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ON A GENERALIZED EULERIAN DISTRIBUTION

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Abstract. The distribution with probability function $p_k(n, \alpha, \beta) = A_{n,k}(\alpha, \beta)/(\alpha+\beta)^{[n]}$, $k = 0, 1, 2, \ldots, n$, where the parameters α and β are positive real numbers, $A_{n,k}(\alpha, \beta)$ is the generalized Eulerian number and $(\alpha+\beta)^{[n]} = (\alpha+\beta)(\alpha+\beta+1)\cdots(\alpha+\beta+n-1)$, introduced and discussed by Janardan (1988, *Ann. Inst. Statist. Math.*, **40**, 439–450), is further studied. The probability generating function of the generalized Eulerian distribution is expressed by a generalized Eulerian polynomial which, when expanded suitably, provides the factorial moments in closed form in terms of non-central Stirling numbers. Further, it is shown that the generalized Eulerian distribution is unimodal and asymptotically normal.

Key words and phrases: Eulerian numbers, Eulerian polynomials, Stirling numbers, random permutations, unimodality, asymptotic normality.

1. Introduction

Carlitz and Scoville (1974) introduced a generalized (symmetric) Eulerian number $A(r, s \mid \alpha, \beta)$ in connection with the problem of enumerating (α, β)-sequences (generalized permutations). Recurrence relations and other algebraic properties of these numbers were developed. It can be shown that

$$p_k(n, \alpha, \beta) = A_{n,k}(\alpha, \beta)/(\alpha+\beta)^{[n]}, \qquad k = 0, 1, 2, \ldots, n, \tag{1.1}$$

where the parameters α and β are positive real numbers and $A_{n,k}(\alpha, \beta) = A(k, n-k \mid \alpha, \beta)$ is a legitimate probability function. The distribution with probability function (1.1) may be called a generalized Eulerian distribution.

Morisita (1971), after a series of experimental studies with ant lions, suggested a model in which each ant lion was allowed to settle in fine sand (or in coarse sand) with a probability proportional to the environmental density. He then provided a recurrence relation for the probability that k out of n ant lions settled in fine sand. Janardan (1988), in an interesting mathematical and statistical treatment of Morisita's model, proved that an explicit solution of this recurrence can be given in terms of the generalized Eulerian number $E_{n,k}(a, b)$. This number is related to the generalized (symmetric) Eulerian number $A(r, s \mid \alpha, \beta)$ of Carlitz and Scoville (1974) by $E_{n,k}(a, b) = A(n-k, k \mid a, b) = A(k, n-k \mid b, a)$.

In the present paper, which is motivated by Janardan's work, it is shown that the problem of deriving the probability function of the number L_n of ant lions choosing fine sand to settle (in Morisita's model) when the environmental densities are positive integers is equivalent to the problem of deriving the probability function of the number X_{n+1} of rises in a random (α, β)-sequence of $Z_{n+1} = \{1, 2, \ldots, n+1\}$. The notion of an (α, β)-sequence, introduced by Carlitz and Scoville (1974), constitutes a generalization of the notion of a permutation. From the above equivalence the probability function of L_n is deduced (Section 2). The probability generating function of this distribution, when the environmental densities are positive real numbers, is expressed in terms of a generalized Eulerian polynomial. Further, this polynomial, if suitably expanded, provides an explicit expression of the factorial moments in terms of non-central Stirling numbers (Section 3). It is also shown that the n-th generalized Eulerian polynomial has n distinct non-positive real roots. Using this result, one can prove that the generalized Eulerian distribution is unimodal and asymptotically normal (Section 4).

2. Morisita's model, random permutations and (α, β)-sequences

Morisita (1971) considered n ant lions, each of which was allowed to choose to settle either in fine or coarse sand, and postulated that

(2.1) Pr(the first ant lion to choose coarse sand)

$$= 1 - \Pr(\text{the first ant lion to choose fine sand}) = a/(a+b),$$

(2.2) Pr(the n-th ant lion to choose coarse sand given that k

$$\text{ant lions are in find sand}) = (a+k)/(a+b+n-1),$$

(2.3) Pr(the n-th ant lion to choose fine sand given that $n-k-1$

$$\text{ant lions are in coarse sand}) = (b+n-k-1)/(a+b+n-1)$$

where the parameters a and b are positive real numbers.

Consider an arbitrary permutation $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_{n+1})$ of the set $Z_{n+1} = \{1, 2, \ldots, n+1\}$. A pair of consecutive elements (σ_i, σ_{i+1}) in σ is called a rise if $\sigma_i < \sigma_{i+1}$ and a fall if $\sigma_i > \sigma_{i+1}$. If $k(\sigma)$ is the number of rises of the permutation σ, then clearly $0 \leq k(\sigma) \leq n$ and the number of falls of the same permutation is $n - k(\sigma)$. The number of permutations of Z_{n+1} with k rises (and $n-k$ falls) is equal to the Eulerian number $A_{n+1,k+1}$.

Suppose that a permutation is randomly chosen from the set of the $(n+1)!$ permutations of Z_{n+1} and let X_{n+1} be the number of its rises. Then the probability function of the random variable X_{n+1} is given by

$$p_k(n) = \Pr(X_{n+1} = k) = A_{n+1,k+1}/(n+1)!, \qquad k = 0, 1, 2, \ldots, n. \tag{2.4}$$

In order to relate Morisita's model with a random permutation model, consider the following construction of a random permutation of Z_{n+1}. Starting with the number 1, the remaining n numbers, $2, 3, \ldots, n+1$, are placed one after the other in all possible ways. There are two possible ways of placing 2: either to the left or to the right of 1, inducing a fall: (2,1) or a rise: (1,2). Thus,

$$\begin{aligned}(2.5)\qquad &\Pr(\text{placement of 2 induces a fall})\\ &\quad = \Pr(\text{placement of 2 induces a rise}) = 1/2.\end{aligned}$$

Further, there are $n+1$ possible ways of placing $n+1$. If it is placed between the two elements of a rise or to the left of the elements already placed, the number of rises remains unchanged while the number of falls is increased by one. If it is placed between the two elements of a fall or to the right of the elements already placed, the number of rises is increased by one while the number of falls remains unchanged. Thus,

$$\begin{aligned}(2.6)\qquad &\Pr(\text{placement of } n+1 \text{ induces a fall given that } k \text{ rises}\\ &\quad \text{are already induced}) = (k+1)/(n+1),\\ (2.7)\qquad &\Pr(\text{placement of } n+1 \text{ induces a rise given that } k \text{ falls}\\ &\quad \text{are already induced}) = (n-k+1)/(n+1).\end{aligned}$$

It is apparent from the preceding analysis that there is a one-to-one correspondence between the set of different choices of the n ant lions to settle, when $a = b = 1$, and the set of the different choices of the n numbers $2, 3, \ldots, n+1$ to be placed. More specifically, if the j-th ant lion chooses fine (or coarse) sand to settle, then the number $j+1$ is inserted in a place inducing a rise (or a fall) and vice versa. This correspondence implies that in the particular case of Morisita's model with $a = b = 1$, the probability function $\Pr(L_n = k)$, $k = 0, 1, 2, \ldots, n$ of the number L_n of ant lions choosing fine sand to settle is given by (2.4).

The notion of an (α, β)-sequence, introduced by Carlitz and Scoville (1974) and constituting a generalization of the notion of a permutation, can be related to Morisita's model when the parameters a and b are positive integers. An (α, β)-sequence, of Z_{n+1}, in addition to the $n+1$ elements of Z_{n+1}, includes α symbols 0 and β symbols $0'$ subject to the conditions that there is at least one symbol 0 on the extreme left and at least one symbol $0'$ on the extreme right and that the number 1 has all α symbols 0 to its left and all β symbols $0'$ to its right. There is one (α, β)-sequence of Z_1: $(0, \ldots, 0, 1, 0', \ldots, 0')$ and ths (α, β)-sequences of Z_{n+1} can be obtained by inserting the numbers $2, 3, \ldots, n+1$ one after the other in all different ways. Since there are $\alpha + \beta + j - 2$ different ways of inserting the number j for $j = 2, 3, \ldots, n+1$, it follows that the number of (α, β)-sequences of Z_{n+1} is equal to $(\alpha+\beta)^{[n]}$. Consider an arbitrary (α, β)-sequence $s = (s_1, s_2, \ldots, s_{n+\alpha+\beta+1})$ of Z_{n+1}. A rise is defined as a pair of consecutive elements (s_i, s_{i+1}) with $s_i < s_{i+1}$ where s_i may be 0. Similarly a fall is defined as a pair of consecutive elements (s_i, s_{i+1}) with $s_i > s_{i+1}$ where s_{i+1} may be $0'$. If $k(s)+1$ is the number of rises of an (α, β)-sequence s of Z_{n+1}, then $0 \le k(s) \le n$ and the number of falls of the same (α, β)-sequence is $n-k(s)+1$. The number of (α, β)-sequences of Z_{n+1} with $k+1$ rises (and $n-k+1$ falls) is equal to $A(k, n-k \mid \alpha, \beta) = A(n-k, k \mid \beta, \alpha)$, the generalized symmetric Eulerian number studied by Carlitz and Scoville (1974).

Thus, putting $A_{n,k}(\alpha,\beta) = A(k, n-k \mid \alpha, \beta)$ it follows that

$$(2.8)\quad A_{n,k}(\alpha,\beta) = \sum_{j=0}^{k}(-1)^j \binom{\alpha+\beta+n}{j}\binom{\alpha+\beta+k-j-1}{k-j}(\beta+k-j)^n.$$

Note that

$$(2.9)\qquad A_{n,k}(1,1) = A_{n+1,k+1}, \qquad A_{n,k}(1,0) = A_{n,k}.$$

Suppose that an (α,β)-sequence is randomly chosen from the set of the $(\alpha+\beta)^{[n]}$ (α,β)-sequences of Z_{n+1} and let X_{n+1} be the number of its rises. Then, the probability function $p_k(n,\alpha,\beta) = \Pr(X_{n+1} = k+1)$, $k = 0, 1, 2, \dots, n$ is given by (1.1), where the parameters α and β are, in this case, positive integers. Further, the preceding analysis of the construction of an (α,β)-sequence, by virtue of Morisita's postulates (2.1), (2.2) and (2.3), with $\alpha = b$ and $\beta = a$ positive integers, implies $\Pr(L_n = k) = \Pr(X_{n+1} = k+1) = p_k(n,\alpha,\beta)$, $k = 0, 1, 2, \dots, n$.

In the general case of Morisita's model where the parameters a and b are positive real numbers, not necessarily integers, the probability function $\Pr(L_n = k)$, $k = 0, 1, 2, \dots, n$, of the number L_n of ant lions choosing fine sand to settle can be obtained as (1.1) with $\alpha = b$ and $\beta = a$, by comparing the recurrence relation for $\Pr(L_n = k)$ deduced from postulates (2.1), (2.2) and (2.3) with the recurrence relation for the ratio $A_{n,k}(\alpha,\beta)/(\alpha+\beta)^{[n]}$ deduced from the following recurrence relation of the generalized Eulerian numbers $A_{n,k}(\alpha,\beta) = A(k, n-k \mid \alpha,\beta)$ (Carlitz and Scoville (1974))

$$(2.10)\qquad A_{n+1,k}(\alpha,\beta) = (\beta+k)A_{n,k}(\alpha,\beta) + (\alpha+n+1-k)A_{n,k-1}(\alpha,\beta)$$

$$k = 0, 1, 2, \dots, n+1, \qquad n = 0, 1, 2, \dots$$

with $A_{0,0}(\alpha,\beta) = 1$, $A_{0,k}(\alpha,\beta) = 0$, $k \neq 0$ (see also Janardan (1988) where $A_{n,k}(\alpha,\beta) = E_{n,k}(\beta,\alpha)$).

3. Generating functions, factorial moments and generalized Eulerian polynomials

The probability generating function of the generalized Eulerian distribution (1.1), on using the generalized Eulerian polynomials

$$(3.1)\qquad A_n(t;\alpha,\beta) = \sum_{k=0}^{n} A_{n,k}(\alpha,\beta)t^k, \qquad n = 0, 1, 2, \dots,$$

may be obtained as

$$(3.2)\qquad G_n(t;\alpha,\beta) = \sum_{k=0}^{n} p_k(n,\alpha,\beta)t^k = A_n(t;\alpha,\beta)/(\alpha+\beta)^{[n]}.$$

The factorial moment generating function is then given by

$$F_n(t;\ \alpha,\ \beta) = \sum_{r=0}^{\infty} \mu_{(r)}(n,\ \alpha,\ \beta)t^r/r! = A_n(t+1;\ \alpha,\ \beta)/(\alpha+\beta)^{[n]} \tag{3.3}$$

where

$$\mu_{(r)}(n,\ \alpha,\ \beta) = E[L_n^{(r)}], \qquad r = 1,\ 2, \ldots\ , \qquad \mu_{(0)}(n,\ \alpha,\ \beta) = 1,$$

with $L_n^{(r)} = L_n(L_n - 1)(L_n - 2)\cdots(L_n - r + 1)$.

The derivation of the factorial moments by expanding the right-hand side of (3.3) is facilitated by the following brief study of the generalized Eulerian polynomials.

Introducing in (3.1) the expression (2.8) of the generalized Eulerian numbers $A_{n,k}(\alpha,\ \beta)$ and since $A_{n,k}(\alpha,\ \beta) = 0$ for $k > n$, it follows that

$$\begin{aligned} A_n(t;\ \alpha,\ \beta) & \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{k}(-1)^j \binom{\alpha+\beta+n}{j}\binom{\alpha+\beta+k-j-1}{k-j}(\beta+k-j)^n t^k \\ &= \sum_{j=0}^{\infty}(-1)^j \binom{\alpha+\beta+n}{j} t^j \sum_{k=j}^{\infty}\binom{\alpha+\beta+k-j-1}{k-j}(\beta+k-j)^n t^{k-j} \\ &= \sum_{j=0}^{\infty}(-1)^j \binom{\alpha+\beta+n}{j} t^j \sum_{r=0}^{\infty}\binom{\alpha+\beta+r-1}{r}(\beta+r)^n t^r. \end{aligned}$$

Thus,

$$A_n(t;\ \alpha,\ \beta) = (1-t)^{\alpha+\beta+n}\sum_{r=0}^{\infty}\binom{\alpha+\beta+r-1}{r}(\beta+r)^n t^r. \tag{3.4}$$

The generating function of the generalized Eulerian polynomials

$$A(t,\ u;\ \alpha,\ \beta) = \sum_{n=0}^{\infty} A_n(t;\ \alpha,\ \beta)u^n/n!,$$

on using the expression (3.4) may be obtained as

$$\begin{aligned} A(t,\ u;\ \alpha,\ \beta) &= \sum_{n=0}^{\infty}(1-t)^{\alpha+\beta+n}\sum_{r=0}^{\infty}\binom{\alpha+\beta+r-1}{r}(\beta+r)^n t^r u^n/n! \\ &= (1-t)^{\alpha+\beta}\sum_{r=0}^{\infty}\binom{\alpha+\beta+r-1}{r} t^r \sum_{n=0}^{\infty}[(\beta+r)u(1-t)]^n/n! \\ &= e^{\beta u(1-t)}(1-t)^{\alpha+\beta}\sum_{r=0}^{\infty}\binom{\alpha+\beta+r-1}{r}[te^{u(1-t)}]^r \\ &= e^{\beta u(1-t)}(1-t)^{\alpha+\beta}[1-te^{u(1-t)}]^{-\alpha-\beta}. \end{aligned}$$

Note, in passing, that

$$\lim_{t\to 1} A(t,\ u;\ \alpha,\ \beta) = (1-u)^{-\alpha-\beta},$$

implying

$$\sum_{k=0}^{n} A_{n,k}(\alpha,\ \beta) = (\alpha+\beta)^{[n]}, \tag{3.5}$$

in agreement with the result in Theorem 3.2 of Janardan (1988).

The generating function of the generalized Eulerian polynomials may be rewritten in the form

$$A(t,\ u;\ \alpha,\ \beta) = e^{\alpha u(t-1)}\{1 - [e^{u(t-1)} - 1]/(t-1)\}^{-\alpha-\beta}$$

which, on using the non-central Stirling numbers (Koutras (1982))

$$S(n,\ r \mid \alpha) = \frac{1}{r!}\sum_{k=0}^{r}(-1)^k \binom{r}{k}(\alpha+k)^n, \qquad \begin{array}{l} r = 0,\ 1,\ 2, \ldots,\ n, \\ n = 0,\ 1,\ 2, \ldots \end{array} \tag{3.6}$$

with the generating function

$$\sum_{n=r}^{\infty} S(n,\ r \mid \alpha)u^n/n! = e^{\alpha u}(e^u - 1)^r/r!, \qquad r = 0,\ 1,\ 2, \ldots\ ,$$

can be expanded as

$$\begin{aligned} A(t,\ u;\ \alpha,\ \beta) &= \sum_{r=0}^{\infty}\binom{\alpha+\beta+r-1}{r} e^{\alpha u(t-1)}[e^{u(t-1)} - 1]^r (t-1)^{-r} \\ &= \sum_{r=0}^{\infty}\binom{\alpha+\beta+r-1}{r}\sum_{n=r}^{\infty} r!S(n,\ r \mid \alpha)(t-1)^{n-r}u^n/n! \\ &= \sum_{n=0}^{\infty}\left\{\sum_{r=0}^{\infty}(\alpha+\beta)^{[r]}S(n,\ r \mid \alpha)(t-1)^{n-r}\right\}u^n/n!, \end{aligned}$$

yielding for the generalized Eulerian polynomials the expression

$$A_n(t;\ \alpha,\ \beta) = \sum_{r=0}^{n}(\alpha+\beta)^{[r]}S(n,\ r \mid \alpha)(t-1)^{n-r}. \tag{3.7}$$

Returning to the generalized Eulerian distribution, its factorial moment generating function (3.3), by virtue of (3.7), may be expanded in powers of t as

$$\begin{aligned} F_n(t;\ \alpha,\ \beta) &= \sum_{r=0}^{n}\{(\alpha+\beta)^{[n-r]}/(\alpha+\beta)^{[n]}\}S(n,\ n-r \mid \alpha)t^r \\ &= \sum_{r=0}^{n}\left\{S(n,\ n-r \mid \alpha)\Big/\binom{\alpha+\beta+n-1}{r}\right\}t^r/r!. \end{aligned}$$

Thus,

$$(3.8)\quad \mu_{(r)}(n,\,\alpha,\,\beta) = S(n,\,n-r \mid \alpha) \Big/ \binom{\alpha+\beta+n-1}{r}, \qquad r = 0,\,1,\,2,\ldots,\,n$$

and $\mu_{(r)}(n,\,\alpha,\,\beta) = 0$ for $r = n+1,\,n+2,\ldots$.

The computation of the factorial moments (3.8) is facilitated by the following expression of the non-central Stirling numbers as polynomials of the non-centrality parameter:

$$(3.9)\qquad S(n,\,n-r \mid \alpha) = \sum_{k=0}^{r} \binom{n}{k} S(n-k,\,n-r)\alpha^k$$

where $S(n,\,r) = S(n,\,r \mid 0)$ is the usual Stirling numbers.

From (3.8) with $r=1$ and $r=2$, on using (3.9) and

$$S(n-1,\,n-1) = S(n-2,\,n-2) = 1, \qquad S(n,\,n-1) = \binom{n}{2},$$
$$S(n,\,n-2) = 3\binom{n}{4} + \binom{n}{3}$$

it follows that

$$(3.10)\quad \mu(n,\,\alpha,\,\beta) \equiv \mu_{(1)}(n,\,\alpha,\,\beta) = \left[\binom{n}{2} + n\alpha\right] \Big/ (\alpha+\beta+n-1),$$

$$\mu_{(2)}(n,\,\alpha,\,\beta) = \frac{2\left[3\binom{n}{4} + \binom{n}{3}(3\alpha+1) + \binom{n}{2}\alpha^2\right]}{(\alpha+\beta+n-1)(\alpha+\beta+n-2)}$$

and

$$(3.11)\quad \sigma^2(n,\,\alpha,\,\beta) = \frac{2\binom{n}{4} + \binom{n}{3}(2\alpha+2\beta+1) + \binom{n}{2}(\alpha+\beta)^2 + n\alpha\beta(\alpha+\beta-1)}{(\alpha+\beta+n-1)^2(\alpha+\beta+n-2)}$$

in agreement with the expressions obtained by Janardan (1988).

Before concluding this section it is worth noting that the expression (3.4) has the following direct probabilistic application.

The n-th (power) moment about an arbitrary point β of a random variable X obeying a negative binomial, or binomial distribution, can be expressed in terms of the generalized Eulerian polynomials.

4. The asymptotic behavior of the distribution

The probability generating functions $G_n(t;\,\alpha,\,\beta)$ of the generalized Eulerian distribution (1.1) which is given by (3.2) in terms of the generalized Eulerian polynomials $A_n(t;\,\alpha,\,\beta)$ can be written as

$$(4.1)\qquad G_n(t;\,\alpha,\,\beta) = \prod_{j=1}^{n}(q_j + p_j t), \quad q_j = 1 - p_j, \quad j = 1,\,2,\ldots,\,n,$$

$0 < p_1 < p_2 < \cdots < p_n = 1$. This representation is shown by proving that the generalized Eulerian polynomial $A_n(t;\ \alpha,\ \beta)$ has n distinct real non-positive roots for all $n = 1, 2, \ldots$. This proof may be carried out by induction as follows:

The generalized Eulerian polynomials $A_n(t;\ \alpha,\ \beta)$ satisfy the difference-differential equation

$$A_{n+1}(t;\ \alpha,\ \beta) = t(1-t)\frac{d}{dt}A_n(t;\ \alpha,\ \beta) + t(\alpha+\beta+n)A_n(t;\ \alpha,\ \beta), \qquad n = 0, 1, 2, \ldots \tag{4.2}$$

with $A_0(t;\ \alpha,\ \beta) = 1$, which may be deduced from (3.1) on using the triangular recurrence relation (2.10) of the generalized Eulerian numbers $A_{n,k}(\alpha,\ \beta)$. From (4.2) it follows that

$$A_1(t;\ \alpha,\ \beta) = (\alpha+\beta)t, \qquad A_2(t;\ \alpha,\ \beta) = (\alpha+\beta)t[(\alpha+\beta)t+1]$$

that is the statement holds for $n = 1, 2$. Now suppose that $A_n(t;\ \alpha,\ \beta)$ has n distinct real non-positive roots and consider the function

$$B_n(t;\ \alpha,\ \beta) = (1-t)^{-(\alpha+\beta+n)}A_n(t;\ \alpha,\ \beta).$$

Then $B_n(t;\ \alpha,\ \beta)$ has exactly the same finite zeroes as those of $A_n(t;\ \alpha,\ \beta)$ and the identity (4.2) becomes

$$B_{n+1}(t;\ \alpha,\ \beta) = t\frac{d}{dt}B_n(t;\ \alpha,\ \beta), \qquad n = 0, 1, 2, \ldots .$$

$B_n(t;\ \alpha,\ \beta)$ also has a zero at $t = -\infty$, and by Rolle's theorem, between any two zeroes of $B_n(t;\ \alpha,\ \beta)$, the derivative $dB_n(t;\ \alpha,\ \beta)/dt$ has a zero. This implies that $B_{n+1}(t;\ \alpha,\ \beta)$ has n distinct zeroes on the negative axis; obviously $t = 0$ is another zero, making $n+1$ altogether. Since $A_{n+1}(t;\ \alpha,\ \beta)$ is of degree $n+1$ by induction, we have found all roots and the statement is proved.

As a consequence of this property of the generalized Eulerian polynomials, the generalized Eulerian number $A_{n,k}(\alpha,\ \beta)$ is a strictly logarithmic concave function of k, that is

$$[A_{n,k}(\alpha,\ \beta)]^2 > A_{n,k+1}(\alpha,\ \beta)A_{n,k-1}(\alpha,\ \beta). \tag{4.3}$$

Further, it follows that the generalized Eulerian distribution (1.1) is *unimodal* either with a peak or with a plateau of two points (see, for example, Comtet (1974), p. 270).

Let us, now, consider the sequence of independent zero-one random variables $X_{n,j}$, $j = 1, 2, \ldots, n$, with $P(X_{n,j} = 0) = q_j$, $P(X_{n,j} = 1) = p_j$, $j = 1, 2, \ldots, n$. Then the probability generating function of the sum $S_n = \sum_{j=1}^{n} X_{n,j}$ is given by (4.1) and hence

$$P(S_n = k) = A_{n,k}(\alpha,\ \beta)/(\alpha+\beta)^{[n]}, \qquad k = 0, 1, 2, \ldots, n.$$

From (3.11) it follows that $\mathrm{Var}(S_n) \to \infty$ as $n \to \infty$. Therefore, letting

$$Y_{n,j} = [\mathrm{Var}(S_n)]^{-1/2}[X_{n,j} - E(X_{n,j})], \qquad j = 1, 2, \ldots, n$$

and $F_{n,j}(y) = \Pr(Y_{n,j} \leq y)$, it follows that for a given $\epsilon > 0$ there exists n_0 such that $|Y_{n,k}| < \epsilon$ for all $n > n_0$, implying that the Lindeberg condition

$$\lim_{n\to\infty} \sum_{j=1}^{n} \int_{|y|>\epsilon} y^2 dF_{n,j}(y) = 0,$$

of the bounded variance normal convergence criterion is fulfilled. Hence

$$\lim_{n\to\infty} P\left[\frac{S_n - E(S_n)}{\sqrt{\mathrm{Var}(S_n)}} \leq x\right] = \Phi(x) \tag{4.4}$$

with Φ being the distribution function of the standard normal distribution.

Note that (4.4) still holds when $E(S_n)$ and $\mathrm{Var}(S_n)$ are replaced by approximate values (as $n \to \infty$).

The r-th factorial moment (3.8) may be approximated as follows:

Introducing the expression

$$S(n-k,\, n-r) = \sum_{j=0}^{r-k} S_2(r+j-k,\, j) \binom{n-k}{r-k+j}$$

where $S_2(m, j)$ is the associated Stirling number of second kind (see Comtet (1974), p. 226) into (3.9), (3.8) may be written as

$$\mu_{(r)}(n, \alpha, \beta) = \sum_{j=0}^{r}\sum_{k=0}^{r-j} S_2(r+j-k,\, j)\binom{n}{k}\binom{n-k}{r+j-k}\alpha^k \Big/ \binom{\alpha+\beta+n-1}{r}.$$

Since $S_2(2r, r) = (2r)!/(r!2^r)$, $S_2(2r-1, r-1) = (2r-1)!/[3(r-2)!2^{r-1}]$, it follows that

$$\mu_{(r)}(n, \alpha, \beta) = \frac{n^{(2r)}/2^r + [r\alpha + r(r-1)/3]n^{(2r-1)}/2^{r-1}}{(\alpha+\beta+n-1)^{(r)}} + o(n^{-r+2}).$$

A further approximation of the factorial polynomials of n leads to

$$\mu_{(r)}(n, \alpha, \beta) = (n/2)^r + [r\alpha + r(r-1)/3](n/2)^{r-1} + o(n^{-r+2}).$$

Therefore,

$$\mu(n, \alpha, \beta) = n/2 + o(1), \qquad \mu_{(2)}(n, \alpha, \beta) = (n/2)^2 + (\alpha + 1/3)n + o(1).$$

Instead of using these approximate values to find an approximate value of the variance, it is better to derive such a value by approximating its exact value (3.11). In this way, we find

$$\sigma^2(n, \alpha, \beta) = n/12 + o(1).$$

Thus in (4.4) we may use

$$E(S_n) = n/2, \qquad \mathrm{Var}(S_n) = n/12. \tag{4.5}$$

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