Ann. Inst. Statist. Math.
Vol. 43, No. 2, 287–295 (1991)

# ON CHARACTERIZATIONS OF DISTRIBUTIONS BY MEAN ABSOLUTE DEVIATION AND VARIANCE BOUNDS

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(Received October 2, 1989; revised March 24, 1990)

**Abstract.** In this paper we present a bound for the mean absolute deviation of an arbitrary real-valued function of a discrete random variable. Using this bound we characterize a mixture of two Waring (hence geometric) distributions by linearity of a function involved in the bound. A double Lomax distribution is characterized by linearity of the same function involved in the analogous bound for a continuous distribution. Finally, we characterize the Pearson system of distributions and the generalized hypergeometric distributions by a quadratic function involved in a similar bound for the variance of a function of a random variable.

*Key words and phrases*: Characterizations, geometric, hypergeometric and Pearson distributions, mean absolute deviation, mixtures.

## 1. Introduction

Bounds for the mean absolute deviation (MAD) are of very recent origin. Freimer and Mudholkar (1989) gave one such bound for the MAD. More specifically, let $X$ be a continuous random variable (r.v.) with distribution function (d.f.) $F$, density $f$, a median $\delta$, and further let there exist a function $\eta(x)$ such that $\eta(x)f(x) = F(x)$ for $x \leq \delta$ and $\eta(x)f(x) = 1 - F(x)$ for $x > \delta$. Then for any absolutely continuous real-valued function $g$, $MD[g(X)] \leq E[\eta(x)|g'(x)|]$, where $g'$ is the derivative of $g$. They used this bound to characterize the double exponential distribution by $\eta(x) \equiv 1$. In this paper we first extend their results to a discrete r.v. We then use these bounds to characterize the double Lomax and mixtures of two Waring distributions (which include the geometric as a special case) by linearity of $\eta$.

When it comes to bounds for the variance of an r.v., related characterizations and other applications, there is an extensive literature. Much work in this area was stimulated by Chernoff's result (1981): If $X$ has a normal distribution with variance $\sigma^2 > 0$ and $g$ is an absolutely continuous function such that $E[g(X)]^2 < \infty$, then $\mathrm{Var}[g(X)] \leq \sigma^2 E[g'(X)]^2$, equality holding if and only if $g$ is linear. Borovkov and Utev (1983) let $U(X) = \sup\{\mathrm{Var}[g(X)]/\sigma^2 E[g'(X)]^2\}$, where the supremum is taken over the class of absolutely continuous functions $g$ and proved $U(X) \geq 1$ and if $U(X) = 1$ then $X$ must have a normal distribution. Chen (1982), using different tools, also proved Chernoff's result and extended it to the multivariate normal distribution. Chen (1988) used the Borovkov-Utev result to prove the sufficiency and necessity of the Lindeberg condition in the central limit theorem. Chen and Lou (1987) used Poincaré-type inequalities to characterize infinitely divisible distributions and obtain some related results. More recently, Cacoullos and Papathanasiou (1989) derived a lower bound for the variance of a real-valued function of an r.v. and showed that the function $W$ involved in this bound characterized the r.v. The function $W$, for example, is constant for the normal and Poisson. Earlier, Srivastava and Sreehari (1987) used an upper bound for the variance of a real-valued function of a discrete r.v. to arrive at the same characterization of the r.v. They cited as particular cases the characterizations of the binomial, Poisson and negative binomial distributions.

In Section 4 we show that the linearity of $W$ characterizes the above three distributions in the discrete case, and the normal and shifted gamma distributions in the continuous case. Our final results characterize the Pearson system of distributions and the generalized hypergeometric by a quadratic $W$.

## 2. The bound for the mean absolute deviation

Let $X$ be a discrete r.v. taking values $\{\ldots, -1, 0, 1, 2, \ldots\}$ with a median $\delta$, d.f. $F$ and probability function (p.f.) $f$. We can always take $\delta$ to be an integer. Suppose there is a function $\eta$ such that

$$\eta(x)f(x) = \begin{cases} F(x), & x < \delta \\ 1 - F(x), & x \geq \delta. \end{cases} \tag{2.1}$$

Then $\eta(x) \geq 0$ for all $x$ and

$$MD(X) = E(\eta(X)) \tag{2.2}$$

where

$$MD(X) = E(|X - \delta|) \tag{2.3}$$

is the mean absolute deviation of $X$. We then prove

THEOREM 2.1. *For any real-valued function $g$ defined on $\{\ldots, -1, 0, 1, \ldots\}$,*

$$MD(g(X)) \leq E(\eta(X)|\Delta g(X)|), \tag{2.4}$$

*where $\Delta g(x) = g(x+1) - g(x)$.*

The proof parallels the proof of Theorem 2.1 in Freimer and Mudholkar (1989).

PROOF. First suppose $MD(g(X)) < \infty$. Then

$$\begin{aligned}
MD(g(X)) &\leq E(|g(X) - g(\delta)|) \\
&= \sum_{x=-\infty}^{\infty} \left| \sum_{y=x}^{\delta-1} \Delta g(y) \right| f(x) \\
&\leq \sum_{x=-\infty}^{\delta-1} \sum_{y=x}^{\delta-1} |\Delta g(y)| f(x) + \sum_{x=\delta+1}^{\infty} \sum_{y=\delta}^{x-1} |\Delta g(y)| f(x) \\
&= \sum_{y=-\infty}^{\delta-1} |\Delta g(y)| \sum_{x=-\infty}^{y} f(x) + \sum_{y=\delta}^{\infty} |\Delta g(y)| \sum_{x=y+1}^{\infty} f(x) \\
&= \sum_{y=-\infty}^{\infty} |\Delta g(y)| \eta(y) f(y) = E(\eta(X) |\Delta g(x)|).
\end{aligned}$$

If $MD(g(X)) = \infty$, then $E(|g(X) - g(\delta)|) = \infty$. Also, for any $A$, $B > 0$

$$\sum_{x=-A}^{B} \left| \sum_{y=x}^{\delta} \Delta g(y) \right| f(x) \leq \sum_{y=-A}^{B} |\Delta g(y)| \eta(y) f(y)$$

and since the left-hand side $\to \infty$ as $A$, $B \to \infty$, so does the right-hand side. □

We next prove the converse of Theorem 2.1 which will be useful for characterizing distributions.

Define for each integer $y$, $h_y(x) = 0$ for $x \leq y$ and $h_y(x) = 1$ for $x \geq y + 1$. Also define for $\theta > -1$, $g_{\theta,y}(x) = x + \theta h_y(x)$. Then $g_{\theta,y}$ is increasing and we can take $g_{\theta,y}(\delta)$ to be a median of $g_{\theta,y}(X)$. Lemma 2.1, Theorem 2.2 and their proofs are similar to Lemma 3.1 and Theorem 3.2 of Freimer and Mudholkar (1989).

LEMMA 2.1. $MD(g_{\theta,y}(X)) - MD(X) = \theta E(|h_y(X) - h_y(\delta)|)$.

PROOF. This follows from the definition of $h_y(x)$, $g_{\theta,y}(x)$ and the fact that $g_{\theta,y}(x)$ is increasing. □

THEOREM 2.2. *If for a function $\eta$ satisfying* (2.2) *the inequality* (2.4) *holds for all real-valued $g$ defined on the integers, then $\eta$ must satisfy* (2.1).

PROOF. First apply (2.4) to $g_{\theta,y}$ and use (2.2) to get $MD(g_{\theta,y}(X)) - MD(X) \leq \theta E(\eta(X) \Delta h_y(X))$. Next use Lemma 2.1 to arrive at the result $E(|h_y(X) - h_y(\delta)|) = E(\eta(X) \Delta h_y(X))$. Now a direct evaluation shows the right-hand side of this equality to be $\eta(y) f(y)$ and the left-hand side to be $F(y)$ for $y < \delta$ and $1 - F(y)$ for $y \geq \delta$. □

## 3. Characterizations by linearity of $\eta$

In this section we characterize a mixture of two Waring (and hence of two geometric) distributions by (2.4) and linearity of $\eta$. We also characterize a symmetric double Lomax distribution by linearity of $\eta$ and the continuous analogue of (2.4):

THEOREM 3.1. (Freimer and Mudholkar (1989)) *Let $X$ be a continuous r.v. with a median $\delta$, density function $f$ and d.f. $F$. Let $\eta$ be defined by* (2.1) *and satisfy* (2.2). *Then for an absolutely continuous function $g$*

$$MD(g(X)) \leq E(\eta(X)|g'(X)|). \tag{3.1}$$

A continuous r.v. $X$ has a symmetric double Lomax distribution or Pareto distribution of the second kind if its density is given by

$$f(x) = (\alpha/2c)(1 + |x - \delta|/c)^{-1-\alpha}, \qquad -\infty < x < \infty, \quad \alpha, c > 0. \tag{3.2}$$

Note that (3.2) tends to the double exponential distribution as $\alpha \to 0$, $c \to 0$ such that $c/\alpha \to$ const. This latter distribution is characterized by (3.1) with $\eta \equiv 1$ by Freimer and Mudholkar (1989). Suppose now that

$$\eta(x) = a + b|x - \delta|, \qquad -\infty < x < \infty, \tag{3.3}$$

a linear function in $|x - \delta|$. We now prove

THEOREM 3.2. *Let $X$ be a continuous r.v. with a median $\delta$, density function $f$ and d.f. $F$. Let $\eta$ be defined by* (2.1) *and satisfy* (2.2). *Then for an absolutely continuous function $g$* (3.1) *holds with $\eta$ given by* (3.3) *for some constants $a$ and $b$ if and only if $X$ has the double Lomax distribution* (3.2) *with $\alpha = 1/b$ and $c = a/b$.*

PROOF. Suppose first $X$ has the density (3.2). Then it follows that $\eta(x) = c/\alpha + |x - \delta|/\alpha$ which is (3.3) with $\alpha = 1/b$ and $c = a/b$. Note that $MD(X) = c/(\alpha - 1) = E(\eta(X))$, $\alpha > 1$. Thus $\eta$ satisfies (2.1) and (2.2). Conversely, suppose next (3.1) and (3.3) hold. Then by Theorem 3.3 stated below, we get that $f(x)/F(x) = \{a + b(\delta - x)\}^{-1}$ for $x < \delta$ and $f(x)/\{1 - F(x)\} = \{a + b(x - \delta)\}^{-1}$ for $x \geq \delta$. Since $\delta$ is the median and since we must have $\lim_{x\to\infty}[1 - F(x)] = 0 = \lim_{x\to-\infty} F(x)$, we conclude that $a > 0$, $b > 0$, and $f(x) = (1/2a)\{1 + b|x - \delta|\}^{-1-1/b}$, $-\infty < x < \infty$, which is (3.2) with $\alpha = 1/b$ and $c = a/b$. In order that $MD(X)$ be finite we must have $1/b > 1$. In this case $MD(X) = a/(1 - b) = E(\eta(X))$. □

THEOREM 3.3. (Freimer and Mudholkar (1989)) *Suppose* (3.1) *holds for all absolutely continuous function $g$. Suppose also $\eta$ satisfies* (2.2). *Then $\eta$ must satisfy* (2.1).

Next we prove an analogue of Theorem 3.2 for the discrete case. Let $X$ be a discrete r.v. on the integers with p.f.

$$f(x) = \begin{cases} (1-p)(\lambda_1 - c_1)c_1^{[\delta-1-x]}/\lambda_1^{[\delta-x]}, & x < \delta \\ p(\lambda_2 - c_2)c_2^{[x-\delta]}/\lambda_2^{[x-\delta+1]}, & x \geq \delta \end{cases} \tag{3.4}$$

where $\delta$ is an integer, $0 < p < 1$, $\gamma_1 > c_1 > 0$, $\lambda_2 > c_2 > 0$ and

$$a^{[r]} = a^{[r]}(\gamma) = a(a+\gamma)\cdots(a+\gamma(r-1)), \qquad r = 1, 2, \ldots; \quad a^{[0]} = 1. \tag{3.5}$$

Here $\gamma$ is a real number. Let $\delta$ be a median of $X$. Note that (3.4) is a mixture of two Waring distributions. For (3.4), we have $\eta(x) = \lambda_1 - \gamma + \gamma(\delta - x)$ for $x < \delta$ and $\eta(x) = c_2 + \gamma(x - \delta)$ for $x \geq \delta$. Thus $\eta(x) = a + b|x - \delta|$, $-\infty < x < \infty$, if and only if $c_2 = \lambda_1 - \gamma = a$ and $\gamma = b$. Further $\delta$ is a median if

$$1/2 \leq p \leq \lambda_2/2c_2 \tag{3.6}$$

and $MD(X) = (1-p)(\lambda_1 - \gamma)/(\lambda_1 - c_1 - \gamma) + pc_2/(\lambda_2 - c_2 - \gamma)$, $\lambda_1 - c_1 > \gamma$, $\lambda_2 - c_2 > \gamma$. Now $MD(X) = E(\eta(X)) = a + bMD(X)$ if

$$\lambda_1 = a+b, \quad c_1 = a+b-1, \quad \lambda_2 = a+1, \quad c_2 = a, \quad \gamma = b \quad \text{and} \quad b < 1. \tag{3.7}$$

In this case $MD(X) = E(\eta(X)) = a/(1-b)$. We summarize the above in

PROPOSITION 3.1. *If $X$ has the distribution* (3.4) *with restrictions* (3.6) *and* (3.7) *on its parameters, then* (2.4) *holds with $\eta$ satisfying* (2.1), (2.2) *and* (3.3) *and for all real-valued functions $g$ defined on the integers.*

*Remark* 3.1. Proposition 3.1 covers a mixture of two geometric distributions as a special case when $b = 0$.

We close this section by proving the converse of Proposition 3.1 thus characterizing a mixture of two Waring distributions.

THEOREM 3.4. *Let $X$ be a discrete r.v. on the integers with p.f. $f$, d.f. $F$ and a median $\delta$. If for a function $\eta$ satisfying* (2.2) *and* (3.3) *for some constants $a$ and $b$,* (2.4) *holds for all real-valued functions $g$ defined on the integers, then $X$ must have the distribution* (3.4) *with restrictions* (3.6) *and* (3.7).

PROOF. By Theorem 2.2, $\eta$ must satisfy (2.1). Since $\eta$ also satisfies (3.3) we have $f(x)\{a + b(\delta - x)\} = F(x)$, for $x < \delta$ and $f(x)\{a + b(x - \delta)\} = 1 - F(x)$ for $x \geq \delta$. From these it follows that

$$f(x) = \begin{cases} f(\delta-1)(a+b)(a+b-1)^{[\delta-1-x]}/(a+b)^{[\delta-x]}, & x < \delta \\ f(\delta)(a+1)a^{[x-\delta]}/(a+1)^{[x-\delta+1]}, & x \geq \delta \end{cases} \tag{3.8}$$

where $a^{[r]} = a^{[r]}(b)$ is given by (3.5). Since $\delta$ is a median, we must have $f(\delta - 1)(a+b) + f(\delta)(a+1) = 1$, $f(\delta - 1)(a+b) + f(\delta) \geq 1/2$ and $f(\delta)(a+1) \geq 1/2$. Thus letting $p = f(\delta)(a+1) = 1 - F(\delta - 1) > 0$, we see that (3.8) satisfies (3.4) with restrictions (3.6) and (3.7). The fact that $b < 1$ follows from $MD(X) = a/(1-b)$ and $f(\delta)a = 1 - F(\delta) > 0$. □

## 4. Characterizing distributions by variance bounds

In this section we characterize some distributions by bounds on variances of functions of random variables. Cacoullos and Papathanasiou (1989) proved the following theorem:

THEOREM 4.1. (Cacoullos and Papathanasiou (1989)) *Let $X$ be a continuous r.v. with density $f$, and mean $\mu$ and let $g$ be real and absolutely continuous. Then*

$$\inf_g \{\mathrm{Var}[g(X)]\,\mathrm{Var}(X)/E^2[W(X)g'(X)]\} = 1 \tag{4.1}$$

*holds for some function $W$ with $E\{W(X)\} = \mathrm{Var}(X)$ if and only if $W$ satisfies*

$$\int_{-\infty}^{x} (\mu - t)f(t)dt = W(x)f(x), \qquad -\infty < x < \infty. \tag{4.2}$$

Thus $W$ (4.2) characterizes the distribution of $X$. As examples of $W$ they cite $W(x) = \sigma^2$ and $W(x) = x$ as characterizing the normal and exponential distributions respectively. Here we will show that $W(x) = a + bx$, with $a$, $b$ constants, characterizes precisely the normal and gamma. Similarly we show that "$W(x) =$ a quadratic in $x$" characterizes the Pearson system of continuous distributions.

Cacoullos and Papathanasiou (1989) also prove the discrete analogue of Theorem 4.1:

THEOREM 4.2. (Cacoullos and Papathanasiou (1989)) *Let $X$ be a nonnegative integer-valued random variable with p.f. $f$ and mean $\mu$, and let $g$ be a real-valued function defined on the nonnegative integers. Assume $f(0) > 0$. Then*

$$\inf_g \{\mathrm{Var}[g(X)]\,\mathrm{Var}(X)/E^2[W(X)\Delta g(X)]\} = 1 \tag{4.3}$$

*for some function $W$ with $E(W(X)) = \mathrm{Var}(X)$ if and only if $W$ satisfies*

$$\sum_{y=0}^{x} (\mu - y)f(y) = W(x)f(x), \qquad x = 0, 1, \ldots\ . \tag{4.4}$$

$W$, as before, characterizes the distribution of $X$. They cite as examples $W(x) = \lambda > 0$ characterizing the Poisson and $W(x) = c(1 - x/n)$, with $n$ a positive integer and $c > 0$, characterizing the binomial. Srivastava and Sreehari (1987) arrive at (4.4) through a different variance bound and characterize the negative binomial by $W(x) = c(1 + x/\gamma)$, $c$, $\gamma > 0$, in addition to characterizing the Poisson and the binomial as above. Here we show that "$W(x) =$ a linear function of $x$" characterizes these three distributions. Also, "$W(x) =$ a quadratic function in $x$" characterizes the generalized hypergeometric distribution of Kemp and Kemp (1956).

The following theorem characterizes the normal and the gamma distributions by linearity of $W$:

THEOREM 4.3. *Let $X$, $g$ and $W$ be as in Theorem* 4.1. *Then* (4.1) *holds with $W(x) = a + bx$ for some constants $a$ and $b$ if and only if $X$ has either a normal distribution or a shifted gamma distribution.*

PROOF. Suppose $X - \theta$ has the gamma distribution $G(\alpha, \beta)$ with density

(4.5) $f(x;\, \theta,\, \alpha,\, \beta) = \{\beta^\alpha/\Gamma(\alpha)\}(x-\theta)^{\alpha-1}\exp\{-(x-\theta)\beta\}, \quad x > \theta \quad \alpha, \beta > 0,$

where $\theta$ is real. Then (4.2) holds with $W(x) = -\theta/\beta + x/\beta$ which is linear in $x$. If $\theta - X$ is $G(\alpha, \beta)$, then (4.2) again holds with $W(x) = \theta/\beta - x/\beta$. If $X$ is $N(\mu, \sigma^2)$, then (4.2) holds with $W(x) = \sigma^2$.

Conversely, suppose (4.1) holds for all absolutely continuous $g$ and $W(x) = a + bx$ for some constants $a$ and $b$. Then since $E(W(X)) = a + b\mu = \mathrm{Var}(X) > 0$, either (i) $b = 0$ and $a > 0$ or (ii) $b \neq 0$, and $\mu + a/b$ and $b$ have the same sign. By Theorem 4.1 and $W(x) = a + bx$ it follows that

(4.6) $$f'(x)/f(x) = (\mu - b - x)/(a + bx).$$

Now in case (i) this leads to $N(\mu, a)$, $a > 0$. In case (ii) (4.6) leads to the result that $X - \theta$ is gamma (4.5) if $b > 0$ and $\theta - X$ is gamma (4.5) if $b < 0$, where $\alpha = (\mu b + a)/b$, $\beta = 1/|b|$ and $\theta = -a/b$. □

We next characterize the Pearson system of continuous distributions by "$W(x)$ = a quadratic in $x$".

THEOREM 4.4. *Let $X$, $g$ and $W$ be as in Theorem* 4.1. *Then* (4.1) *holds with $W(x) = a + bx + cx^2$ for some constants $a$, $b$ and $c$, if and only if the density of $X$ belongs to the Pearson system of continuous distributions.*

PROOF. "Only if" part: Writing $W(x) = b_0 + b_1(x - \mu) + b_2(x - \mu)^2$, we get by Theorem 4.1 that $g'(x)/g(x) = -\{b_1 + (1 + 2b_2)x\}/(b_0 + b_1x + b_2x^2)$ for the density $g(x)$ of $X - \mu$. If $1 + 2b_2 = 0$, then the differential equation leads to the beta density $g(x) = \{1/2B(1+\alpha, 1-\alpha)\}(x+d)^\alpha(d-x)^{-\alpha}$, $-d < x < d$, where $d = \sqrt{b_1^2 + 2b_0} - b_1$ and $\alpha = b_1/(d + b_1)$. (It can be shown that $d > 0$ and $|\alpha| < 1$.) If $1 + 2b_2 \neq 0$, then the above differential equation reduces to $g(x)/g(x) = -(x + c_1)/(c_0 + c_1x + c_2x^2)$, where $c_0 = b_0k$, $c_1 = b_1k$, $c_2 = b_2k$ and $k = (1 + 2b_2)^{-1}$. This is an equation defining the Pearson system.

"If" part: It is known that all the distributions in the Pearson system satisfy the above equation (see, for example, Johnson and Kotz (1970), p. 9). □

We next take up the characterization of discrete distributions.

THEOREM 4.5. *Let $X$, $g$ and $W$ be as in Theorem* 4.2. *Then* (4.3) *holds with $W(x) = a + bx$ for some constants $a$ and $b$ if and only if $X$ has either a Poisson, binomial or a negative binomial distribution.*

PROOF. "Only if" part: To avoid trivialities we assume $X$ is nondegenerate. Then from (4.4) it follows that $a = \mu > 0$, $b + 1 > 0$ and

(4.7) $$f(x) = f(0)a^{[x]}/(b+1)^x x!,$$

where $a^{[x]} = a^{[x]}(b)$ is given by (3.5). Now note that

$$X \text{ is bounded if and only if } b < 0 \tag{4.8}$$

which is easy to see. Now, we, accordingly, distinguish three cases: (i) $-1 < b < 0$, (ii) $b = 0$ and (iii) $b > 0$. Now (4.7) and (4.8) show $X$ to be $B(m, -b)$ (for some positive integer $m$) in case (i), Poisson with mean $a > 0$ in case (ii) and negative binomial with parameters $r = a/b$ and $p = 1/(b+1)$ in case (iii).

"If" part follows from Cacoullos and Papathanasiou (1989) and Srivastava and Sreehari (1987). □

Our final characterization is of the generalized hypergeometric distribution (as defined by Kemp and Kemp (1956)) by "$W(x) =$ a quadratic in $x$".

THEOREM 4.6. *Let $X$, $g$ and $W$ be as in Theorem 4.2. Then (4.3) holds with $W(x) = b_0 + b_1 x + b_2 x(x-1)$ for some constants $b_0$, $b_1$ and $b_2$ satisfying $(b_1 - b_2)^2 - 4b_0 b_2 \geq 0$ if and only if $X$ has a generalized hypergeometric distribution with p.f. of the form*

$$f(x) = \binom{a}{x}\binom{b}{n-x} \Big/ \binom{a+b}{n}, \qquad x = 0, 1, \dots . \tag{4.9}$$

PROOF. "Only if" part: By Theorem 4.2 it follows that $b_0 = \mu > 0$, $b_1 + 1 > 0$ and for $b_2 \neq 0$

$$f(x) = f(0)\alpha^{[x]}\beta^{[x]}/x!\gamma^{[x]}, \tag{4.10}$$

where $a^{[x]} = a^{[x]}(1)$ is given by (3.5), $-\alpha$, $-\beta$ are the (real) roots of $x^2 + \{(b_1 - b_2)/b_2\}x + b_0/b_2 = 0$ and $\gamma = (b_1+1)/b_2$. Now (4.10) can be written in the form (4.9) (see Kemp and Kemp (1956)).

"If" part: Suppose $X$ has the distribution (4.9). Then it is easy to check that (4.4) holds with $\gamma = an/(a+b)$ and $W(x) = (a-x)(n-x)/(a+b)$. □

*Remark* 4.1. Kemp and Kemp (1956) gave eight sets of conditions under which (4.9) represents a discrete distribution. Type II(B) and Type III(B) in their classification scheme have no integral order moments and hence are ineligible as distributions for $X$ which is assumed to have first two moments finite.

## Acknowledgements

The author wishes to thank the Editor and the referees for constructive suggestions which improved the paper.

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