-1)/2m), q(j/2m) < x_2 < q((j-1)/2m)\}, \quad i, j = 1, 2.$$

Let g be a function such that

$$g(x_1, x_2) = \begin{cases} \min\{(1 - \alpha m), \alpha m\} & \text{if} \quad (x_1, x_2) \in B_{11} \cup B_{22}, \\ -\min\{(1 - \alpha m), \alpha m\} & \text{if} \quad (x_1, x_2) \in B_{12} \cup B_{21}, \\ 0 & \text{otherwise.} \end{cases}$$

Since g satisfies the condition of Lemma 3.1, the test function $\varphi_1 + g$ is uniformly more powerful than φ_1. Thus φ_1 is neither α-admissible nor d-admissible.

Although we do not have unbiased and *admissible* tests for all $\alpha \neq 1/m$, we will present an example of such tests. When $\alpha = 2/3$, we define a test function φ_2 with

$$\varphi_2(x_1, x_2) = \begin{cases} 0 & \text{if } (x_1, x_2) \in A_{13} \cup A_{22} \cup A_{31}, \\ 1 & \text{otherwise,} \end{cases}$$

where $A_{ij} = \{(x_1, x_2); q(i/3) < x_1 < q((i-1)/3), q(j/3) < x_2 < q((j-1)/3)\}$. Since it holds that

$$\varphi_2(x_1, x_2) = 2/3 + \sum_{1 \leq i < j \leq 3} h_{ij}(x_1, x_2),$$

where

$$h_{ij}(x_1, x_2) = \begin{cases} 1/3 & \text{if } (x_1, x_2) \in A_{i(4-j)} \cup A_{j(4-i)}, \\ -1/3 & \text{if } (x_1, x_2) \in A_{i(4-i)} \cup A_{j(4-j)}, \\ 0 & \text{otherwise,} \end{cases}$$

from Lemma 3.1 φ_2 is a level 2/3 unbiased test. The d-admissibility of φ_2 is shown in the same technique as Theorem 3.2. If ψ is any competitor, $\varphi_2 - \psi$ satisfies the conditions of Lemma 3.2 except the set of measure zero.

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