

## ON THE ESTIMATION OF ORDERED MEANS OF TWO EXPONENTIAL POPULATIONS

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**Abstract.** Let random samples of equal sizes be drawn from two exponential distributions with ordered means  $\lambda_i$ . The maximum likelihood estimator  $\lambda_i^*$  of  $\lambda_i$  is shown to have a smaller mean square error than that of the usual estimator  $\bar{X}_i$ , for each  $i = 1, 2$ . The asymptotic efficiency of  $\lambda_i^*$  relative to  $\bar{X}_i$  has also been found.

*Key words and phrases:* Asymptotic efficiency, exponential distribution, isotonic regression, maximum likelihood estimation, mean square error.

### 1. Introduction

The maximum likelihood estimator (MLE)  $\hat{\mu}$  of a nondecreasing regression function has been studied in detail (see Barlow *et al.* (1972), Robertson *et al.* (1988)). In terms of its coordinates, little was known about its quadratic loss until Lee (1981) showed that the mean square error (MSE) of  $\hat{\mu}_i$  is less than the usual estimator  $\bar{X}_i$  for each  $i$ , when  $\bar{X}_1, \bar{X}_2, \dots, \bar{X}_k$  are independent normal variates. Robertson *et al.* (1988, p. 44) have mentioned that it is of interest to know if such a result is valid under other conditions, and in particular, whether it holds true for samples from other types of populations. In this article we give a partial answer to their question with regard to sampling from exponential distributions and show that the result holds for the case of samples of equal sizes from two exponential distributions having ordered means.

The results of this article will prove useful in reliability when one frequently comes across situations where the estimation of means of two ordered exponential life distributions is desired. For example, there are situations where one is interested in the estimation of the mean lives of two mechanical devices having exponential life distributions, of which one is an improvement of the other, and naturally the improved device should not have a mean life length less than that of the original device. In yet another situation, the interest may be in the estimation of the mean lives of two components having exponential life distributions, in which one is produced by a standard company whereas the other is manufactured by a local company, and where it is known, a priori, that the mean life of the component



Table 2. Efficiency  $e(\lambda_2^*, \bar{X}_2)$  of  $\lambda_2^*$  relative to  $\bar{X}_2$ .

| $\lambda_2/\lambda_1$ | $n$    |        |        |        |        |        |        |        |
|-----------------------|--------|--------|--------|--------|--------|--------|--------|--------|
|                       | 2      | 5      | 10     | 20     | 30     | 50     | 100    | 1000   |
| 1                     | 1.0666 | 1.1450 | 1.1931 | 1.2304 | 1.2479 | 1.2660 | 1.2850 | 1.3176 |
| 1.3                   | 1.1186 | 1.1627 | 1.1642 | 1.1360 | 1.1086 | 1.0693 | 1.0242 | 1.0000 |
| 1.5                   | 1.1167 | 1.1319 | 1.1087 | 1.0656 | 1.0397 | 1.0152 | 1.0015 | 1.0000 |
| 2.0                   | 1.0880 | 1.0670 | 1.0339 | 1.0088 | 1.0023 | 1.0001 | 1.0000 | 1.0000 |
| 3.0                   | 1.0448 | 1.0182 | 1.0037 | 1.0001 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 5.0                   | 1.0150 | 1.0023 | 1.0001 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 10.00                 | 1.0027 | 1.0001 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |

## 2. Co-ordinatewise loss of MSE of $\lambda_1^*$ and $\lambda_2^*$

By using the max-min formula of isotonic regression (see Barlow *et al.* (1972)), we have

$$(2.1) \quad \lambda_1^* = \min(\bar{X}_1, (\bar{X}_1 + \bar{X}_2)/2),$$

$$(2.2) \quad \lambda_2^* = \max(\bar{X}_2, (\bar{X}_1 + \bar{X}_2)/2).$$

We introduce the following notations

$$(2.3) \quad R(\lambda_i^*) = E[(\lambda_i^* - \lambda_i)^2], \quad i = 1, 2.$$

$$(2.4) \quad R(\bar{X}_i) = E[(\bar{X}_i - \lambda_i)^2] = \lambda_i^2/n, \quad i = 1, 2.$$

$$(2.5) \quad e(\lambda_i^*, \bar{X}_i) = R(\bar{X}_i)/R(\lambda_i^*), \quad i = 1, 2.$$

We shall call  $e(\lambda_i^*, \bar{X}_i)$  the efficiency of  $\lambda_i^*$  relative to  $\bar{X}_i$ ,  $i = 1, 2$ . For convenience, we denote  $\lambda_i/(\lambda_1 + \lambda_2)$  by  $Z_i$ ,  $i = 1, 2$ ,  $(\lambda_2/\lambda_1)$  by  $y_1$  and  $(\lambda_1/\lambda_2)$  by  $y_2$ . Since  $\lambda_1 \leq \lambda_2$ , we have  $y_1 \geq 1$ ,  $0 < y_2 \leq 1$ ,  $0 < Z_1 \leq (1/2)$  and  $(1/2) \leq Z_2 < 1$ .

Since  $\bar{X}_i$  has a gamma distribution with pdf

$$(2.6) \quad g_i(y) = (1/(n-1)!) (n/\lambda_i)^n y^{n-1} \exp(-ny/\lambda_i), \quad y \geq 0$$

for  $i = 1, 2$  and the relationship

$$(2.7) \quad (1/(m-1)!) \alpha^m \int_x^\infty y^{m-1} \exp(-\alpha y) dy = \exp(-\alpha x) \sum_{i=0}^{m-1} (\alpha x)^i / i!$$

holds for any positive integer  $m$  and for any  $\alpha > 0$ , it easily follows that

$$\begin{aligned}
 (2.8) \quad E[(\lambda_1^* - \lambda_1)^2 / \bar{X}_1 = x] &= [(n+1)/4n]\lambda_2^2 + \lambda_1^2 - \lambda_1\lambda_2 + [(\lambda_2/2) - \lambda_1]x + (x^2/4) \\
 &+ \exp(-nx/\lambda_2) \left\{ (3/4)x^2 \sum_{i=0}^{n-1} [(nx/\lambda_2)^i / i!] \right. \\
 &\quad - (\lambda_2/2)x \sum_{i=0}^n [(nx/\lambda_2)^i / i!] \\
 &\quad - [(n+1)/4n]\lambda_2^2 \sum_{i=0}^{n+1} [(nx/\lambda_2)^i / i!] \\
 &\quad - \lambda_1 x \sum_{i=0}^{n-1} [(nx/\lambda_2)^i / i!] \\
 &\quad \left. + \lambda_1 \lambda_2 \sum_{i=0}^n [(nx/\lambda_2)^i / i!] \right\}.
 \end{aligned}$$

By taking the expectation of (2.8) with respect to the distribution of  $\bar{X}_1$ , we have, after simplification,

$$\begin{aligned}
 (2.9) \quad R(\lambda_1^*) &= (\lambda_1^2 + \lambda_2^2)[(n+1)/4n] - (\lambda_1\lambda_2)/2 \\
 &+ (1 - Z_1)^n \left\{ [3(n+1)/4n]\lambda_2^2 Z_1^2 \sum_{i=0}^{n-1} \binom{n+i+1}{i} Z_1^i \right. \\
 &\quad - (1/2)\lambda_2^2 Z_1 \sum_{i=0}^n \binom{n+1}{i} Z_1^i \\
 &\quad - [(n+1)/4n]\lambda_2^2 \sum_{i=0}^{n+1} \binom{n+i-1}{i} Z_1^i \\
 &\quad - \lambda_1 \lambda_2 Z_1 \sum_{i=0}^{n-1} \binom{n+i}{i} Z_1^i \\
 &\quad \left. + \lambda_1 \lambda_2 \sum_{i=0}^n \binom{n+i-1}{i} Z_1^i \right\}.
 \end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
 (2.10) \quad R(\lambda_2^*) &= (\lambda_2^2/n) \\
 &+ (1 - z_2)^n \left\{ -[3(n+1)/4n]\lambda_1^2 Z_2^2 \sum_{i=0}^{n-1} \binom{n+i+1}{i} Z_2^i \right. \\
 &\quad + (\lambda_1^2/2) Z_2 \sum_{i=0}^n \binom{n+i}{i} Z_2^i \\
 &\quad + [(n+1)/4n]\lambda_1^2 \sum_{i=0}^{n+1} \binom{n+i-1}{i} Z_2^i \\
 &\quad + \lambda_1 \lambda_2 Z_2 \sum_{i=0}^{n-1} \binom{n+i}{i} Z_2^i \\
 &\quad \left. - \lambda_1 \lambda_2 \sum_{i=0}^n \binom{n+i-1}{i} Z_2^i \right\}.
 \end{aligned}$$

THEOREM 2.1.

- (a)  $R(\lambda_1^*) < R(\bar{X}_1)$ ,  
 (b)  $\lim_{(\lambda_2/\lambda_1) \rightarrow \infty} e(\lambda_1^*, \bar{X}_1) = 1$ ,  
 (c)  $\lim_{n \rightarrow \infty} e(\lambda_1^*, \bar{X}_1) = \begin{cases} 1 & \text{if } \lambda_2 > \lambda_1, \\ 4/3 & \text{if } \lambda_2 = \lambda_1. \end{cases}$

PROOF. Since  $(1 - Z_1)^{-m} = \sum_{k=0}^{\infty} \binom{m+k-1}{k} Z_1^k$ , for any positive integer  $m$ , it follows from (2.9) that

$$\begin{aligned}
 (2.11) \quad [R(\lambda_1^*) - R(\bar{X}_1)]/R(\bar{X}_1) &= n \left\{ -[3(n+1)/4n] \sum_{k=n}^{\infty} \binom{n+k+1}{k} Z_1^k (1 - Z_1)^{n+2} \right. \\
 &\quad + (y_1/2) \sum_{k=n+1}^{\infty} \binom{n+k}{k} Z_1^k (1 - Z_1)^{n+1} \\
 &\quad + [(n+1)/4n] y_1^2 \sum_{k=n+2}^{\infty} \binom{n+k-1}{k} Z_1^k (1 - Z_1)^n \\
 &\quad + \sum_{k=n}^{\infty} \binom{n+k}{k} Z_1^k (1 - Z_1)^{n+1} \\
 &\quad \left. - y_1 \sum_{k=n+1}^{\infty} \binom{n+k-1}{k} Z_1^k (1 - Z_1)^n \right\}.
 \end{aligned}$$

Using the relationship between binomial and negative binomial distributions

(see Meyer (1970), p. 174) we have from (2.11)

$$\begin{aligned}
 (2.12) \quad & [R(\lambda_1^*) - R(\bar{X}_1)]/R(\bar{X}_1) \\
 & = n\{-[3(n+1)/4n]B(n+1, 2n+1, 1-Z_1) \\
 & \quad + (y_1/2)B(n, 2n+1, 1-Z_1) \\
 & \quad + [(n+1)/4n]y_1^2B(n-1, 2n+1, 1-Z_1) \\
 & \quad + B(n, 2n, 1-Z_1) - y_1B(n-1, 2n, 1-Z_1)\},
 \end{aligned}$$

where

$$(2.13) \quad B(k, n, p) = \sum_{r=0}^k \binom{n}{r} p^r (1-p)^{n-r}$$

$$(2.14) \quad = (n-k) \binom{n}{k} \int_0^{1-p} t^{n-k-1} (1-t)^k dt,$$

for  $0 \leq k < n$

(see Feller (1957), p. 163 for (2.14)). Using (2.14) in (2.12), we obtain

$$\begin{aligned}
 (2.15) \quad & [R(\lambda_1^*) - R(\bar{X}_1)]/R(\bar{X}_1) \\
 & = n \binom{2n}{n} \int_0^{1/(1+y_1)} t^{n-1} (1-t)^{n-1} [A(y_1)t^2 + B(y_1)t + C] dt,
 \end{aligned}$$

where

$$\begin{aligned}
 A(y_1) &= [(2n+1)/4](y_1^2 - 2y_1 - 3), \\
 B(y_1) &= (1/2)(y_1 + 4n + 3), \\
 C &= -(2n+3)/4.
 \end{aligned}$$

It can be shown that

$$A(y_1)t^2 + B(y_1)t + C < 0 \quad \text{for } t \in [0, 1/(1+y_1)].$$

By (2.15), part (a) of Theorem 2.1 holds.

Since  $4t(1-t) \leq 1$ , we have from (2.15)

$$\begin{aligned}
 |[R(\lambda_1^*)/R(\bar{X}_1)] - 1| &\leq n \binom{2n}{n} (1/2)^{2n-2} \int_0^{1/(1+y_1)} |A(y_1)t^2 + B(y_1)t + C| dt \\
 &\leq n \binom{2n}{n} (1/2)^{2n-2} \{ [|A(y_1)|/3][1/(1+y_1)]^3 \\
 & \quad + [|B(y_1)|/2][1/(1+y_1)]^2 \\
 & \quad + |C|[1/(1+y_1)] \}.
 \end{aligned}$$

The right-hand side of the above inequality tends to zero as  $y_1 \rightarrow \infty$  and part (b) of Theorem 2.1 has been established.

Consider the case when  $y_1 > 1$ . Then

$$(2.16) \quad nB(n+1, 2n+1, 1-Z_1) = n^2 \binom{2n+1}{n+1} \int_0^{Z_1} t^{n-1} (1-t)^{n+1} dt \\ \leq n^2 \binom{2n+1}{n+1} [Z_1(1-Z_1)]^{n-1} \int_0^1 (1-t)^2 dt.$$

Using Stirling's approximation formula, the right-hand side of (2.16) is asymptotically equivalent to

$$n^{(3/2)} [4Z_1(1-Z_1)]^{n-1} (8/3\sqrt{\pi})$$

and it tends to zero as  $n \rightarrow \infty$ . It follows that the first term of (2.12) tends to zero as  $n \rightarrow \infty$ , so are the remaining terms of (2.12). Part (c) of Theorem 2.1 for the case  $\lambda_2 > \lambda_1$  has been established.

For  $y_1 = 1$ , we have from (2.15)

$$\begin{aligned} & [R(\lambda_1^*) - R(\bar{X}_1)] - 1 \\ &= n \binom{2n}{n} \int_0^{1/2} [-(2n+1)t^2 + 2(n+1)t - (2n+3)/4] t^{n-1} (1-t)^{n-1} dt \\ &= n \binom{2n}{n} \left\{ (2n+2) \int_0^{1/2} t^n (1-t)^n dt - [(2n+3)/4] \int_0^{1/2} t^{n-1} (1-t)^{n-1} dt \right. \\ &\quad \left. + \int_0^{1/2} t^{n+1} (1-t)^{n-1} dt \right\} \\ &= n \binom{2n}{n} \left\{ (n+1) \int_0^1 t^n (1-t)^n dt - [(2n+3)/8] \int_0^1 t^{n-1} (1-t)^{n-1} dt \right. \\ &\quad \left. + [(n+1)/2n] \int_0^1 t^n (1-t)^n dt - (1/n)(1/2)^{2n+1} \right\}. \end{aligned}$$

After simplification, the above gives

$$[R(\lambda_1^*)/R(\bar{X}_1)] - 1 = -\binom{2n}{n} (1/2)^{2n+1} - (1/4).$$

As  $n \rightarrow \infty$ , the first term on the right-hand side tends to zero by using Stirling's approximation formula. Thus for  $\lambda_2 = \lambda_1$ ,  $\lim_{n \rightarrow \infty} [R(\lambda_1^*)/R(\bar{X}_1)] = 3/4$ , which proves part (c) of Theorem 2.1.

#### THEOREM 2.2.

(a)  $R(\lambda_2^*) < R(\bar{X}_2)$ , for  $n \geq 2$ ,  $\lambda_1 \leq \lambda_2$  and for  $n = 1$ ,  $\lambda_1 < \lambda_2$ .  $R(\lambda_2^*) = R(\bar{X}_2)$ , for  $n = 1$ ,  $\lambda_1 = \lambda_2$ .

(b)  $\lim_{(\lambda_2/\lambda_1) \rightarrow \infty} e(\lambda_2^*, \bar{X}_2) = 1$ .

(c)  $\lim_{n \rightarrow \infty} e(\lambda_2^*, \bar{X}_2) = \begin{cases} 1 & \text{if } \lambda_2 > \lambda_1, \\ (4/3) & \text{if } \lambda_2 = \lambda_1. \end{cases}$

PROOF. Proceeding on similar lines as those for Theorem 2.1 and by using the relationship

$$(2.17) \quad 1 - B(k, n, p) = n \binom{n-1}{k} \int_0^p t^k (1-t)^{n-k-1} dt, \quad 0 \leq k < n.$$

(see Feller (1957), p. 163), we get

$$(2.18) \quad [R(\lambda_2^*) - R(\bar{X}_2)]/R(\bar{X}_2) \\ = n \binom{2n}{n} \int_0^{y_2/(1+y_2)} [A_1(y_2)t^2 + B_1(y_2)t + C_1(y_2)]t^{n-1}(1-t)^{n-1} dt,$$

where

$$A_1(y_2) = -[(2n+1)/4](3+2y_2-y_2^2), \\ B_1(y_2) = [(2n+1)/2](y_2-y_2^2) + n(1+y_2), \\ C_1(y_2) = [(2n+1)/4]y_2^2 - ny_2.$$

We consider  $[R(\lambda_2^*) - R(\bar{X}_2)]/R(\bar{X}_2)$  given by (2.18) as a function of  $y_2$  and denote it by  $\phi(y_2)$ . Thus

$$(2.19) \quad \phi(y_2) = n \binom{2n}{n} \int_0^{y_2/(1+y_2)} [A_1(y_2)t^2 + B_1(y_2)t + C_1(y_2)]t^{n-1}(1-t)^{n-1} dt.$$

It can be shown that if  $0 < y_2 \leq n/(n+1)$ , then  $y_2/(1+y_2)$  is the smallest positive root of  $A_1(y_2)t^2 + B_1(y_2)t + C_1(y_2) = 0$  and  $A_1(y_2)t^2 + B_1(y_2)t + C_1(y_2) < 0$  for  $t \in [0, y_2/(1+y_2)]$ . Thus, from (2.19) it follows that

$$(2.20) \quad \phi(y_2) < 0 \quad \text{for } y_2 \in (0, n/(n+1)].$$

Differentiating (2.19) twice with respect to  $y_2$  gives

$$(2.21) \quad \phi''(y_2) = n \binom{2n}{n} \left\{ [(2n+1)/2] \int_0^{y_2/(1+y_2)} t^{n-1}(1-t)^{n+1} dt \right. \\ \left. + [y_2/(1+y_2)]^{n-1} [1/(1+y_2)]^{n+1} [(n+1)y_2 - n] \right\}.$$

Thus,  $\phi''(y_2) > 0$  for  $y_2 \geq n/(n+1)$ , which implies that  $\phi(y_2)$  is convex in  $[n/(n+1), 1]$ . From (2.19) we also have

$$\phi(1) = n \binom{2n}{n} \int_0^{1/2} [-(2n+1)t^2 + 2nt - (2n-1)/4]t^{n-1}(1-t)^{n-1} dt \\ = n \binom{2n}{n} \left\{ n \int_0^1 t^n(1-t)^n dt - [(2n-1)/8] \int_0^1 t^{n-1}(1-t)^{n-1} dt \right. \\ \left. - [(n+1)/2n] \int_0^1 t^n(1-t)^n dt + (1/n)(1/2)^{2n+1} \right\}.$$

After simplification, the above gives

$$(2.22) \quad \phi(1) = (1/2) \left[ \binom{2n-1}{k} (1/2)^{2n-1} - (1/2) \right].$$



It is easily seen from (2.17) that

$$(2.23) \quad \sum_{k=n}^{2n-1} \binom{2n-1}{k} (1/2)^{2n-1} = (1/2).$$

Thus, from (2.22) and (2.23) we have

$$(2.24) \quad \begin{cases} \phi(1) < 0 & \text{for } n \geq 2, \\ \phi(1) = 0 & \text{for } n = 1. \end{cases}$$

Since  $\phi(y_2)$  is convex in  $[n/(n+1), 1]$ , it follows that

$$(2.25) \quad \phi(y_2) < 0, \quad y_2 \in (n/(n+1), 1).$$

Part (a) of Theorem 2.2 follows from (2.20), (2.24) and (2.25). The proofs of part (b) and (c) of Theorem 2.2 similarly follow from their counterparts in Theorem 2.1.

From the expressions (2.9) and (2.10), it immediately follows that  $e(\lambda_i^*, \bar{X}_i)$  is a function of  $n$  and  $\lambda_2/\lambda_1$ . By using these expressions, the values of  $e(\lambda_i^*, \bar{X}_i)$  for  $i = 1, 2$  have been calculated by computer for some values of  $n$  and  $\lambda_2/\lambda_1$ . These have been tabulated in Tables 1 and 2.

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