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ON THE DETERMINATION AND CONSTRUCTION OF OPTIMAL ROW-COLUMN DESIGNS HAVING UNEQUAL ROW AND COLUMN SIZES

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Abstract. In this paper we consider experimental situations requiring usage of a row-column design where v treatments are to be applied to experimental units arranged in b_1 rows and b_2 columns where row i has size k_{1i}, $i = 1, \ldots, b_1$ and column j has size k_{2j}, $j = 1, \ldots, b_2$. Conditions analogous to those given in Kunert (1983, *Ann. Statist.*, **11**, 247–257) and Cheng (1978, *Ann. Statist.*, **6**, 1262–1272) are given which can often be used to establish the optimality of a given row-column design from the optimality of an associated block design. In addition, sufficient conditions are derived which guarantee the existence of an optimal row-column design which can be constructed by appropriately arranging treatments within blocks of an optimal block design.

Key words and phrases: Row-column design, block design, incidence matrix, balanced unequal block design.

1. Introduction

In this paper we consider experimental settings in which v treatments are to be tested using a row-column design where n experimental units are arranged in b_1 rows and b_2 columns where row i has size k_{1i}, $i = 1, \ldots, b_1$ and column j has size k_{2j}, $j = 1, \ldots, b_2$. If we let d denote some design which can be used in such an experimental setting, then we shall let $N_{d_1} = (n_{dij}^{(1)})$, $N_{d_2} = (n_{dij}^{(2)})$ and $N_{d_3} = (n_{dij}^{(3)})$ denote, respectively, the $v \times b_1$ treatment-row incidence matrix, the $v \times b_2$ treatment-column incidence matrix, and the $b_1 \times b_2$ row-column incidence matrix, i.e., $n_{dij}^{(1)}$ = the number of times treatment i occurs in row j, $n_{dij}^{(2)}$ = the number of times treatment i occurs in column j, and $n_{dij}^{(3)}$ = the number of observations obtained in row i and column j. The model assumed here for the data obtained from some design d specifies that an observation Y_{ijkl} obtained after

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applying treatment i to an experimental unit occurring in row j and column k can be expressed as

$$(1.1) \qquad Y_{ijkl} = \mu + \alpha_i + \beta_j + \gamma_k + E_{ijkl}, \quad 1 \le i \le v, \quad 1 \le j \le b_1, \quad 1 \le k \le b_2, \quad 0 \le l \le n_{djk}^{(3)}$$

where u = the overall mean effect, α_i = the effect of treatment i, β_j = the effect of row j, γ_k = the effect of column k and the E_{ijkl}'s are all uncorrelated random variables having expectation zero and constant variance σ^2. Under this model, using $\operatorname{diag}(a_1, \dots, a_n)$ to denote an $n \times n$ diagonal matrix, A' to denote the transpose of a matrix A, and A^- to denote a generalized inverse of A, the coefficient matrix of the reduced normal equations for obtaining the least squares estimates of the treatment effects in d can be written as

$$(1.2) \qquad C_d = R_d - N_{d_1}K_1^{-1}N'_{d_1} - (N_{d_2} - N_{d_1}K_1^{-1}N_{d_3})C_{d_3}^{-}(N_{d_2} - N_{d_1}K_1^{-1}N_{d_3})'$$

where

$$\begin{aligned}
&R_d = \operatorname{diag}(r_{d1}, \dots, r_{dv}),\\
&r_{di} = \text{the number of replications of treatment } i \text{ under } d,\\
&K_1 = \operatorname{diag}(k_{11}, k_{12}, \dots, k_{1b_1}),\\
&C_{d_3} = K_2 - N'_{d_3}K_1^{-1}N_{d_3},\\
&K_2 = \operatorname{diag}(k_{21}, k_{22}, \dots, k_{2b_2}).
\end{aligned}$$

An equivalent formula for C_d is

$$(1.3) \qquad C_d = R_d - N_{d_2}K_2^{-1}N'_{d_2} - (N_{d_1} - N_{d_2}K_2^{-1}N'_{d_3})C_{d_4}^{-}(N_{d_1} - N_{d_2}K_2^{-1}N'_{d_3})'$$

where

$$C_{d_4} = K_1 - N_{d_3}K_2^{-1}N'_{d_3}.$$

The matrix C_d is usually called the C-matrix of d and is well known to be positive semi-definite with zero row-sums.

Throughout this paper we shall only be considering treatment connected designs, i.e. designs whose C-matrices have rank $v-1$. Henceforth, $D(v;\ b_1, \mathbf{k}'_1, b_2, \mathbf{k}'_2)$ ($\mathbf{k}'_1 = (k_{11}, \dots, k_{1b_1})$ and $\mathbf{k}'_2 = (k_{21}, \dots, k_{2b_2})$) is used to denote the class of treatment connected row-column designs having v treatments assigned to experimental units arranged in b_1 rows of size k_{1i}, $i = 1, \dots, b_1$ and b_2 columns of size k_{2i}, $i = 1, \dots, b_2$.

With each row-column design $d \in D(v;\ b_1, \mathbf{k}'_1, b_2, \mathbf{k}'_2)$ we associate two block designs d_1 and d_2 having incidence matrices N_{d_1} and N_{d_2}, respectively, i.e., d_1 is that block design which can be obtained from d by treating the rows of d as blocks and ignoring column effects whereas d_2 is that block design which can be obtained from d by treating the columns of d as blocks and ignoring the row effects. We say that d_i is a binary design if $n_{dpq}^{(i)} = 0$ or 1 for all p, q, otherwise we say that d_i is nonbinary for $i = 1,\ 2$. The matrices $N_{d_1}N'_{d_1} = (\lambda_{dij}^{(1)})$ and $N_{d_2}N'_{d_2} =$

$(\lambda_{dij}^{(2)})$ are called the concurrence matrices of d_1 and d_2. We will henceforth let $D_1(v;\ b_1, k_1')$ and $D_2(v;\ b_2, k_2')$ denote the classes of block designs d_1 and d_2, respectively, corresponding to $d \in D(v;\ b_1, k_1',\ b_2, k_2')$ and note that the coefficient matrices of the reduced normal equations for estimating the treatment effects in d_1 and d_2 are, under the appropriate two-way model,

$$(1.4) \qquad C_{d_1} = R_d - N_{d_1} K_1^{-1} N_{d_1}' \quad \text{and} \quad C_{d_2} = R_d - N_{d_2} K_2^{-1} N_{d_2}'.$$

These matrices are called the C-matrices of d_1 and d_2, respectively, and possess the same properties as C_d.

In this paper we consider the determination and construction of optimal designs in classes $D(v;\ b_1, k_1',\ b_2, k_2')$. In Section 2 we define the optimality criteria being considered and give our main optimality results. The results obtained are similar to those given in Cheng (1978) for equally replicated row-column designs and similar to those given in Kunert (1983) for repeated measurements designs. In particular, it is shown that finding an optimal row-column design having unequal row or column sizes can often be accomplished by finding a corresponding optimal block design and then arranging treatments appropriately within blocks of the block design. In Section 3 we discuss conditions under which optimal row-column designs can actually be constructed from the corresponding optimal block designs.

2. Optimality results

In this section we give a method for determining optimal row-column designs from their corresponding block designs. To this end, we shall let ϕ denote any optimality function which is nonincreasing in the sense that $\phi(C) \leq \phi(D)$ for any positive semi-definite matrices C and D such that $C - D$ is positive semi-definite. A design d^* is said to be ϕ-optimal over a given class of designs if for any other design d within the class,

$$(2.1) \qquad \phi(C_{d^*}) \leq \phi(C_d).$$

With this definition, we now give a result which is analogous to Theorem 3.1 of Cheng (1978).

THEOREM 2.1. *Suppose $d^* \in D(v;\ b_1, k_1',\ b_2, k_2')$ has d_1^* which is ϕ-optimal in $D_1(v;\ b_1, k_1')$ or d_2^* which is ϕ-optimal in $D_2(v;\ b_2, k_2')$. If $\phi(C_{d^*}) = \phi(C_{d_1^*})$ or $\phi(C_{d_2^*})$, then d^* is also ϕ-optimal in $D(v;\ b_1, k_1',\ b_2, k_2')$.*

PROOF. We shall only consider the case where $\phi(C_{d^*}) = \phi(C_{d_2^*})$ since the case where $\phi(C_{d^*}) = \phi(C_{d_1^*})$ is similar. So let $d \in D(v;\ b_1, k_1',\ b_2, k_2')$ be arbitrary. Then by (1.3) and (1.4), we see that

$$C_d = C_{d_2} - (N_{d_1} - N_{d_2} K_2^{-1} N_{d_3}') C_{d_4}^{-} (N_{d_1} - N_{d_2} K_2^{-1} N_{d_3}')'.$$

Now, it is easily verified that C_{d_4} is positive semi-definite, hence

$$(N_{d_1} - N_{d_2} K_2^{-1} N_{d_3}') C_{d_4}^{-} (N_{d_1} - N_{d_2} K_2^{-1} N_{d_3}')'$$

is also positive semi-definite and $\phi(C_d) \geq \phi(C_{d_2})$. But then

$$\phi(C_d) \geq \phi(C_{d_2}) \geq \phi(C_{d_2^*}) = \phi(C_{d^*})$$

and we have the desired result.

With Theorem 2.1 in mind, we now give a useful corollary.

COROLLARY 2.1. *Suppose* $d^* \in D(v;\ b_1, \mathbf{k}_1', b_2, \mathbf{k}_2')$.

(a) *If* d_1^* *is* ϕ*-optimal in* $D_1(v;\ b_1, \mathbf{k}_1')$ *and the rows of* $N_{d_2^*} - N_{d_1^*}K_1^{-1}N_{d_3^*}$ *are all multiples of* J_{1,b_2} *where* $J_{m,n}$ *denotes the* $m \times n$ *matrix of ones, then* d^* *is* ϕ*-optimal in* $D(v;\ b_1, \mathbf{k}_1', b_2, \mathbf{k}_2')$.

(b) *If* d_2^* *is* ϕ*-optimal in* $D_2(v;\ b_2, \mathbf{k}_2')$ *and the rows of* $N_{d_1^*} - N_{d_2^*}K_2^{-1}N_{d_3^*}'$ *are all multiples of* J_{1,b_1}, *then* d^* *is* ϕ*-optimal in* $D(v;\ b_1, \mathbf{k}_1', b_2, \mathbf{k}_2')$.

PROOF. We shall only give the proof for (b) since the proof of (a) is similar. So to begin with, we note that the row and column sums of $C_{d_4^*}$ are zero. Thus, if we let $C_{d_4^*}^+$ denote the Moore-Penrose inverse of $C_{d_4^*}$, it follows that $C_{d_4^*}^+ J_{b_1,1} = 0$. Hence, since by assumption the rows of $N_{d_1^*} - N_{d_2^*}K_2^{-1}N_{d_3^*}'$ are all multiples of J_{1,b_1}, it follows that

$$(N_{d_1^*} - N_{d_2^*}K_2^{-1}N_{d_3^*}')C_{d_4^*}^+(N_{d_1^*} - N_{d_2^*}K_2^{-1}N_{d_3^*}')' = 0.$$

Thus $C_{d^*} = C_{d_2^*}$ and $\phi(C_{d^*}) = \phi(C_{d_2^*})$ in this case and the result follows from Theorem 2.1.

Comment. We note that a design d having N_{d_3} with all positive entries will have corresponding C_{d_3} and C_{d_4} matrices of rank $v-1$. Thus the only way such a design can have $C_d = C_{d_1}$ or C_{d_2} is if and only if the rows of $N_{d_2} - N_{d_1}K_1^{-1}N_{d_3}$ or $N_{d_1} - N_{d_2}K_2^{-1}N_{d_3}'$ are all multiples of J_{1,b_2} or J_{1,b_1}, respectively. The situation which arises most often in practice where d has $C_d = C_{d_1}$ or C_{d_2} is when $N_{d_2} - N_{d_1}K_1^{-1}N_{d_3} = 0$ or $N_{d_1} - N_{d_2}K_2^{-1}N_{d_3}' = 0$. In the next lemma, we give some sufficient conditions for a row-column design having different column sizes to have $C_d = C_{d_2}$.

LEMMA 2.1. *Suppose* d *is a row-column design having* d_2 *such that* $N_{d_2} = (N_{d_{21}}, \ldots, N_{d_{2t}})$ *where* $N_{d_{2j}}$ *corresponds to a block design* d_{2j} *having* b_{2j} *blocks of size* $\bar{k}_{2j}$ *for* $j = 1, \ldots, t$. *Further assume that the following conditions hold;*

(i) *Each treatment is replicated* $b_{2j}\bar{k}_{2j}/v = r_{2j}$ *times in* d_{2j} *for* $j = 1, \ldots, t$,

(ii) $n_{dij}^{(1)} = k_{1j}/v$ *for* $i = 1, \ldots, v$ *and* $j = 1, \ldots, b_1$,

(iii) *For* $i = 1, \ldots, b_1$,

$$n_{dij}^{(3)} = \begin{cases} a_{i1} & \text{for } 1 \leq j \leq b_{21}, \\ a_{ip} & \text{for } \sum_{x=1}^{p-1} b_{2x} + 1 \leq j \leq \sum_{x=1}^{p} b_{2x} \end{cases}$$

where the a_{ip}, $p = 1, \ldots, t$ *are nonnegative integers. Then* $C_d = C_{d_2}$.

PROOF. We shall prove the result by showing that $N_{d_1} - N_{d_2}K_2^{-1}N'_{d_3} = 0$. To this end, we note that the $(i,\ j)$-th entry of $N_{d_2}K_2^{-1}N'_{d_3}$ is given by

$$\begin{aligned}
\sum_{p=1}^{b_2} n_{dip}^{(2)} k_{2p}^{-1} n_{djp}^{(3)} &= \sum_{p=1}^{b_{21}} n_{dip}^{(2)} \bar{k}_{21}^{-1} n_{djp}^{(3)} + \sum_{s=1}^{t-1} \sum_{p=b_{21}+\cdots+b_{2s}+1}^{b_{21}+\cdots+b_{2,\,s+1}} n_{dip}^{(2)} \bar{k}_{2,\,s+1}^{-1} n_{djp}^{(3)} \\
&= \sum_{p=1}^{b_{21}} n_{dip}^{(2)} \bar{k}_{21}^{-1} a_{j1} + \sum_{s=1}^{t-1} \sum_{p=b_{21}+\cdots+b_{2s}+1}^{b_{21}+\cdots+b_{2,\,s+1}} n_{dip}^{(2)} \bar{k}_{2,\,s+1}^{-1} a_{j,\,s+1} \\
&= r_{21}\bar{k}_{21}^{-1} a_{j1} + \sum_{s=1}^{t-1} r_{2,\,s+1} \bar{k}_{2,\,s+1}^{-1} a_{j,\,s+1} \\
&= (b_{21}a_{j1}/v) + \sum_{s=1}^{t-1} (b_{2,\,s+1} a_{j,\,s+1}/v) = k_{1j}/v = n_{dij}^{(1)}.
\end{aligned}$$

Since the $(i,\ j)$-th entry of $N_{d_2}K_2^{-1}N'_{d_3}$ is equal to $n_{dij}^{(1)}$, the result follows from (1.3) and (1.4).

Comment. Suppose $d \in D(v;\ b_1, \mathbf{k}'_1,\ b_2, \mathbf{k}'_2)$ is a row-column design which satisfies the conditions of Lemma 2.1. Then it is easily seen that any row-column design $\bar{d}$ obtained by combining the rows of d into $\bar{b}_1$ rows, $a \le \bar{b}_1 \le b_1$, again satisfies the conditions of Lemma 2.1 and has $C_{\bar{d}} = C_{\bar{d}_2}$.

3. Construction of optimal designs

In this section we derive some sufficient conditions which can often be used to establish the existence of a ϕ-optimal row-column design d which satisfies the conditions of the previous section and which can be constructed from its corresponding ϕ-optimal block design d_2 by arranging treatments appropriately within the blocks of d_2. However, we begin our discussion by introducing some notation and definitions which are useful in the sequel and which are analogous to some given in Agrawal (1966).

DEFINITION 3.1. If $S_1, \ldots,\ S_n$ are n subsets of a given set S, then $(O_1, \ldots,\ O_n)$ is called an $(m_1, \ldots,\ m_n)$ system of distinct representatives (SDR) if the following conditions hold;

(i) $O_i \subseteq S_i$,
(ii) $n(O_i) = m_i$, where $n(O_i)$ denotes the number of elements in O_i,
(iii) $O_i \cap O_j = \phi$ for $i \neq j,\ i,\ j = 1, \ldots,\ n$.

If $m_1 = \cdots = m_n = m$, the sets are called an m-ple SDR.

Agrawal (1966) gives the following theorem which can be used to establish the existence of an $(m_1, \ldots,\ m_n)$ SDR.

THEOREM 3.1. *Let $S_1, \ldots, S_n$ be subsets of a given finite set S. Then a necessary and sufficient condition for an $(m_1, \ldots, m_n)$ SDR to exist is that*

$$n(S_{i_1} \cup S_{i_2} \cup \cdots \cup S_{i_k}) \geq \sum_{j=1}^{k} m_{i_j}, \qquad 1 \leq i_1 < i_2 < \cdots < i_k \leq n, \qquad 1 \leq k \leq n.$$

Using Theorem 3.1, we now extend a result of Agrawal's to the case of nonbinary designs.

THEOREM 3.2. *Let d_2 be a block design having v treatments arranged in $b = mv$ blocks of size k such that each treatment is replicated $bk/v = \gamma$ times. Further assume that each treatment occurs $[k/v]$ or $[k/v]+1$ times in each block. Then the treatments can be rearranged within the blocks of d_2 to construct a row column design $\hat{d}_2 \in D(v;\ k,\ mvJ_{1,k},\ b,\ kJ_{1,b})$ such that every treatment occurs m times in each row.*

PROOF. The proof given here for this result is similar to the proof of Theorem 3.1 given in Agrawal (1966). However, because the proof demonstrates the basic problems which arise when trying to construct a row-column design satisfying the condition given, we include the proof in its entirety. So to start with, let us assume that $k = v[k/v] + \beta$ and $r = b[k/v] + \alpha$. Since $bk = vr$ and $b = mv$, it easily follows that $\alpha = m\beta$. We now form the sets $S_1, S_2, \ldots, S_v$ where S_i is the set of all the block numbers of the design d_2 containing treatment i $[k/v] + 1$ times. Now

$$n(S_{i_1} \cup S_{i_2} \cup \cdots \cup S_{i_h}) \geq h\alpha/\beta = hm, \qquad 1 \leq i_1 < \cdots < i_h \leq v, \qquad 1 \leq h \leq v,$$

as any treatment occurs $[k/v]+1$ times in exactly α blocks of d_2 and each particular block of d_2 contains exactly β treatments $[k/v] + 1$ times. Hence by Theorem 3.1 we can choose an m-ple SDR, say $(O_1, \ldots, O_v)$ from $S_1, \ldots, S_v$. Using this SDR, we now write down the first row of $\hat{d}_2$ by placing treatment i in those columns of the first row corresponding to the block numbers in O_i, $i = 1, \ldots, v$. Now let $\bar{S}_i = S_i - \{O_i\}$, $i = 1, \ldots, v$. Then every $\bar{S}_i$ contains $m(\beta - 1)$ different block numbers and each block number appears once in $(\beta - 1)$ of the $\bar{S}_i$'s. Thus

$$n(\bar{S}_{i_1} \cup \bar{S}_{i_2} \cup \cdots \cup \bar{S}_{i_{h'}}) \geq h'm(\beta - 1)/(\beta - 1) = h'm,$$
$$1 \leq i_1 < \cdots < i_{h'} \leq v, \qquad 1 \leq h' \leq v,$$

and by Theorem 3.1 it follows that we can choose an m-ple SDR $(\bar{O}_1, \ldots, \bar{O}_v)$ from $\bar{S}_1, \ldots, \bar{S}_v$. We now write out the second row of $\hat{d}_2$ by placing treatment i in those columns of the second row corresponding to those block numbers in $\bar{O}_i$, $i = 1, \ldots, v$. Continuing in this manner we obtain the first β rows of the row-column design $\hat{d}_2$. Finally, the row-column design $\hat{d}_2$ is obtained by adjoining to the β rows of $\hat{d}_2$ thus far constructed the rows of the row-column design $\hat{d}_0$ given by

$$\hat{d}_0 = \begin{bmatrix} \hat{d}_{11} & \hat{d}_{12} & \cdots & \hat{d}_{1m} \\ \hat{d}_{21} & \hat{d}_{22} & \cdots & \hat{d}_{2m} \\ \vdots & \vdots & & \vdots \\ \hat{d}_{p1} & \hat{d}_{p2} & \cdots & \hat{d}_{pm} \end{bmatrix}$$

where $p = [k/v]$ and

$$\hat{d}_{ij} = \begin{bmatrix} 1 & 2 & 3 & \cdots & v-2 & v-1 & v \\ 2 & 3 & 4 & \cdots & v-1 & v & 1 \\ 3 & 4 & 5 & \cdots & v & 1 & 2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v & 1 & 2 & \cdots & v-3 & v-2 & v-1 \end{bmatrix}$$

for $i = 1, \ldots, p$, $j = 1, \ldots, m$.

THEOREM 3.3. *Let $d_{21} \in D_2(v;\ b_{21},\ k_{21}J_{1,b_{21}})$ where $b_{21} = m_{21}v$ be an equireplicate block design which has each treatment occurring $[k_{21}/v]$ or $[k_{21}/v]+1$ times in each block. Let $d_{22} \in D_2(v;\ b_{22},\ k_{22}J_{1,b_{22}})$ where $b_{22} = m_{22}v$ be another block design which has each treatment occurring $[k_{22}/v]$ or $[k_{22}/v]+1$ times in each block. Furthermore, suppose $k_{21} \le k_{22}$ and let $d_2 \in D_2(v;\ b_2, \mathbf{k}_2')$ have $N_{d_2} = (N_{d_{21}},\ N_{d_{22}})$ where $b_2 = b_{21} + b_{22}$ and $\mathbf{k}_2' = (k_{21}J_{1,b_{21}},\ k_{22}J_{1,b_{22}})$. Then there exists a row-column design $\hat{d}_2 \in D(v;\ b_1, \mathbf{k}_1', b_2, \mathbf{k}_2')$ corresponding to d_2 which has $C_{\hat{d}_2} = C_{d_2}$ where b_2 and $\mathbf{k}_2'$ are defined as above, $b_1 = k_{22}$ and $\mathbf{k}_1' = ((m_{21}+m_{22})vJ_{1,k_{21}},\ m_{22}vJ_{1,b_1-k_{21}})$.*

PROOF. The proof is by construction. It follows from Theorem 3.2 that a row-column design $\hat{d}_{21} \in D(v;\ k_{21},\ m_{21}vJ_{1,k_{21}},\ b_{21},\ k_{21}J_{1,b_{21}})$ can be constructed from d_{21} such that every treatment appears m_{21} times in each row of $\hat{d}_{21}$. Similarly a row-column design $\hat{d}_{22} \in D(v;\ k_{22},\ m_{22}vJ_{1,k_{22}},\ b_{22},\ k_{22}J_{1,b_{22}})$ can be constructed from d_{22} such that every treatment appears m_{22} times in each row of $\hat{d}_{22}$. Finally, we obtain $\hat{d}_2$ by letting $\hat{d}_2 = (\hat{d}_{21},\ \hat{d}_{22})$. The result now follows from Lemma 2.1.

THEOREM 3.4. *Assume the conditions of Theorem 3.3 hold and let d_{21}^*, d_{22}^* and d_2^* be block designs such as described in Theorem 3.3 with d_2^* being ϕ-optimal in $D_2(v;\ b_2, \mathbf{k}_2')$. Now let $\hat{d}_2^* \in D(v;\ b_1, \mathbf{k}_1', b_2, \mathbf{k}_2')$ be a row-column design which can be obtained from d_2^* such that $C_{\hat{d}_2^*} = C_{d_2^*}$. Then $\hat{d}_2^*$ is ϕ-optimal in $D(v;\ b_1, \mathbf{k}_1', b_2, \mathbf{k}_2')$.*

PROOF. This directly follows from Theorem 3.3 and Theorem 2.1.

For the remainder of this paper, we shall concentrate on the construction of optimal row-column designs which can be constructed from balanced unequal block designs (BUBD's).

DEFINITION 3.2. A design $d_2 \in D_2(v;\ b_2, \mathbf{k}_2')$ is called a BUBD if $\bar{k}C_{d_2} = (\bar{r} - \bar{\lambda})I_v + \bar{\lambda}J_{vv}$ where $\bar{k} = \prod_{j=1}^{b_2} k_{2j}$ and $\bar{r}$ and $\bar{\lambda}$ are appropriate constants. If all the elements of $\mathbf{k}_2'$ are equal, then d_2 is called a balanced block design (BBD) whereas if d_2 is a binary BBD, then d_2 is called a balanced incomplete block design (BIBD).

It is well known from the results of Kiefer (1975) that if $d_2^* \in D_2(v;\ b_2, k_2')$ is a BUBD such that

$$n^{(2)}_{d_2^* ij} = [k_{2j}/v] \quad \text{or} \quad [k_{2j}/v]+1 \quad \text{for} \quad i = 1, \ldots, v, \quad j = 1, \ldots, b_2, \tag{3.1}$$

then d_2^* is ϕ-optimal in $D(v;\ b_2, k_2')$ under most widely used optimality criteria including the A, D, and E-optimality criteria. There are several techniques available for constructing BUBD's. For example, if d_{21} and d_{22} are two BBD's having different block sizes but based on v treatments each, then d_2 having $N_{d_2} = (N_{d_{21}}, N_{d_{22}})$ is a BUBD. Gupta and Jones (1983) give a method for constructing BUBD's using partially balanced incomplete block (PBIB) designs. The reader is referred to Raghavarao (1971) for more information on PBIB designs.

Example 3.1. Consider the row column design $\hat{d}_2$ having 4 rows and 18 columns which is given by

$$\hat{d}_2 = \begin{bmatrix} 1 & 3 & 5 & 4 & 2 & 6 & 1 & 5 & 3 & 2 & 6 & 4 & 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 5 & 3 & 1 & 5 & 3 & 1 & 6 & 4 & 2 & 4 & 5 & 6 & 1 & 2 & 3 \\ & & & & & & & & & & & & 2 & 3 & 1 & 5 & 6 & 4 \\ & & & & & & & & & & & & 5 & 6 & 4 & 2 & 3 & 1 \end{bmatrix}.$$

The BUBD d_2 corresponding to $\hat{d}_2$ is constructed as described in Gupta and Jones (1983) by combining the PBIB designs S2 and R18 given in Clatworthy (1973). Since d_2 is a BUBD which satisfies the conditions given in (3.1) and $\hat{d}_2$ satisfies the conditions of Theorem 3.4, it follows that $\hat{d}_2$ is ϕ-optimal under most optimality criteria in $D(6;\ 4,\ (18,\ 18,\ 6,\ 6),\ 18,\ (2J_{1,12},\ 4J_{1,6}))$. We also note that any row-column design which can be obtained from $\hat{d}_2$ by combining the rows of $\hat{d}_2$ in some manner will satisfy, as mentioned in the comment following Lemma 2.1, the conditions of Lemma 2.1. Hence any such row-column design will also be ϕ-optimal in its corresponding class of row-column designs. For example, the row-column designs $\bar{d}_2$ and $\tilde{d}_2$ given by

$$\bar{d}_2 = \begin{bmatrix} 1 & 3 & 5 & 4 & 2 & 6 & 1 & 5 & 3 & 2 & 6 & 4 & 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 5 & 3 & 1 & 5 & 3 & 1 & 6 & 4 & 2 & 4 & 5 & 4 & 1 & 2 & 3 \\ & & & & & & & & & & & & 2,3 & 3,6 & 1,4 & 5,2 & 6,3 & 4,1 \end{bmatrix}$$

and

$$\tilde{d}_2 = \begin{bmatrix} 1 & 3 & 5 & 4 & 2 & 6 & 1 & 5 & 3 & 2 & 6 & 4 & 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 5 & 3 & 1 & 5 & 3 & 1 & 6 & 4 & 2 & 4,2,5 & 5,3,6 & 6,1,4 & 1,5,2 & 2,6,3 & 3,4,1 \end{bmatrix}$$

are both ϕ-optimal in their corresponding classes of row-column designs since they satisfy the conditions of Lemma 2.1. In the designs $\bar{d}_2$ and $\tilde{d}_2$ given above, treatment sets written like 4, 2, 5 indicate that these are the treatments assigned to experimental units occurring in the appropriate row and column, i.e., in $\tilde{d}_2$, the treatment set 4, 2, 5 indicates that treatments 4, 2 and 5 are the treatments assigned to experimental units occurring in row 2 and column 13 of $\tilde{d}_2$. We note that d_2 for both $\bar{d}_2$ and $\tilde{d}_2$ is the same as d_2 described above corresponding to $\hat{d}_2$.

Example 3.2. Consider the row-column design $\hat{d}_2$ with 14 rows and 10 columns given by

$$\hat{d}_2 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 \\ & & & & & 1 & 2 & 3 & 4 & 5 \\ & & & & & 2 & 3 & 4 & 5 & 1 \\ & & & & & 3 & 4 & 5 & 1 & 2 \\ & & & & & 4 & 5 & 1 & 2 & 3 \\ & & & & & 5 & 1 & 2 & 3 & 4 \end{bmatrix}.$$

Here d_2 is a BUBD obtained by combining the BBD's $d_{21} \in D_2(5;\ 5,\ 9J_{1,5})$ and $d_{22} \in D_2(5;\ 5,\ 14J_{1,5})$. Since d_2 satisfies the conditions of (3.1) and since $\hat{d}_2$ satisfies Theorem 3.4, it follows that $\hat{d}_2$ is ϕ-optimal in $D(5;\ 14,\ (10J_{1,9},\ 5J_{1,5}),\ 10,\ (9J_{1,5},\ 14J_{1,5}))$ under most optimality criteria. We note again, as with Example 3.1, that any row-column design $\bar{d}_2$ obtained by combining the rows of $\hat{d}_2$ into $\bar{b}_1$ rows, $1 \leq \bar{b}_1 \leq 14$, will again satisfy the conditions of (3.1) and Theorem 3.4. Thus any such row-column design $\bar{d}_2$ will be ϕ-optimal in its appropriate class of row-column designs under most optimality criteria.

Comment. The results given in this section, such as Theorem 3.4, only establish the existence of optimal row-column designs which can be constructed from corresponding optimal block designs. Specific methods for constructing such row-column designs are not given. The problem of actually constructing optimal row-column designs from optimal block designs essentially reduces, as indicated in the proof of Theorem 3.2, to the problem of choosing SDR's. Hall (1945) gives a specific algorithm for choosing SDR's which can be implemented to construct row-column designs such as those described in this section when trial and error methods fail. For more information on this algorithm, the reader is referred to Hall (1945).

THEOREM 3.5. *Let d_0 be a binary block design having v treatments arranged in b_0 blocks of size k such that each treatment is replicated $b_0k/v = r_0$ times. Further assume that $b_0 = mv + t$, $t(v-k) = v$ and d_0 has $N_{d_0} = (N_{d_{01}}, N_{d_{02}})$ where $N_{d_{02}}$ is the incidence matrix of a block design d_{02} which has t blocks with every treatment occurring once in $tk/v = r_{02}$ blocks of d_{02}. Then the treatments within the blocks of d_0 can be arranged to form a row-column design $\hat{d}_0$ which has $t-1$ rows with every treatment appearing the same number of times in each row.*

PROOF. We shall prove the result by constructing row-column designs $\hat{d}_{01}$

and $\hat{d}_{02}$ from d_{01} and d_{02}, respectively, such that $\hat{d}_{01}$ and $\hat{d}_{02}$ each have $t-1$ rows with each treatment being equally replicated in each row. The final design $\hat{d}_0$ is then obtained by letting $\hat{d}_0 = (\hat{d}_{01}, \hat{d}_{02})$. To begin, we note that from the conditions given, $k = (t-1)(v-k)$ and d_{01} is an equi-replicate binary block design with $b = mv$ blocks. Thus, by Theorem 3.2, we see that we can arrange the treatments within the blocks of d_{01} to form a row-column design $\bar{d}_{01}$ having k rows such that every treatment appears m times in each row of $\bar{d}_{01}$. We now obtain the desired design $\hat{d}_{01}$ by combining succeeding sets of $v-k$ rows in $\bar{d}_{01}$. Thus $\hat{d}_{01}$ is a row-column design with $t-1$ rows having each treatment occurring $m(v-k)$ times in each row. Now, to construct the appropriate design from d_{02}, let us define B_j, $j = 1, \ldots, t$, to be the set of treatments which occur in the j-th block of d_{02}. Then each treatment occurs in $r_{02} = t-1$ of the sets $B_1, \ldots, B_t$ and $n(B_{i_1} \cup B_{i_2} \cup \cdots \cup B_{i_h}) \geq hk/(t-1) = h(t-1)(v-k)/(t-1) = h(v-k)$, $1 \leq h \leq t$. Hence, by Theorem 3.1, a $(v-k)$-ple SDR $(O_1, \ldots, O_t)$ exists for the sets $B_1, \ldots, B_t$. We now form the first $v-k$ rows of a row-column design $\bar{d}_{02}$ by choosing one treatment out of each set O_j, $j = 1, \ldots, t$, to appear in each row. Now define $\bar{B}_j = B_j - \{O_j\}$, $j = 1, \ldots, t$. Then $\bar{B}_j$ contains $\bar{k} = 2k - v = (t-2)(v-k)$ treatments and each treatment occurs in $t-2$ of the $\bar{B}_j$'s. Thus $n(\bar{B}_{i_1} \cup \bar{B}_{i_2} \cup \cdots \cup \bar{B}_{i_h}) \geq h\bar{k}/(t-2) = h(v-k)$, $1 \leq i_1 < \cdots < i_h \leq t$, $1 < h \leq t$ and by Theorem 3.1 we can select a $(v-k)$-ple SDR $(\bar{O}_1, \ldots, \bar{O}_t)$ for the sets $\bar{B}_1, \ldots, \bar{B}_t$. We now write down the second set of $v-k$ rows of $\bar{d}_{02}$ by choosing from each set $\bar{O}_j$ one treatment to appear in each row. Continuing in this manner, we can construct the remaining rows of $\bar{d}_{02}$ so that $\bar{d}_{02}$ has $t-1$ sets of $v-k$ rows with each treatment occurring once in each set. To obtain the desired design corresponding to d_{02} from $\bar{d}_{02}$, we combine the $v-k$ rows contained in each of the $t-1$ sets given above to form the row-column design $\hat{d}_{02}$ which possesses $t-1$ rows with each treatment occurring once in each row. The final design $\hat{d}_0$ is given by $\hat{d}_0 = (\hat{d}_{01}, \hat{d}_{02})$.

COROLLARY 3.1. *Let $d_0 \in D_2(v;\ b_0, kJ_{1,b_0})$ be a block design such that each treatment is replicated $b_0k/v = r_0$ times. Further assume that each treatment occurs $[k/v]$ or $[k/v]+1$ times in any block, $b_0 = mv+t$, $k = v[k/v]+\beta$, $t(v-\beta) = v$ and d_0 has $N_{d_0} = (N_{d_{01}}, N_{d_{02}})$ where $N_{d_{02}}$ is the incidence matrix of a block design d_{02} which has t blocks such that every treatment occurs $tk/v = r_{02}$ times. Then the treatments within the blocks of d_0 can be arranged to form a row-column design $\hat{d}_0$ having $[k/v]t + (t-1)$ rows with each row containing every treatment the same number of times.*

PROOF. To begin with, we note that d_{01} is a block design which has $b_{01} = mv$ blocks with each treatment replicated $r_{01} = mvk/v = mk$ times. Thus, by Theorem 3.2 we can construct a row-column design $\bar{d}_{01}$ by arranging treatments within the blocks of d_{01} so that $\bar{d}_{01}$ has k rows with each treatment occurring the same number of times in each row. Now, from $\bar{d}_{01}$, we can form the row-column design $\hat{d}_{01}$ by combining successive sets of $(v-\beta)$ rows so that $\hat{d}_{01}$ is a row-column design having $[k/v]t + (t-1)$ rows with each treatment occurring the same number of times in each row. We now show that a row-column design $\hat{d}_{02}$

can be constructed from d_{02} which has properties similar to those of $\hat{d}_{01}$. To this end, let B_j, $j = 1, \ldots, t$, be the set of all treatments which occur $[k/v]+1$ times in the j-th block of d_{02} and let $\underline{d}_{02}$ be that block design having B_j, $j = 1, \ldots, t$ as its blocks. Then, as in the proof of Theorem 3.5, we can construct a row-column design $\dot{d}_{02}$ from $\underline{d}_{02}$ such that each succeeding set of $v-\beta$ rows in $\dot{d}_{02}$ contains each treatment once and there are $(t-1)$ such sets. Now consider the row-column design

$$\bar{d}_{02} = \begin{bmatrix} \tilde{d}_{02} \\ \tilde{d}_{02} \\ \vdots \\ \tilde{d}_{02} \\ \dot{d}_{02} \end{bmatrix}$$

where $\tilde{d}_{02}$ appears $[k/v]$ times in $\bar{d}_{02}$ and is given by

$$\tilde{d}_{02} = \begin{bmatrix}
1 & (v-\beta)+1 & 2(v-\beta)+1 & \cdots & (t-1)(v-\beta)+1 \\
2 & (v-\beta)+2 & 2(v-\beta)+2 & \cdots & (t-1)(v-\beta)+2 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
(v-\beta) & 2(v-\beta) & 3(v-\beta) & \cdots & t(v-\beta) \\
 & & & & \\
(v-\beta)+1 & 2(v-\beta)+1 & 3(v-\beta)+1 & \cdots & 1 \\
(v-\beta)+2 & 2(v-\beta)+2 & 3(v-\beta)+2 & \cdots & 2 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
2(v-\beta) & 3(v-\beta) & 4(v-\beta) & \cdots & v-\beta \\
 & & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
 & & & & \\
(t-1)(v-\beta)+1 & 1 & (v-\beta)+1 & \cdots & (t-2)(v-\beta)+1 \\
(t-1)(v-\beta)+2 & 2 & (v-\beta)+2 & \cdots & (t-2)(v-\beta)+2 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
t(v-\beta) & v-\beta & 2(v-\beta) & \cdots & (t-1)(v-\beta)
\end{bmatrix}.$$

Now, by combining succeeding sets of $(v-\beta)$ rows in $\bar{d}_{02}$ we obtain a design $\hat{d}_{02}$ which has $[k/v]t+(t-1)$ rows with each treatment occurring exactly once in each row. The final design $\hat{d}_0$ is given by $\hat{d}_0 = (\hat{d}_{01}, \hat{d}_{02})$ and has $[k/v]t+(t-1)$ rows with each treatment occurring the same number of times in each row.

THEOREM 3.6. *Let* $d_{21} \in D_2(v;\ b_{21}k_{21}J_{1,b_{21}})$ *and* $d_{22} \in D_2(v;\ b_{22}, k_{22}J_{1,b_{22}})$ *be two block designs which satisfy the conditions of Corollary* 3.1 *with* $b_{21} = m_{21}v + t_1$, $b_{22} = m_{22}v + t_2$, $k_{21} = v[k_{21}/v] + \beta_1$ *and* $k_{22} = v[k_{22}/v] + \beta_2$. *Further suppose* $b_{11} = [k_{21}/v]t_1 + (t_1 - 1)$, $b_{12} = [k_{22}/v]t_2 + (t_2 - 1)$ *and* $b_{11} \le b_{12}$. *Let* $d_2 \in D_2(v;\ b_2, \boldsymbol{k}_2')$ *have* $N_{d_2} = (N_{d_{21}}, N_{d_{22}})$ *where* $b_2 = b_{21} + b_{22}$ *and* $\boldsymbol{k}_2' = (k_{21}J_{1,b_{21}}, k_{22}J_{1,b_{22}})$. *Then there exists a row-column design* $\hat{d}_2$ *having* $C_{\hat{d}_2} = C_{d_2}$ *in* $D(v;\ b_1, \boldsymbol{k}_1', b_2, \boldsymbol{k}_2')$ *where* b_2 *and* $\boldsymbol{k}_2'$ *are as defined above,* $b_1 = b_{12}$ *and* $\boldsymbol{k}_1' = (v(m_{21}(v-\beta_1)+1+m_{22}(v-\beta_2)+1)J_{1,b_{11}}, v(m_{22}(v-\beta_2)+1)J_{1,b_{12}-b_{11}})$.

PROOF. The proof follows by construction. By applying Corollary 3.1 to the block design d_{21}, we see that we can construct a row-column design $\hat{d}_{21} \in D(v;\ b_{11},\ v(m_{21}(v-\beta_1)+1)J_{1,b_{11}},\ b_{21},\ k_{21}J_{1,b_{21}})$ such that every treatment appears $m_{21}(v-\beta_1)+1$ times in each row of $\hat{d}_{21}$. Similarly, it follows from Corollary 3.1 that we can construct from d_{22} a row-column design $\hat{d}_{22} \in D(v;\ b_{12},\ v(m_{22}(v-\beta_2)+1)J_{1,b_{12}},\ b_{22},\ k_{22}J_{1,b_{22}})$ such that every treatment appears $m_{22}(v-\beta_2)+1$ times in each row of $\hat{d}_{22}$. The desired row-column design $\hat{d}_2$ is now given by $\hat{d}_2 = (\hat{d}_{21},\ \hat{d}_{22})$. It is easily verified that $\hat{d}_2$ satisfies the conditions of Lemma 2.1, hence that $C_{\hat{d}_2} = C_{d_2}$.

COROLLARY 3.2. *Assume the conditions of Theorem* 3.6 *hold and let* d^*_{21}, d^*_{22} *and* d^*_2 *be block designs such as described in Theorem* 3.6 *where* d^*_2 *is* ϕ*-optimal in* $D_2(v;\ b_2, k_2')$. *Now let* $\hat{d}^*_2 \in D(v;\ b_1, k_1',\ b_2, k_2')$ *be a row-column design that can be obtained from* d^*_2 *such that* $C_{\hat{d}^*_2} = C_{d^*_2}$. *Then* $\hat{d}^*_2$ *is* ϕ*-optimal in* $D(v;\ b_1, k_1',\ b_2, k_2')$.

PROOF. This result follows directly from Theorem 2.1.

In the remaining portion of this paper we shall show the existence of several classes of BBD's that satisfy the conditions of Corollary 3.1. To begin with, let d_0 be a BBD with b blocks of size k and having each treatment occur $[k/v]$ or $[k/v]+1$ times in each block. Further, as in the proof of Corollary 3.1, let B_j, $j = 1, \ldots, b$, be the set of all treatments which occur $[k/v]+1$ times in the j-th block of d_0 and let $\underline{d}_0$ be the block design having B_j, $j = 1, \ldots, b$ as its blocks. We note that $\underline{d}_0$ is a BIBD and we shall henceforth refer to $\underline{d}_0$ as the component BIBD of d_0.

COROLLARY 3.3. *Let* d_0 *be that BBD based on* $v = 2t$ *treatments,* $t \geq 1$, *arranged in* b *blocks of size* $k = v[k/v] + (v-2)$ *which satisfies* (3.1) *and is such that* $\underline{d}_0$ *is that irreducible BIBD whose blocks consist of all the* $(v-2)$*-ples that are possible to choose from the* v *available treatments. Then* d_0 *satisfies the conditions of Corollary* 3.1.

PROOF. To begin with, we note that since the blocks of $\underline{d}_0$ consist of all $(v-2)$-ples that can be formed from the treatments $1, \ldots, v$, we can find blocks $B_1, \ldots, B_t$ in $\underline{d}_0$ such that B_j does not contain treatments $2(j-1)+1$ and $2j$. Now let $N_{d_0} = (N_{d_{01}},\ N_{d_{02}})$ where $N_{d_{02}}$ is that set of blocks in N_{d_0} corresponding to $B_1, \ldots, B_t$. The result now follows since all treatments in $N_{d_{02}}$ are equally replicated and $t(v-\beta) = t(2t - 2(t-1)) = 2t = v$.

Comment. It follows from our discussion so far that if we start out with a block design $d_2 = (d_{21},\ d_{22})$ where $d_{21} \in D_2(v;\ b_{21},\ k_{21}J_{1,b_{21}})$ is a BBD satisfying the conditions of Corollary 3.1 and $d_{22} \in D_2(v;\ b_{22},\ k_{22}J_{1,b_{22}})$ is a BBD satisfying the conditions of Theorem 3.2, then we can construct a row-column design which is ϕ-optimal in the appropriate class of row-column designs.

Example 3.3. Consider the row-column design $\hat{d}_2^*$ arranged in 11 rows and 21 columns given by

$$\hat{d}_2^* = \begin{bmatrix} 123454612345 6135 & 123456 \\ 234561234561246 & 234561 \\ 345612345612351 & 345612 \\ 456123456123462 & 456123 \\ 561234561234513 & 561234 \\ 612345612345624 & 612345 \\ 112233554466124 & 123456 \\ 224411663355356 & 234561 \\ 335644115622213 & 345612 \\ 561156342234465 & 456123 \\ & 561234 \end{bmatrix}.$$

Here $d_2^* = (d_{21}^*, d_{22}^*)$ is a BUBD where $d_{21}^* \in D_2(6;\ 15,\ 10J_{1,15})$ is a BBD satisfying the conditions of Corollary 3.3 and $d_{22}^* \in D_2(6;\ 6,\ 11J_{1,6})$ is a BBD satisfying the conditions of Theorem 3.2. Thus, applying Corollary 3.1, a row-column design $\hat{d}_{21}^* \in D(6;\ 10,\ 15J_{1,10},\ 15,\ 10J_{1,15})$ is obtained from d_{21}^* and a row-column design $\hat{d}_{22}^* \in D(6;\ 11,\ 6J_{1,11},\ 6,\ 11J_{1,6})$ is obtained from d_{22}^*. Now, the row-column design $\hat{d}_2^*$ given by $\hat{d}_2^* = (\hat{d}_{21}^*, \hat{d}_{22}^*)$ satisfies the conditions of Theorem 3.4 and hence is ϕ-optimal in $D(6;\ 11,\ (21J_{1,10},\ 6J_{1,1}),\ 21,\ (10J_{1,15},\ 11J_{1,6}))$ under most optimality criteria.

We now give two additional classes of block designs which can be used to construct optimal row-column designs of the types given in Corollary 3.1.

DEFINITION 3.3. Let d_0 be a binary block design based on v treatments arranged in b blocks of size k. The complement of d_0, denoted by d_0', is that design whose incidence matrix is given by $N_{d_0'} = J_{vb} - N_{d_0}$.

DEFINITION 3.4. Let d_0 be a binary block design based on v treatments arranged in b blocks of size k. We say d_0 is resolvable if its blocks can be divided into classes $S_1, \ldots, S_t$ each containing the same number of blocks, such that in each class, every treatment is replicated once.

A number of families of BIBD's can be constructed using finite geometries. In particular, if s is a prime number or a power of a prime number, then two families of BIBD's having the following parameters can be constructed using finite geometries (see Raghavarao (1971), p. 78):

$$\begin{aligned} &\text{family } 1: v = s^2+s+1 = b, \quad r = s+1 = k, \quad \lambda = bk(k-1)/v(v-1) = 1 \\ &\text{family } 2: v = (s+1)(s^2+1), \quad b = (s^2+1)(s^2+s+1), \quad r = s^2+s+1, \\ &\qquad k = s+1, \quad \lambda = bk(k-1)/v(v-1) = 1. \end{aligned} \tag{3.2}$$

Using the families of BIBD's described in (3.2), we now give our final result.

COROLLARY 3.4. *Let d_0 be that BBD based on v treatments arranged in b blocks of size k which satisfies* (3.1) *and which has $\underline{d}_0$ which is the complement of one of the BIBD's described in* (3.2) *for an appropriate value of s. Then d_0 satisfies the conditions of Corollary* 3.1.

PROOF. We shall consider two cases.

Case 1: $\underline{d}_0$ is the complement of some design from family 1 of (3.2).

In this case, for an appropriate value of s, $\underline{d}_0$ is a BIBD which has parameters $v = s^2$, $b = s(s+1)$ and $k' = k - v[k/v] = s^2 - s$. Also, it follows from Raghavarao (1971) that $\underline{d}_0$ is resolvable. Thus we can find blocks $B_1, \ldots, B_t$ in $\underline{d}_0$ where $t = s$ such that each treatment occurs in $B_1, \ldots, B_t$ exactly once. Now let $N_{d_0} = (N_{d_{01}}, N_{d_{02}})$ where $N_{d_{02}}$ is that set of blocks in N_{d_0} corresponding to $B_1, \ldots, B_t$ of $\underline{d}_0$. The result now follows since all treatments in $N_{d_{02}}$ are equally replicated and $t(v - k') = s^2 = v$.

Case 2: $\underline{d}_0$ is the complement of some design from family 2 of (3.2).

Here, for an appropriate value of s, $\underline{d}_0$ is a resolvable BIBD having parameters $v = s^3$, $b = s^2(s^2 + s + 1)$ and $k' = k - v[k/v] = s^3 - s$ (see Raghavarao (1971)). Thus we can find blocks $B_1, \ldots, B_t$ in $\underline{d}_0$ where $t = s^2$ such that each treatment occurs in $B_1, \ldots, B_t$ exactly once. The proof now follows as in Case 1 since $t(v - k') = s^3 = v$.

Comment. The comment made following Corollary 3.3 holds for the designs given in Corollary 3.4 above as well and designs such as given in Example 3.3 can be constructed.

REFERENCES

Agrawal, H. (1966). Some generalizations of distinct representatives with applications to statistical designs, *Ann. Math. Statist.*, **37**, 525–526.

Clatworthy, W. H. (1973). Tables of two-associate class partially balanced designs, *National Bureau of Standards Applied Mathematics Series 6*, U.S. Department of Commerce.

Cheng, C. S. (1978). Optimal designs for the elimination of multiway heterogeneity, *Ann. Statist.*, **6**, 1262–1272.

Gupta, S. C. and Jones, B. (1983). Equireplicate balanced block designs with unequal block sizes, *Biometrika*, **70**, 433–440.

Hall, M. (1945). An algorithm for distinct representatives, *Amer. Math. Monthly*, **63**, 716–717.

Kiefer, J. (1975). Construction and optimality of generalized Youden designs, *A Survey of Statistical Design and Linear Models* (ed. J. N. Srivastava), 333–353, North Holland, Amsterdam.

Kunert, J. (1983). Optimal design and refinement of the linear model with applications to repeated measurements designs, *Ann. Statist.*, **11**, 247–257.

Raghavarao, D. (1971). *Constructions and Combinatorial Problems in Design of Experiments*, Wiley, New York.