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SEARCH DESIGNS FOR SEARCHING FOR ONE AMONG THE TWO- AND THREE-FACTOR INTERACTION EFFECTS IN THE GENERAL SYMMETRIC AND ASYMMETRIC FACTORIALS

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Abstract. This paper describes the construction of search designs which permit the estimation of the general mean and main-effects, and allow the search for and estimation of one possibly unknown non-zero effect among the two- and three-factor interactions in the general symmetric and asymmetric factorial set-up.

Key words and phrases: Main-effect, two- and three-factor interaction.

1. Introduction

The notion of search designs is originally due to Srivastava (1975) who gave the basic mathematical formulation of the problem. Srivastava (1975, 1976*a*) and Ghosh (1975) gave comprehensive construction procedures for the 2^n series. Srivastava (1976*b*) also considered optimality criteria, bias and some interesting applications of search models. A good number of construction procedures for the 2^n series were suggested by Srivastava and Ghosh (1977) and Srivastava and Gupta (1979). Anderson and Thomas (1980) suggested construction procedures for search designs for the p^n series, where p is a prime or a prime power. The problems of constructing search designs were investigated by Gupta (1977, 1981, 1984), Ghosh (1980, 1981), Gupta and Ramirez Carvajal (1981) and Ohnishi and Shirakura (1985). Chatterjee and Mukerjee (1986) obtained some results on search designs which permit the estimation of the general mean and main-effects, and allow the search and estimation of one possibly unknown non-zero effect among the two-factor interactions in the general symmetric and asymmetric factorial set-up.

The present paper deals with the construction of search designs for a general factorial set-up with m factors that allow estimation of the general mean and main-effects, and detect and estimate at most one among two- and three-factor interactions. This problem was earlier considered by Ghosh (1981) and he gave construction procedures for the above problem in the 2^n factorial set-up. Here our primary objective is the construction in the general asymmetric factorial set-up. To motivate, we begin with the symmetric factorial set-up and then extend the same to the general asymmetric factorials. The suggested designs keep the number of runs considerably low.

The linear model considered here is similar to that of Srivastava (1975). Let $Y^{(N\times 1)}$ be the vector of observations with $E(Y) = X\tau$, Disp $(Y) = \sigma^2 I$, where

$$X = [X_1^{(N\times n_1)}, X_2^{(N\times n_2)}] \tag{1.1}$$

is the known design matrix and $\tau = (\tau_1', \tau_2')'$, τ_1 being an $n_1 \times 1$ vector of unknown fixed parameters and τ_2 being an $n_2 \times 1$ vector of fixed parameters, which may be partitioned as:

$$\tau_2 = (\boldsymbol{\beta}_1', \boldsymbol{\beta}_2', \ldots, \boldsymbol{\beta}_\eta')' . \tag{1.2}$$

Here τ_2 is not completely unknown. It is known that at most z among $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \ldots, \boldsymbol{\beta}_\eta$ are non-null where z is small relative to "η". The problem here is to search for the non-null vectors of (1.2) and to draw inference regarding them and τ_1. In the general symmetric and asymmetric factorial set-up, at least one of $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \ldots, \boldsymbol{\beta}_\eta$ involves more than a single component, for in such settings a factorial effect may be represented by more than one independent parameters; on the other hand, Srivastava (1975) considered a set-up where each $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \ldots, \boldsymbol{\beta}_\eta$ involves a single component. In the present paper we take $z = 1$.

Let the partitioning of X_2 corresponding to (1.2) be $X_2 = [X_{21}, X_{22}, \ldots, X_{2\eta}]$ and assume that the error variance is negligible (i.e., $\sigma^2 = 0$). Then the following theorem holds along with that of Srivastava (1975) (see also Chatterjee and Mukerjee (1986)).

THEOREM 1.1. *A necessary and sufficient condition that the above search design problem will be completely solved is that for arbitrary* $g_1, g_2, \ldots, g_{2z}$ $(1 \le g_1 < g_2 < \cdots < g_{2z} \le \eta)$, *the matrix* $[X_1, X_{2g_1}, X_{2g_2}, \ldots, X_{2g_{2z}}]$ *has full column rank.*

2. Notation and preliminaries

Consider a factorial set-up involving m (≥ 3) factors $F_1, F_2, \ldots, F_m$ at $s_1, s_2, \ldots, s_m$ levels, respectively, $(s_j \ge 2,\ 1 \le j \le m)$ where,

$$s_1 \le s_2 \le \cdots \le s_m \, . \tag{2.1}$$

A typical level combination would be $(i_1, i_2, \ldots, i_m)$ $(0 \le i_j \le s_j - 1;\ 1 \le j \le m)$. Throughout this paper, the $v = \Pi s_j$ level combinations will be assumed to be lexicographically ordered. The main-effect of factor F_i carrying $(s_i - 1)$ degrees of freedom will also be denoted by F_i, while F_iF_j and $F_iF_jF_t$ will represent typical two- and three-factor interactions carrying $(s_i - 1)(s_j - 1)$ and $(s_i - 1)(s_j - 1)(s_t - 1)$ degrees of freedom, respectively $(1 \le i,\ j, t \le m,\ i < j < t)$.

Suppose prior information is available regarding the absence of all interactions involving four or more factors (these interactions will be assumed to be absent throughout). The following notation will be helpful in this context. Let for $1 \le j \le m$, $\mathbf{1}_j^{(s_j \times 1)} = (1, 1, \ldots, 1)'$ and P_j be an $s_j \times (s_j - 1)$ matrix of full column rank such that $[s_j^{-1/2}\mathbf{1}_j, P_j]$ is orthogonal. Define for $1 \le j \le m$,

$$P_j(x_j) = \begin{cases} \mathbf{1}_j & \text{if} \quad x_j = 0 \, ; \\ P_j & \text{if} \quad x_j = 1 \, . \end{cases}$$

Let for $1 \le i \le m$, $Z_i = \bigtimes_{u=1}^{m} P_u(\delta_{ui})$, for $1 \le i < j \le m$, $Z_{ij} = \bigtimes_{u=1}^{m} P_u(\delta_{ui} + \delta_{uj})$ and for $1 \le i < j < t \le m$, $Z_{ijt} = \bigtimes_{u=1}^{m} P_u(\delta_{ui} + \delta_{uj} + \delta_{ut})$ and $\mathbf{1} = \bigtimes_{u=1}^{m} \mathbf{1}_u$ where $\times$ denotes the Kronecker product and δ's are Kronecker deltas. It is easy to observe that if one observation is made for each of the v level combinations of $F_1, F_2, \ldots, F_m$, the resulting design matrix will be of the form

$$[\mathbf{1}, Z_1, \ldots, Z_m, Z_{12}, Z_{13}, \ldots, Z_{m-1,m}, Z_{123}, Z_{124}, \ldots, Z_{m-2,m-1,m}] \, , \tag{2.2}$$

where $\mathbf{1}$ corresponds to the general mean effect and the columns of Z_i, Z_{ij}, Z_{ijt} correspond to the main-effect F_i $(1 \le i \le m)$, interaction effects F_iF_j $(1 \le i < j \le m)$ and $F_iF_jF_t$ $(1 \le i < j < t \le m)$, respectively. Of course, (2.2) incorporates Z_{ij} and Z_{ijt} only when the associated interaction effects F_iF_j and $F_iF_jF_t$ are included in the model. Let $H_1 = [\mathbf{1}, Z_1, Z_2, \ldots, Z_m]$; $H_2 = [Z_{12}, Z_{13}, \ldots, Z_{m-1,m}]$ and $H_3 = [Z_{123}, Z_{124}, \ldots, Z_{m-2,m-1,m}]$.

Let S be a subset of N level combinations of $F_1, F_2, \ldots, F_m$ with the corresponding submatrices of H_1, H_2 and H_3 as, say, $T_1 = [\varepsilon, L_1, L_2, \ldots, L_m]$, $T_2 = [G_{12}, G_{13}, \ldots, G_{m-1,m}]$ and $T_3 = [V_{123}, V_{124}, \ldots, V_{m-2,m-1,m}]$, respectively. Then from Theorem 1.1, one observes that S represents a search design of the above problem if and only if the following matrices

i) $[T_1, G_{g_1g_3}, G_{g_2g_4}]$ where $1 \le g_1 < g_3 \le m$, $1 \le g_2 < g_4 \le m$, $g_3 \le g_4$, $(g_1, g_3) \ne (g_2, g_4)$,

ii) $[T_1, G_{g_1g_2}, V_{h_1h_2h_3}]$ where $1 \le g_1 < g_2 \le m$, $1 \le h_1 < h_2 < h_3 \le m$ and

iii) $[T_1, V_{h_1h_3h_5}, V_{h_2h_4h_6}]$ where $1 \le h_1 < h_3 < h_5 \le m$, $1 \le h_2 < h_4 < h_6 \le m$, $h_5 \le h_6$ and $(h_1, h_3, h_5) \ne (h_2, h_4, h_6)$, have full column rank.

3. Main result in the symmetric case

In order to construct search designs for the above set-up, first consider the case where $s_1 = s_2 = \cdots = s_m = s$, say. The following notation will be helpful in describing the search designs considered here.

Define the $m \times 1$ vectors $\boldsymbol{a}_j$ $(0 \le j \le s-1)$ such that in $\boldsymbol{a}_j$, all the elements are j; let A be an $m \times s$ array whose columns are given by $\boldsymbol{a}_j$ for all possible j.

For $0 \le j \le s-1$, define the $m \times 1$ vectors $\boldsymbol{b}_{j,u,i_u}$ $(1 \le u \le m,\ 0 \le i_u \le s-1,\ j \ne i_u)$ such that in $\boldsymbol{b}_{j,u,i_u}$, the u-th element is i_u and the remaining elements all equal j; let B_j be an $m \times m(s-1)$ array whose columns are $\boldsymbol{b}_{j,u,i_u}$ for all possible u and i_u.

Define the $m \times 1$ vectors $\boldsymbol{c}_{j,u_1,u_2}$ $(1 \le j \le s-1,\ 1 \le u_1 < u_2 \le m)$ such that in $\boldsymbol{c}_{j,u_1,u_2}$, the u_1-th and u_2-th elements are zero and the remaining elements all equal j; let C be an $m \times m(m-1)(s-1)/2$ array whose columns are given by $\boldsymbol{c}_{j,u_1,u_2}$ for all possible j, u_1 and u_2.

For $1 \le j \le s-2$, define the $m \times 1$ vectors $\boldsymbol{d}_{j,u_1,u_2,i_{u_1},i_{u_2}}$ $(1 \le u_1 < u_2 \le m,\ 0 \le \min(i_{u_1}, i_{u_2}) < j < \max(i_{u_1}, i_{u_2}) \le s-1)$ such that in $\boldsymbol{d}_{j,u_1,u_2,i_{u_1},i_{u_2}}$, the u_1-th element is i_{u_1}, the u_2-th element is i_{u_2} and the remaining elements equal j; let D_j be an $m \times [m(m-1)j(s-j-1)]$ array whose columns are $\boldsymbol{d}_{j,u_1,u_2,i_{u_1},i_{u_2}}$ for all possible u_1, u_2, i_{u_1} and i_{u_2}.

For $1 \le j \le s-2$, define the $m \times 1$ vectors $\boldsymbol{e}_{j,u_1,u_2,i_{u_1},i_{u_2}}$ $(1 \le u_1 < u_2 \le m,\ j+1 \le i_{u_1}, i_{u_2} \le s-1)$ such that in $\boldsymbol{e}_{j,u_1,u_2,i_{u_1},i_{u_2}}$, the u_1-th and u_2-th elements are i_{u_1} and i_{u_2} and the remaining elements equal j; let E_j be an $m \times [m(m-1)(s-j-1)^2/2]$ array whose columns are $\boldsymbol{e}_{j,u_1,u_2,i_{u_1},i_{u_2}}$ for all possible u_1, u_2, i_{u_1} and i_{u_2}.

Let

$$
\begin{aligned}
n &= \text{total number of runs} \\
&= s + ms(s-1) + m(m-1)(s-1)/2 + m(m-1)s(s-1)(s-2)/6 \\
&\quad + m(m-1)(s-1)(s-2)(2s-3)/12
\end{aligned}
$$

and define the following $m \times n$ array:

$$
Q = [A, B_0, B_1, \ldots, B_{s-1}, C, D_1, D_2, \ldots, D_{s-2}, E_1, E_2, \ldots, E_{s-2}]\,. \tag{3.1}
$$

Example 3.1. Let $m = 6$ and $s = 4$.

$$A:\begin{array}{c}0123\\0123\\0123\\0123\\0123\\0123\end{array};\quad B_0:\begin{array}{c}123000000000000000\\000123000000000000\\000000123000000000\\000000000123000000\\000000000000123000\\000000000000000123\end{array};\quad B_1:\begin{array}{c}023111111111111111\\111023111111111111\\111111023111111111\\111111111023111111\\111111111111023111\\111111111111111023\end{array};$$

$$B_2:\begin{array}{c}013222222222222222\\222013222222222222\\222222013222222222\\222222222013222222\\222222222222013222\\222222222222222013\end{array};\quad B_3:\begin{array}{c}012333333333333333\\333012333333333333\\333333012333333333\\333333333012333333\\333333333333012333\\333333333333333012\end{array};$$

$$C:\begin{array}{c}000001111111111000002222222222000003333333333\\011110000111111022220000222222033330000333333\\101110111000111202220222000222303330333000333\\110111011011001220222022022002330333033033003\\111011101101010222022202202020333033303303030\\111101110110100222202220220200333303330330300\end{array};$$

$$D_1:\begin{array}{c}002300230023002300231111111111111111111111111111111111111111\\230011111111111111110023002300230023111111111111111111111111\\111123001111111111112300111111111111002300230323111111111111\\111111112300111111111111230011111111230011111111002300231111\\111111111111230011111111111123001111111123001111230011110023\\111111111111111123001111111111112300111111112300111123002300\end{array};$$

$$D_2:\begin{array}{c}013301330133013301332222222222222222222222222222222222222222\\330122222222222222220133013301330133222222222222222222222222\\222233012222222222223301222222222222013301330133222222222222\\222222223301222222222222330122222222330122222222013301332222\\222222222222330122222222222233012222222233012222330122220133\\222222222222222233012222222222223301222222223301222233013301\end{array};$$

$$E_1:\begin{array}{c}223322332233223322331111111111111111111111111111111111111111\\232311111111111111112233223322332233111111111111111111111111\\111123231111111111112323111111111111223322332233111111111111\\111111112323111111111111232311111111232311111111223322331111\\111111111111232311111111111123231111111123231111232311112233\\111111111111111123231111111111112323111111112323111123232323\end{array};$$

$$E_2:\begin{array}{c}333332222222222\\322223333222222\\232223222333222\\223222322322332\\222322232232323\\222232223223233\end{array}.$$

THEOREM 3.1. *The columns of the matrix Q, interpreted as the level combinations of $F_1, F_2, \ldots, F_m$, give the required search design in n runs.*

The proof of the theorem follows from the following lemmas.

LEMMA 3.1. *Let $P = [\boldsymbol{p}_0, \boldsymbol{p}_1, \ldots, \boldsymbol{p}_{s-1}]'$ be an $s \times (s-1)$ matrix such that $[s^{-1/2}\mathbf{1}, P]$ is orthogonal. If $(\boldsymbol{p}_r' - \boldsymbol{p}_0')\boldsymbol{\varphi} = 0$, $r = 1, 2, \ldots, s-1$, then $\boldsymbol{\varphi} = \mathbf{0}$.*

PROOF. If $(\boldsymbol{p}_r' - \boldsymbol{p}_0')\boldsymbol{\varphi} = 0$, $r = 1, 2, \ldots, s-1$, then it is easy to see that $BP\boldsymbol{\varphi} = \mathbf{0}$, where the $(s-1) \times s$ matrix B is given by

$$B = [\,-\mathbf{1}_{s-1} \quad I_{s-1}\,]\,.$$

Now by the definition of B and P, $BPP'B' = I_{s-1} + J_{s-1,s-1}$, which is non-singular, where I_{s-1} is an identity matrix of order $(s-1)$ and $J_{s-1,s-1}$ is an $(s-1) \times (s-1)$ matrix with all elements unity. So the square matrix BP is non-singular and hence the result is immediate.

It also follows from *Lemma* 3.1, that the columns of P^* with rows $(\boldsymbol{p}_r' - \boldsymbol{p}_0')$, $r = 1, 2, \ldots, s-1$, are linearly independent.

LEMMA 3.2. *The columns of the matrix Q, interpreted as the level combinations of $F_1, F_2, \ldots, F_m$, is a subset of n level combinations of $F_1, F_2, \ldots, F_m$ for which the matrix $[T_1, G_{g_1g_3}, G_{g_2g_4}]$, where $1 \le g_1 < g_3 \le m$, $1 \le g_2 < g_4 \le m$, $g_3 \le g_4$ and $(g_1, g_3) \ne (g_2, g_4)$, has full column rank.*

LEMMA 3.3. *The columns of the matrix Q, interpreted as the level combinations of $F_1, F_2, \ldots, F_m$, are a subset of n level combinations of $F_1, F_2, \ldots, F_m$ for which the matrix $[T_1, G_{g_1g_2}, V_{h_1h_2h_3}]$, where $1 \le g_1 < g_2 \le m$ and $1 \le h_1 < h_2 < h_3 \le m$, has full column rank.*

PROOF. Under the notation of Section 2, consider the matrix $X(g_1, g_2, h_1, h_2, h_3) = [T_1, G_{g_1g_2}, V_{h_1h_2h_3}] = [\boldsymbol{\varepsilon}, L_1, L_2, \ldots, L_m, G_{g_1g_2}, V_{h_1h_2h_3}]$, where

$$1 \le g_1 < g_2 \le m, \qquad 1 \le h_1 < h_2 < h_3 \le m\,. \tag{3.2}$$

To prove the lemma, consider

$$X(g_1, g_2, h_1, h_2, h_3)\boldsymbol{\theta} = \mathbf{0} \tag{3.3}$$

where $\boldsymbol{\theta} = (\theta_0, \boldsymbol{\theta}_1', \ldots, \boldsymbol{\theta}_m', \boldsymbol{\alpha}_1', \boldsymbol{\alpha}_2')'$, say. Recalling the definitions of $L_1, L_2, \ldots, L_m, G_{g_1g_2}, V_{h_1h_2h_3}$, (3.3) together with (3.1) gives the equations (3.4) to (3.8). For example, (3.4) corresponds to the subarray A of Q.

$$\theta_0 + \sum_{\upsilon=1}^{m} \boldsymbol{p}_{\upsilon j}'\boldsymbol{\theta}_\upsilon + (\boldsymbol{p}_{g_1j}' \times \boldsymbol{p}_{g_2j}')\boldsymbol{\alpha}_1 + (\boldsymbol{p}_{h_1j}' \times \boldsymbol{p}_{h_2j}' \times \boldsymbol{p}_{h_3j}')\boldsymbol{\alpha}_2 = 0 \tag{3.4}$$

where $0 \le j \le s-1$ and $\boldsymbol{p}_{\upsilon j}'$ $(1 \le \upsilon \le m,\ 0 \le j \le s-1)$ denote the j-th row of P_υ.

$$\begin{aligned}(3.5)\qquad &\theta_0 + \sum_{\substack{v=1\\ v\neq u}}^{m} \boldsymbol{p}'_{vj}\boldsymbol{\theta}_v + \boldsymbol{p}'_{ui_u}\boldsymbol{\theta}_u \\ &+ [\delta_{ug_1}(\boldsymbol{p}'_{g_1i_{g_1}} \times \boldsymbol{p}'_{g_2j}) + \delta_{ug_2}(\boldsymbol{p}'_{g_1j} \times \boldsymbol{p}'_{g_2i_{g_2}}) \\ &\quad + (1-\delta_{ug_1}-\delta_{ug_2})(\boldsymbol{p}'_{g_1j}\times\boldsymbol{p}'_{g_2j})]\boldsymbol{\alpha}_1 \\ &+ [\delta_{uh_1}(\boldsymbol{p}'_{h_1i_{h_1}}\times\boldsymbol{p}'_{h_2j}\times\boldsymbol{p}'_{h_3j}) + \delta_{uh_2}(\boldsymbol{p}'_{h_1j}\times\boldsymbol{p}'_{h_2i_{h_2}}\times\boldsymbol{p}'_{h_3j}) \\ &\quad + \delta_{uh_3}(\boldsymbol{p}'_{h_1j}\times\boldsymbol{p}'_{h_2j}\times\boldsymbol{p}'_{h_3i_{h_3}}) \\ &\quad + (1-\delta_{uh_1}-\delta_{uh_2}-\delta_{uh_3})(\boldsymbol{p}'_{h_1j}\times\boldsymbol{p}'_{h_2j}\times\boldsymbol{p}'_{h_3j})]\boldsymbol{\alpha}_2 = 0\,, \\ &\qquad\qquad 0\le j\le s-1,\quad 1\le u\le m,\quad 0\le i_u\le s-1,\quad j\neq i_u\,.\end{aligned}$$

Let

$$\begin{aligned}\varphi(\boldsymbol{\theta}) = \theta_0 &+ \sum_{\substack{v=1\\ v\neq u_1,u_2}}^{m} \boldsymbol{p}'_{vj}\boldsymbol{\theta}_v + \boldsymbol{p}'_{u_1i_{u_1}}\boldsymbol{\theta}_{u_1} + \boldsymbol{p}'_{u_2i_{u_2}}\boldsymbol{\theta}_{u_2} \\ &+ [\delta_{u_1g_1}\{\delta_{u_2g_2}(\boldsymbol{p}'_{g_1i_{g_1}}\times\boldsymbol{p}'_{g_2i_{g_2}}) + (1-\delta_{u_2g_2})(\boldsymbol{p}'_{g_1i_{g_1}}\times\boldsymbol{p}'_{gj})\} \\ &\quad + (1-\delta_{u_1g_1})\{\delta_{u_1g_2}(\boldsymbol{p}'_{g_1j}\times\boldsymbol{p}'_{g_2i_{g_2}}) \\ &\quad + (1-\delta_{u_1g_2})\{\delta_{u_2g_1}(\boldsymbol{p}'_{g_1i_{g_1}}\times\boldsymbol{p}'_{g_2j}) + \delta_{u_2g_2}(\boldsymbol{p}'_{g_1j}\times\boldsymbol{p}'_{g_2i_{g_2}}) \\ &\quad + (1-\delta_{u_2g_1}-\delta_{u_2g_2})(\boldsymbol{p}'_{g_1j}\times\boldsymbol{p}'_{g_2j})\}\}]\boldsymbol{\alpha}_1 \\ &+ [\delta_{u_1h_1}\{\delta_{u_2h_2}(\boldsymbol{p}'_{h_1i_{h_1}}\times\boldsymbol{p}'_{h_2i_{h_2}}\times\boldsymbol{p}'_{h_3j}) \\ &\quad + \delta_{u_2h_3}(\boldsymbol{p}'_{h_1i_{h_1}}\times\boldsymbol{p}'_{h_2j}\times\boldsymbol{p}'_{h_3i_{h_3}}) \\ &\quad + (1-\delta_{u_2h_2}-\delta_{u_2h_3})(\boldsymbol{p}'_{h_1i_{h_1}}\times\boldsymbol{p}'_{h_2j}\times\boldsymbol{p}'_{h_3j})\} \\ &\quad + (1-\delta_{u_1h_1})\{\delta_{u_1h_2}\{\delta_{u_2h_3}(\boldsymbol{p}'_{h_1j}\times\boldsymbol{p}'_{h_2i_{h_2}}\times\boldsymbol{p}'_{h_3i_{h_3}}) \\ &\quad + (1-\delta_{u_2h_3})(\boldsymbol{p}'_{h_1j}\times\boldsymbol{p}'_{h_2i_{h_2}}\times\boldsymbol{p}'_{h_3j})\} \\ &\quad + (1-\delta_{u_1h_2})\{\delta_{u_1h_3}(\boldsymbol{p}'_{h_1j}\times\boldsymbol{p}'_{h_2j}\times\boldsymbol{p}'_{h_3i_{h_3}}) \\ &\quad + (1-\delta_{u_1h_3})\{\delta_{u_2h_1}(\boldsymbol{p}'_{h_1i_{h_1}}\times\boldsymbol{p}'_{h_2j}\times\boldsymbol{p}'_{h_3j}) \\ &\quad + \delta_{u_2h_2}(\boldsymbol{p}'_{h_1j}\times\boldsymbol{p}'_{h_2i_{h_2}}\times\boldsymbol{p}'_{h_3j}) + \delta_{u_2h_3}(\boldsymbol{p}'_{h_1j}\times\boldsymbol{p}'_{h_2j}\times\boldsymbol{p}'_{h_3i_{h_3}}) \\ &\quad + (1-\delta_{u_2h_1}-\delta_{u_2h_2}-\delta_{u_3h_3})(\boldsymbol{p}'_{h_1j}\times\boldsymbol{p}'_{h_2j}\times\boldsymbol{p}'_{h_3j})\}\}\}]\boldsymbol{\alpha}_2\,,\end{aligned}$$

then,

$$\begin{aligned}(3.6)\qquad \varphi(\boldsymbol{\theta}) = 0 \quad \text{for}\quad &1\le u_1<u_2\le m,\quad i_{u_1}=0,\quad i_{u_2}=0\quad\text{and}\\ &1\le j\le s-1\,,\end{aligned}$$

(3.7) $\varphi(\boldsymbol{\theta})=0$ for $1 \le u_1 < u_2 \le m, \quad 1 \le j \le s-2,$

$0 \le \min(i_{u_1}, i_{u_2}) < j < \max(i_{u_1}, i_{u_2}) \le s-1,$

(3.8) $\varphi(\boldsymbol{\theta})=0$ for $1 \le u_1 < u_2 \le m, \quad 1 \le j \le s-2,$

$j+1 \le i_{u_1}, i_{u_2} \le s-1.$

Now in view of (3.2), a number of cases may arise. In the above setting, first suppose $g_1 = h_1 < g_2 = h_2 < h_3$. We have to prove that (3.3) has the only solution $\boldsymbol{\theta} = \mathbf{0}$.

For any fixed t $(1 \le t \le m,\ t \ne h_1, h_2, h_3)$, subtracting (3.4) with $j = 0$, from (3.5) with $u = t$ and $j = 0$, one gets

$$(3.9) \qquad (\boldsymbol{p}'_{ti_t} - \boldsymbol{p}'_{t0})\boldsymbol{\theta}_t = 0, \qquad 1 \le i_t \le s-1.$$

From Lemma 3.1 and (3.9), it follows that

$$(3.10) \qquad \boldsymbol{\theta}_t = \mathbf{0}, \qquad 1 \le t \le m, \quad t \ne h_1, h_2, h_3.$$

For any fixed r $(1 \le r \le s-1)$, subtracting (3.4) with $j = 0$ from (3.5) with $u = h_3$, $i_{h_3} = r$ and $j = 0$, one gets

$$(3.11) \qquad (\boldsymbol{p}'_{h_3r} - \boldsymbol{p}'_{h_30})\boldsymbol{\theta}_{h_3} + [\boldsymbol{p}'_{h_10} \times \boldsymbol{p}'_{h_20} \times (\boldsymbol{p}'_{h_3r} - \boldsymbol{p}'_{h_30})]\boldsymbol{\alpha}_2 = 0.$$

For any fixed r $(1 \le r \le s-1)$, subtracting (3.6) with $u_1 = h_1$, $u_2 = h_3$ and $j = r$ from (3.5) with $j = r$, $u = h_1$ and $i_{h_1} = 0$, one gets

$$(3.12) \qquad (\boldsymbol{p}'_{h_3r} - \boldsymbol{p}'_{h_30})\boldsymbol{\theta}_{h_3} + [\boldsymbol{p}'_{h_10} \times \boldsymbol{p}'_{h_2r} \times (\boldsymbol{p}'_{h_3r} - \boldsymbol{p}'_{h_30})]\boldsymbol{\alpha}_2 = 0.$$

For any fixed r $(1 \le r \le s-1)$, subtracting (3.6) with $u_1 = h_2$, $u_2 = h_3$ and $j = r$ from (3.5) with $j = r$, $u = h_2$ and $i_{h_2} = 0$, one gets

$$(3.13) \qquad (\boldsymbol{p}'_{h_3r} - \boldsymbol{p}'_{h_30})\boldsymbol{\theta}_{h_3} + [\boldsymbol{p}'_{h_1r} \times \boldsymbol{p}'_{h_20} \times (\boldsymbol{p}'_{h_3r} - \boldsymbol{p}'_{h_30})]\boldsymbol{\alpha}_2 = 0.$$

For any fixed r $(1 \le r \le s-1)$, subtracting (3.6) with $u_1 = h_1$, $u_2 = h_3$ and $j = q$ from (3.7) with $u_1 = h_1$, $i_{h_1} = 0$, $u_2 = h_3$, $i_{h_3} = r$ and $j = q$, one gets

$$(3.14) \qquad (\boldsymbol{p}'_{h_3r} - \boldsymbol{p}'_{h_30})\boldsymbol{\theta}_{h_3} + [\boldsymbol{p}'_{h_10} \times \boldsymbol{p}'_{h_2q} \times (\boldsymbol{p}'_{h_3r} - \boldsymbol{p}'_{h_30})]\boldsymbol{\alpha}_2 = 0$$

where $1 \le q < r \le s-1$.

For any fixed r $(1 \le r \le s-1)$, subtracting (3.6) with $u_1 = h_2$, $u_2 = h_3$ and $j = k$ from (3.7) with $u_1 = h_2$, $i_{h_2} = 0$, $u_2 = h_3$, $i_{h_3} = r$ and $j = k$, one gets

$$(3.15) \qquad (\boldsymbol{p}'_{h_3r} - \boldsymbol{p}'_{h_30})\boldsymbol{\theta}_{h_3} + [\boldsymbol{p}'_{h_1k} \times \boldsymbol{p}'_{h_20} \times (\boldsymbol{p}'_{h_3r} - \boldsymbol{p}'_{h_30})]\boldsymbol{\alpha}_2 = 0$$

where $1 \le k < r \le s-1$.

For any fixed r $(1 \le r \le s-1)$, subtracting (3.5) with $u = h_2$, $i_{h_2} = q$ and $j = 0$ from (3.7) with $u_1 = h_1$, $i_{h_1} = 0$, $u_2 = h_2$, $i_{h_2} = q$ and $j = r$, one gets

$$(3.16) \qquad (\boldsymbol{p}'_{h_3r} - \boldsymbol{p}'_{h_30})\boldsymbol{\theta}_{h_3} + [\boldsymbol{p}'_{h_10} \times \boldsymbol{p}'_{h_2q} \times (\boldsymbol{p}'_{h_3r} - \boldsymbol{p}'_{h_30})]\boldsymbol{\alpha}_2 = 0$$

where $1 \le r < q \le s-1$.

For any fixed r $(1 \le r \le s-1)$, subtracting (3.5) with $u = h_1$, $i_{h_1} = k$ and $j = 0$ from (3.7) with $u_1 = h_1$, $i_{h_1} = k$, $u_2 = h_2$, $i_{h_2} = 0$ and $j = r$, one gets

$$(3.17) \qquad (\boldsymbol{p}'_{h_3r} - \boldsymbol{p}'_{h_30})\boldsymbol{\theta}_{h_3} + [\boldsymbol{p}'_{h_1k} \times \boldsymbol{p}'_{h_20} \times (\boldsymbol{p}'_{h_3r} - \boldsymbol{p}'_{h_30})]\boldsymbol{\alpha}_2 = 0$$

where $1 \le r < k \le s-1$.

For any fixed r $(1 \le r \le s-1)$, subtracting (3.5) with $u = h_3$, $i_{h_3} = 0$ and $j = r$ from (3.4) with $j = r$, one gets

$$(3.18) \qquad (\boldsymbol{p}'_{h_3r} - \boldsymbol{p}'_{h_30})\boldsymbol{\theta}_{h_3} + [\boldsymbol{p}'_{h_1r} \times \boldsymbol{p}'_{h_2r} \times (\boldsymbol{p}'_{h_3r} - \boldsymbol{p}'_{h_30})]\boldsymbol{\alpha}_2 = 0\,.$$

For any fixed q, r $(1 \le q, r \le s-1,\ q \ne r)$, subtracting (3.5) with $u = h_3$, $i_{h_3} = 0$ and $j = q$ from (3.5) with $u = h_3$, $i_{h_3} = r$ and $j = q$, one gets

$$(3.19) \qquad (\boldsymbol{p}'_{h_3r} - \boldsymbol{p}'_{h_30})\boldsymbol{\theta}_{h_3} + [\boldsymbol{p}'_{h_1q} \times \boldsymbol{p}'_{h_2q} \times (\boldsymbol{p}'_{h_3r} - \boldsymbol{p}'_{h_30})]\boldsymbol{\alpha}_2 = 0\,.$$

For any fixed k, r $(1 \le r < k \le s-1)$, subtracting (3.7) with $j = r$, $u_1 = h_1$, $i_{h_1} = k$, $u_2 = h_3$ and $i_{h_3} = 0$ from (3.5) with $j = r$, $u = h_1$ and $i_{h_1} = k$, one gets

$$(3.20) \qquad (\boldsymbol{p}'_{h_3r} - \boldsymbol{p}'_{h_30})\boldsymbol{\theta}_{h_3} + [\boldsymbol{p}'_{h_1k} \times \boldsymbol{p}'_{h_2r} \times (\boldsymbol{p}'_{h_3r} - \boldsymbol{p}'_{h_30})]\boldsymbol{\alpha}_2 = 0\,.$$

For any fixed q, r $(1 \le r < q \le s-1)$, subtracting (3.7) with $j = r$, $u_1 = h_2$, $i_{h_2} = q$, $u_2 = h_3$ and $i_{h_3} = 0$ from (3.5) with $j = r$, $u = h_2$ and $i_{h_2} = q$, one gets

$$(3.21) \qquad (\boldsymbol{p}'_{h_3r} - \boldsymbol{p}'_{h_30})\boldsymbol{\theta}_{h_3} + [\boldsymbol{p}'_{h_1r} \times \boldsymbol{p}'_{h_2q} \times (\boldsymbol{p}'_{h_3r} - \boldsymbol{p}'_{h_30})]\boldsymbol{\alpha}_2 = 0\,.$$

For any fixed k, q, r $(1 \le r < q < k \le s-1)$, subtracting (3.7) with $j = q$, $u_1 = h_1$, $i_{h_1} = k$, $u_2 = h_3$ and $i_{h_3} = 0$ from (3.7) with $j = q$, $u_1 = h_1$, $i_{h_1} = k$, $u_2 = h_3$ and $i_{h_3} = r$, one gets

$$(3.22) \qquad (\boldsymbol{p}'_{h_3r} - \boldsymbol{p}'_{h_30})\boldsymbol{\theta}_{h_3} + [\boldsymbol{p}'_{h_1k} \times \boldsymbol{p}'_{h_2q} \times (\boldsymbol{p}'_{h_3r} - \boldsymbol{p}'_{h_30})]\boldsymbol{\alpha}_2 = 0\,.$$

For any fixed k, q, r $(1 \le r < k < q \le s-1)$, subtracting (3.7) with $j = k$, $u_1 = h_2$, $i_{h_2} = q$, $u_2 = h_3$ and $i_{h_3} = 0$ from (3.7) with $j = k$, $u_1 = h_2$, $i_{h_2} = q$, $u_2 = h_3$ and $i_{h_3} = r$, one gets

$$(3.23)\qquad (\boldsymbol{p}'_{h_3r} - \boldsymbol{p}'_{h_30})\boldsymbol{\theta}_{h_3} + [\boldsymbol{p}'_{h_1k} \times \boldsymbol{p}'_{h_2q} \times (\boldsymbol{p}'_{h_3r} - \boldsymbol{p}'_{h_30})]\boldsymbol{\alpha}_2 = 0\ .$$

For any fixed k, q, r $(1 \le q \le s-2,\ q < k, r \le s-1)$, subtracting (3.7) with $j = q$, $u_1 = h_1$, $i_{h_1} = k$, $u_2 = h_3$ and $i_{h_3} = 0$ from (3.8) with $j = q$, $u_1 = h_1$, $i_{h_1} = k$, $u_2 = h_3$ and $i_{h_3} = r$, one gets

$$(3.24)\qquad (\boldsymbol{p}'_{h_3r} - \boldsymbol{p}'_{h_30})\boldsymbol{\theta}_{h_3} + [\boldsymbol{p}'_{h_1k} \times \boldsymbol{p}'_{h_2q} \times (\boldsymbol{p}'_{h_3r} - \boldsymbol{p}'_{h_30})]\boldsymbol{\alpha}_2 = 0\ .$$

For any fixed k, q, r $(1 \le k \le s-2,\ k < q, r \le s-1)$, subtracting (3.7) with $j = k$, $u_1 = h_2$, $i_{h_2} = q$, $u_2 = h_3$ and $i_{h_3} = 0$ from (3.8) with $j = k$, $u_1 = h_2$, $i_{h_2} = q$, $u_2 = h_3$ and $i_{h_3} = r$, one gets

$$(3.25)\qquad (\boldsymbol{p}'_{h_3r} - \boldsymbol{p}'_{h_30})\boldsymbol{\theta}_{h_3} + [\boldsymbol{p}'_{h_1k} \times \boldsymbol{p}'_{h_2q} \times (\boldsymbol{p}'_{h_3r} - \boldsymbol{p}'_{h_30})]\boldsymbol{\alpha}_2 = 0\ .$$

Now combining the quations (3.11) to (3.25), one gets for any fixed k, q, r $(1 \le r \le s-1,\ 0 \le k, q \le s-1)$,

$$(3.26)\qquad (\boldsymbol{p}'_{h_3r} - \boldsymbol{p}'_{h_30})\boldsymbol{\theta}_{h_3} + [\boldsymbol{p}'_{h_1k} \times \boldsymbol{p}'_{h_2q} \times (\boldsymbol{p}'_{h_3r} - \boldsymbol{p}'_{h_30})]\boldsymbol{\alpha}_2 = 0\ .$$

Summing the s equations in (3.26), for fixed q, r $(1 \le r \le s-1,\ 0 \le q \le s-1)$, one gets $(\boldsymbol{p}'_{h_3r} - \boldsymbol{p}'_{h_30})\boldsymbol{\theta}_{h_3} = 0$. From Lemma 3.1, it follows that

$$(3.27)\qquad \boldsymbol{\theta}_{h_3} = \mathbf{0}\ .$$

Using (3.27), one gets from (3.26), for any fixed k, q, r $(1 \le r \le s-1,\ 0 \le k, q \le s-1)$,

$$(3.28)\qquad [\boldsymbol{p}'_{h_1k} \times \boldsymbol{p}'_{h_2q} \times (\boldsymbol{p}'_{h_3r} - \boldsymbol{p}'_{h_30})]\boldsymbol{\alpha}_2 = 0\ .$$

Equation (3.28) implies that $[P_{h_1} \times P_{h_2} \times P^*_{h_3}]\boldsymbol{\alpha}_2 = \mathbf{0}$ where $P^*_{h_3}$ is an $(s-1) \times (s-1)$ matrix with rows $(\boldsymbol{p}'_{h_3r} - \boldsymbol{p}'_{h_30})$, $1 \le r \le s-1$. Since P_{h_1}, P_{h_2} and $P^*_{h_3}$ are of full column rank, one obtains $\boldsymbol{\alpha}_2 = \mathbf{0}$. Proceeding in a similar manner, one can obtain $\boldsymbol{\theta}_{h_1} = \mathbf{0}$, $\boldsymbol{\theta}_{h_2} = \mathbf{0}$, $\boldsymbol{\alpha}_1 = \mathbf{0}$ and $\theta_0 = 0$. Hence $\boldsymbol{\theta} = \mathbf{0}$.

The proof of other cases follows in a similar way through considerable algebra. The details of such cases are omitted here to save space.

LEMMA 3.4. *The columns of the matrix Q, interpreted as the level combinations of $F_1, F_2, \ldots, F_m$, are a subset of n level combinations of $F_1, F_2, \ldots, F_m$ for which the matrix $[T_1, V_{h_1h_3h_5}, V_{h_2h_4h_6}]$, where $1 \le h_1 < h_3 < h_5 \le m$, $1 \le h_2 < h_4 < h_6 \le m$, $h_5 \le h_6$ and $(h_1, h_3, h_5) \ne (h_2, h_4, h_6)$, has full column rank.*

PROOF. Under the notation of Section 2, consider the matrix

$X(h_1, h_2, h_3, h_4, h_5, h_6) = [T_1, V_{h_1h_3h_5}, V_{h_2h_4h_6}] = [\boldsymbol{\varepsilon}, L_1, L_2, \ldots, L_m, V_{h_1h_3h_5}, V_{h_2h_4h_6}]$ where

$$(3.29)\qquad \begin{aligned} &1 \le h_1 < h_3 < h_5 \le m, \quad 1 \le h_2 < h_4 < h_6 \le m\,, \\ &h_5 \le h_6 \quad \text{and} \quad (h_1, h_3, h_5) \ne (h_2, h_4, h_6)\,. \end{aligned}$$

To prove the lemma, consider

$$(3.30)\qquad X(h_1, h_2, h_3, h_4, h_5, h_6)\,\boldsymbol{\theta} = \mathbf{0}$$

where $\boldsymbol{\theta} = (\theta_0, \boldsymbol{\theta}_1', \boldsymbol{\theta}_2', \ldots, \boldsymbol{\theta}_m', \boldsymbol{\alpha}_1', \boldsymbol{\alpha}_2')'$. Recalling the definitions of $L_1, L_2, \ldots, L_m$, $V_{h_1h_3h_5}$ and $V_{h_2h_4h_6}$, one gets from (3.30) together with (3.1) the equations (3.31) to (3.35). For example, (3.31) corresponds to the subarray A of Q.

$$(3.31)\qquad \theta_0 + \sum_{v=1}^{m} \boldsymbol{p}_{vj}'\boldsymbol{\theta}_v + (\boldsymbol{p}_{h_1j}' \times \boldsymbol{p}_{h_3j}' \times \boldsymbol{p}_{h_5j}')\,\boldsymbol{\alpha}_1 + (\boldsymbol{p}_{h_2j}' \times \boldsymbol{p}_{h_4j}' \times \boldsymbol{p}_{h_6j}')\,\boldsymbol{\alpha}_2 = 0$$

where $0 \le j \le s-1$ and p_{vj}' are the j-th row of P_v $(1 \le v \le m,\ 1 \le j \le s-1)$.

$$\begin{aligned} (3.32)\qquad \theta_0 &+ \sum_{\substack{v=1\\ v\ne u}}^{m} \boldsymbol{p}_{vj}'\boldsymbol{\theta}_v + \boldsymbol{p}_{ui_u}'\boldsymbol{\theta}_u \\ &+ \sum_{w=1}^{2} [\delta_{uh_w}(\boldsymbol{p}_{h_wi_{h_w}}' \times \boldsymbol{p}_{h_{w+2}j}' \times \boldsymbol{p}_{h_{w+4}j}') + \delta_{uh_{w+2}}(\boldsymbol{p}_{h_wj}' \times \boldsymbol{p}_{h_{w+2}i_{h_{w+2}}}' \times \boldsymbol{p}_{h_{w+4}j}') \\ &\quad + \delta_{uh_{w+4}}(\boldsymbol{p}_{h_wj}' \times \boldsymbol{p}_{h_{w+2}j}' \times \boldsymbol{p}_{h_{w+4}i_{h_{w+4}}}') \\ &\quad + (1 - \delta_{uh_w} - \delta_{uh_{w+2}} - \delta_{uh_{w+4}})(\boldsymbol{p}_{h_wj}' \times \boldsymbol{p}_{h_{w+2}j}' \times \boldsymbol{p}_{h_{w+4}j}')]\,\boldsymbol{\alpha}_w = 0 \end{aligned}$$

where $0 \le j \le s-1$, $1 \le u \le m$, $0 \le i_u \le s-1$, $j \ne i_u$. Next, let us define

$$\begin{aligned} \gamma(\boldsymbol{\theta}) = \theta_0 &+ \sum_{\substack{v=1\\ v\ne u_1, u_2}}^{m} \boldsymbol{p}_{vj}'\boldsymbol{\theta}_v + \boldsymbol{p}_{u_1i_{u_1}}'\boldsymbol{\theta}_{u_1} + \boldsymbol{p}_{u_2i_{u_2}}'\boldsymbol{\theta}_{u_2} \\ &+ \sum_{w=1}^{2} [\delta_{u_1h_w}\{\delta_{u_2h_{w+2}}(\boldsymbol{p}_{h_wi_{h_w}}' \times \boldsymbol{p}_{h_{w+2}i_{h_{w+2}}}' \times \boldsymbol{p}_{h_{w+4}j}') \\ &\quad + \delta_{uh_{w+4}}(\boldsymbol{p}_{h_wi_{h_w}}' \times \boldsymbol{p}_{h_{w+2}j}' \times \boldsymbol{p}_{h_{w+4}i_{h_{w+4}}}') \\ &\quad + (1 - \delta_{u_2h_{w+2}} - \delta_{u_2h_{w+4}})(\boldsymbol{p}_{h_wi_{h_w}}' \times \boldsymbol{p}_{h_{w+2}j}' \times \boldsymbol{p}_{h_{w+4}j}')\} \\ &\quad + (1 - \delta_{u_1h_w})\{\delta_{u_1h_{w+2}}\{\delta_{u_2h_{w+4}}(\boldsymbol{p}_{h_wj}' \times \boldsymbol{p}_{h_{w+2}i_{h_{w+2}}}' \times \boldsymbol{p}_{h_{w+4}i_{h_{w+4}}}') \\ &\quad + (1 - \delta_{u_2h_{w+4}})(\boldsymbol{p}_{h_wj}' \times \boldsymbol{p}_{h_{w+2}i_{h_{w+2}}}' \times \boldsymbol{p}_{h_{w+4}j}')\} \\ &\quad + (1 - \delta_{u_1h_{w+2}})\{\delta_{u_1h_{w+4}}(\boldsymbol{p}_{h_wj}' \times \boldsymbol{p}_{h_{w+2}j}' \times \boldsymbol{p}_{h_{w+4}i_{h_{w+4}}}') \\ &\quad + (1 - \delta_{u_1h_{w+4}})\{\delta_{u_2h_w}(\boldsymbol{p}_{h_wi_{h_w}}' \times \boldsymbol{p}_{h_{w+2}j}' \times \boldsymbol{p}_{h_{w+4}j}') \end{aligned}$$

$$+\ \delta_{u_2 h_{w+2}}(\boldsymbol{p}'_{h_w j} \times \boldsymbol{p}'_{h_{w+2} i_{h_{w+2}}} \times \boldsymbol{p}'_{h_{w+4} j})$$
$$+\ \delta_{u_2 h_{w+4}}(\boldsymbol{p}'_{h_w j} \times \boldsymbol{p}'_{h_{w+2} j} \times \boldsymbol{p}'_{h_{w+4} i_{h_{w+4}}})$$
$$+\ (1 - \delta_{u_2 h_w} - \delta_{u_2 h_{w+2}} - \delta_{u_2 h_{w+4}})(\boldsymbol{p}'_{h_w j} \times \boldsymbol{p}'_{h_{w+2} j} \times \boldsymbol{p}'_{h_{w+4} j})\}\}\}]\boldsymbol{\alpha}_w\ .$$

Then,

$$(3.33)\qquad \gamma(\boldsymbol{\theta}) = 0 \quad \text{for} \quad 1 \le u_1 < u_2 \le m, \quad i_{u_1} = 0, \quad i_{u_2} = 0 \quad \text{and} \quad 1 \le j \le s-1\ ,$$

$$(3.34)\qquad \gamma(\boldsymbol{\theta}) = 0 \quad \text{for} \quad 1 \le u_1 < u_2 \le m, \quad 1 \le j \le s-2\ , \quad 0 \le \min\,(i_{u_1}, i_{u_2}) < j < \max\,(i_{u_1}, i_{u_2}) \le s-1\ ,$$

$$(3.35)\qquad \gamma(\boldsymbol{\theta}) = 0 \quad \text{for} \quad 1 \le u_1 < u_2 \le m, \quad 1 \le j \le s-2\ , \quad j+1 \le i_{u_1}, i_{u_2} \le s-1\ .$$

Now in view of (3.29), a number of cases may arise. In the above setting, first suppose that $h_1 = h_2 < h_3 = h_4 < h_5 < h_6$. Then (3.30) has the only solution $\boldsymbol{\theta} = \mathbf{0}$.

For any fixed t $(1 \le t \le m,\ t \ne h_1, h_3, h_5, h_6)$, subtracting (3.31) with $j = 0$ from (3.32) with $j = 0$ and $u = t$, one gets

$$(3.36)\qquad (\boldsymbol{p}'_{t i_t} - \boldsymbol{p}'_{t0})\,\boldsymbol{\theta}_t = 0, \qquad 1 \le i_t \le s-1\ .$$

From Lemma 3.1 and (3.36), it follows that

$$(3.36')\qquad \boldsymbol{\theta}_t = \mathbf{0}, \qquad 1 \le t \le m, \quad t \ne h_1, h_3, h_5, h_6\ .$$

For any fixed r $(1 \le r \le s-1)$, subtracting (3.31) with $j = 0$ from (3.32) with $j = 0$, $u = h_5$ and $i_{h_5} = r$, one gets

$$(3.37)\qquad (\boldsymbol{p}'_{h_5 r} - \boldsymbol{p}'_{h_5 0})\,\boldsymbol{\theta}_{h_5} + [\boldsymbol{p}'_{h_1 0} \times \boldsymbol{p}'_{h_3 0} \times (\boldsymbol{p}'_{h_5 r} - \boldsymbol{p}'_{h_5 0})]\,\boldsymbol{\alpha}_1 = 0\ .$$

For any fixed r $(1 \le r \le s-1)$, subtracting (3.33) with $u_1 = h_1$, $u_2 = h_5$ and $j = r$ from (3.32) with $u = h_1$, $i_{h_1} = 0$ and $j = r$, one gets

$$(3.38)\qquad (\boldsymbol{p}'_{h_5 r} - \boldsymbol{p}'_{h_5 0})\,\boldsymbol{\theta}_{h_5} + [\boldsymbol{p}'_{h_1 0} \times \boldsymbol{p}'_{h_3 r} \times (\boldsymbol{p}'_{h_5 r} - \boldsymbol{p}'_{h_5 0})]\,\boldsymbol{\alpha}_1 = 0\ .$$

For any fixed r $(1 \le r \le s-1)$, subtracting (3.33) with $u_1 = h_3$, $u_2 = h_5$ and $j = r$ from (3.32) with $u = h_3$, $i_{h_3} = 0$ and $j = r$, one gets

$$(3.39)\qquad (\boldsymbol{p}'_{h_5 r} - \boldsymbol{p}'_{h_5 0})\,\boldsymbol{\theta}_{h_5} + [\boldsymbol{p}'_{h_1 r} \times \boldsymbol{p}'_{h_3 0} \times (\boldsymbol{p}'_{h_5 r} - \boldsymbol{p}'_{h_5 0})]\,\boldsymbol{\alpha}_1 = 0\ .$$

For any fixed q, r $(1 \leq q < r \leq s-1)$, subtracting (3.33) with $u_1 = h_1$, $u_2 = h_5$ and $j = q$ from (3.34) with $u_1 = h_1$, $i_{h_1} = 0$, $u_2 = h_5$, $i_{h_5} = r$ and $j = q$, one gets

$$(3.40) \qquad (\boldsymbol{p}'_{h_5 r} - \boldsymbol{p}'_{h_5 0})\boldsymbol{\theta}_{h_5} + [\boldsymbol{p}'_{h_1 0} \times \boldsymbol{p}'_{h_3 q} \times (\boldsymbol{p}'_{h_5 r} - \boldsymbol{p}'_{h_5 0})]\boldsymbol{\alpha}_1 = 0 .$$

For any fixed k, r $(1 \leq k < r \leq s-1)$, subtracting (3.33) with $u_1 = h_3$, $u_2 = h_5$ and $j = k$ from (3.34) with $u_1 = h_3$, $i_{h_3} = 0$, $u_2 = h_5$, $i_{h_5} = r$ and $j = k$, one gets

$$(3.41) \qquad (\boldsymbol{p}'_{h_5 r} - \boldsymbol{p}'_{h_5 0})\boldsymbol{\theta}_{h_5} + [\boldsymbol{p}'_{h_1 k} \times \boldsymbol{p}'_{h_3 0} \times (\boldsymbol{p}'_{h_5 r} - \boldsymbol{p}'_{h_5 0})]\boldsymbol{\alpha}_1 = 0 .$$

For any fixed q, r $(1 \leq r < q \leq s-1)$, subtracting (3.32) with $u = h_3$, $i_{h_3} = q$ and $j = 0$ from (3.34) with $u_1 = h_1$, $i_{h_1} = 0$, $u_2 = h_3$, $i_{h_3} = q$ and $j = r$, one gets

$$(3.42) \qquad (\boldsymbol{p}'_{h_5 r} - \boldsymbol{p}'_{h_5 0})\boldsymbol{\theta}_{h_5} + (\boldsymbol{p}'_{h_6 r} - \boldsymbol{p}'_{h_6 0})\boldsymbol{\theta}_{h_6} + [\boldsymbol{p}'_{h_1 0} \times \boldsymbol{p}'_{h_3 q} \times (\boldsymbol{p}'_{h_5 r} - \boldsymbol{p}'_{h_5 0})]\boldsymbol{\alpha}_1 + [\boldsymbol{p}'_{h_1 0} \times \boldsymbol{p}'_{h_3 q} \times (\boldsymbol{p}'_{h_6 r} - \boldsymbol{p}'_{h_6 0})]\boldsymbol{\alpha}_2 = 0 .$$

For any fixed k, r $(1 \leq r < k \leq s-1)$, subtracting (3.32) with $u = h_1$, $i_{h_1} = k$ and $j = 0$ from (3.34) with $u_1 = h_1$, $i_{h_1} = k$, $u_2 = h_3$, $i_{h_3} = 0$ and $j = r$, one gets

$$(3.43) \qquad (\boldsymbol{p}'_{h_5 r} - \boldsymbol{p}'_{h_5 0})\boldsymbol{\theta}_{h_5} + (\boldsymbol{p}'_{h_6 r} - \boldsymbol{p}'_{h_6 0})\boldsymbol{\theta}_{h_6} + [\boldsymbol{p}'_{h_1 k} \times \boldsymbol{p}'_{h_3 0} \times (\boldsymbol{p}'_{h_5 r} - \boldsymbol{p}'_{h_5 0})]\boldsymbol{\alpha}_1 + [\boldsymbol{p}'_{h_1 k} \times \boldsymbol{p}'_{h_3 0} \times (\boldsymbol{p}'_{h_6 r} - \boldsymbol{p}'_{h_6 0})]\boldsymbol{\alpha}_2 = 0 .$$

For any fixed r $(1 \leq r \leq s-1)$, subtracting (3.32) with $u = h_5$, $i_{h_5} = 0$ and $j = r$ from (3.31) with $j = r$, one gets

$$(3.44) \qquad (\boldsymbol{p}'_{h_5 r} - \boldsymbol{p}'_{h_5 0})\boldsymbol{\theta}_{h_5} + [\boldsymbol{p}'_{h_1 r} \times \boldsymbol{p}'_{h_3 r} \times (\boldsymbol{p}'_{h_5 r} - \boldsymbol{p}'_{h_5 0})]\boldsymbol{\alpha}_1 = 0 .$$

For any fixed q, r $(1 \leq q, r \leq s-1,\ q \neq r)$, subtracting (3.32) with $u = h_5$, $i_{h_5} = 0$ and $j = q$ from (3.32) with $u = h_5$, $i_{h_5} = r$ and $j = q$, one gets

$$(3.45) \qquad (\boldsymbol{p}'_{h_5 r} - \boldsymbol{p}'_{h_5 0})\boldsymbol{\theta}_{h_5} + [\boldsymbol{p}'_{h_1 q} \times \boldsymbol{p}'_{h_3 q} \times (\boldsymbol{p}'_{h_5 r} - \boldsymbol{p}'_{h_5 0})]\boldsymbol{\alpha}_1 = 0 .$$

For any fixed k, r $(1 \leq r < k \leq s-1)$, subtracting (3.34) with $j = r$, $u_1 = h_1$, $i_{h_1} = k$, $u_2 = h_5$ and $i_{h_5} = 0$ from (3.32) with $j = r$, $u_1 = h_1$ and $i_{h_1} = k$, one gets

$$(3.46) \qquad (\boldsymbol{p}'_{h_5 r} - \boldsymbol{p}'_{h_5 0})\boldsymbol{\theta}_{h_5} + [\boldsymbol{p}'_{h_1 k} \times \boldsymbol{p}'_{h_3 r} \times (\boldsymbol{p}'_{h_5 r} - \boldsymbol{p}'_{h_5 0})]\boldsymbol{\alpha}_1 = 0 .$$

For any fixed q, r $(1 \le r < q \le s-1)$, subtracting (3.34) with $j = r$, $u_1 = h_3$, $i_{h_3} = q$, $u_2 = h_5$ and $i_{h_5} = 0$ from (3.32) with $j = r$, $u = h_3$ and $i_{h_3} = q$, one gets

$$(3.47) \qquad (\boldsymbol{p}'_{h_5 r} - \boldsymbol{p}'_{h_5 0})\boldsymbol{\theta}_{h_5} + [\boldsymbol{p}'_{h_1 r} \times \boldsymbol{p}'_{h_3 q} \times (\boldsymbol{p}'_{h_5 r} - \boldsymbol{p}'_{h_5 0})]\boldsymbol{\alpha}_1 = 0 .$$

For any fixed k, q, r $(1 \le r < q < k \le s-1)$, subtracting (3.34) with $j = q$, $u_1 = h_1$, $i_{h_1} = k$, $u_2 = h_5$ and $i_{h_5} = 0$ from (3.34) with $j = q$, $u_1 = h_1$, $i_{h_1} = k$, $u_2 = h_5$ and $i_{h_5} = r$, one gets

$$(3.48) \qquad (\boldsymbol{p}'_{h_5 r} - \boldsymbol{p}'_{h_5 0})\boldsymbol{\theta}_{h_5} + [\boldsymbol{p}'_{h_1 k} \times \boldsymbol{p}'_{h_3 q} \times (\boldsymbol{p}'_{h_5 r} - \boldsymbol{p}'_{h_5 0})]\boldsymbol{\alpha}_1 = 0 .$$

For any fixed k, q, r $(1 \le r < k < q \le s-1)$, subtracting (3.34) with $j = k$, $u_1 = h_3$, $i_{h_3} = q$, $u_2 = h_5$ and $i_{h_5} = 0$ from (3.34) with $j = k$, $u_1 = h_3$, $i_{h_3} = q$, $u_2 = h_5$ and $i_{h_5} = r$, one gets

$$(3.49) \qquad (\boldsymbol{p}'_{h_5 r} - \boldsymbol{p}'_{h_5 0})\boldsymbol{\theta}_{h_5} + [\boldsymbol{p}'_{h_1 k} \times \boldsymbol{p}'_{h_3 q} \times (\boldsymbol{p}'_{h_5 r} - \boldsymbol{p}'_{h_5 0})]\boldsymbol{\alpha}_1 = 0 .$$

For any fixed k, q, r $(1 \le q \le s-2,\ q < k, r \le s-1)$, subtracting (3.34) with $j = q$, $u_1 = h_1$, $i_{h_1} = k$, $u_2 = h_5$ and $i_{h_5} = 0$ from (3.35) with $j = q$, $u_1 = h_1$, $i_{h_1} = k$, $u_2 = h_5$ and $i_{h_5} = r$, one gets

$$(3.50) \qquad (\boldsymbol{p}'_{h_5 r} - \boldsymbol{p}'_{h_5 0})\boldsymbol{\theta}_{h_5} + [\boldsymbol{p}'_{h_1 k} \times \boldsymbol{p}'_{h_3 q} \times (\boldsymbol{p}'_{h_5 r} - \boldsymbol{p}'_{h_5 0})]\boldsymbol{\alpha}_1 = 0 .$$

For any fixed k, q, r $(1 \le k \le s-2,\ k < q, r \le s-1)$, subtracting (3.34) with $j = k$, $u_1 = h_3$, $i_{h_3} = q$, $u_2 = h_5$ and $i_{h_5} = 0$ from (3.35) with $j = k$, $u_1 = h_3$, $i_{h_3} = q$, $u_2 = h_5$ and $i_{h_5} = r$, one gets

$$(3.51) \qquad (\boldsymbol{p}'_{h_5 r} - \boldsymbol{p}'_{h_5 0})\boldsymbol{\theta}_{h_5} + [\boldsymbol{p}'_{h_1 k} \times \boldsymbol{p}'_{h_3 q} \times (\boldsymbol{p}'_{h_5 r} - \boldsymbol{p}'_{h_5 0})]\boldsymbol{\alpha}_1 = 0 .$$

Now combining the equations (3.37) to (3.41) and the equations (3.44) to (3.51), one gets for any fixed k, q, r $(1 \le r \le s-1,\ 0 \le k, q \le s-1)$, $(k, q, r) \ne (0, q, r)$ for $1 \le r < q \le s-1$ and $(k, q, r) \ne (k, 0, r)$ for $1 \le r < k \le s-1$,

$$(3.52) \qquad (\boldsymbol{p}'_{h_5 r} - \boldsymbol{p}'_{h_5 0})\boldsymbol{\theta}_{h_5} + [\boldsymbol{p}'_{h_1 k} \times \boldsymbol{p}'_{h_3 q} \times (\boldsymbol{p}'_{h_5 r} - \boldsymbol{p}'_{h_5 0})]\boldsymbol{\alpha}_1 = 0 .$$

Summing the s equations in (3.52) one gets, for any fixed q, r $(1 \le q \le s-1,\ 1 \le r \le s-1,\ q \le r)$, $(\boldsymbol{p}'_{h_5 r} - \boldsymbol{p}'_{h_5 0})\boldsymbol{\theta}_{h_5} = 0$. Then from Lemma 3.1, it follows that

$$(3.53) \qquad \boldsymbol{\theta}_{h_5} = \boldsymbol{0} .$$

Using (3.53), one gets from (3.42), for any fixed r, q $(1 \le r < q \le s-1)$,

$$(3.54)\qquad (\boldsymbol{p}'_{h_6r}-\boldsymbol{p}'_{h_60})\boldsymbol{\theta}_{h_6}+[\boldsymbol{p}'_{h_10}\times\boldsymbol{p}'_{h_3q}\times(\boldsymbol{p}'_{h_5r}-\boldsymbol{p}'_{h_50})]\boldsymbol{\alpha}_1 + [\boldsymbol{p}'_{h_10}\times\boldsymbol{p}'_{h_3q}\times(\boldsymbol{p}'_{h_6r}-\boldsymbol{p}'_{h_60})]\boldsymbol{\alpha}_2=0\,.$$

Using (3.53), one gets from (3.43), for any fixed r,k $(1\le r<k\le s-1)$,

$$(3.55)\qquad (\boldsymbol{p}'_{h_6r}-\boldsymbol{p}'_{h_60})\boldsymbol{\theta}_{h_6}+[\boldsymbol{p}'_{h_1k}\times\boldsymbol{p}'_{h_30}\times(\boldsymbol{p}'_{h_5r}-\boldsymbol{p}'_{h_50})]\boldsymbol{\alpha}_1 + [\boldsymbol{p}'_{h_1k}\times\boldsymbol{p}'_{h_30}\times(\boldsymbol{p}'_{h_6r}-\boldsymbol{p}'_{h_60})]\boldsymbol{\alpha}_2=0\,.$$

Using (3.53), one gets from (3.52), for any fixed k,q,r $(1\le r\le s-1,$ $0\le k,q\le s-1)$, $(k,q,r)\ne(0,q,r)$ for $1\le r<q\le s-1$ and $(k,q,r)\ne(k,0,r)$ for $1\le r<k\le s-1$,

$$(3.56)\qquad [\boldsymbol{p}'_{h_1k}\times\boldsymbol{p}'_{h_3q}\times(\boldsymbol{p}'_{h_5r}-\boldsymbol{p}'_{h_50})]\boldsymbol{\alpha}_1=0\,.$$

For any fixed r,q $(1\le r<q\le s-1)$, summing the s equations that (3.54) and (3.56), and then subtracting from (3.54), one gets

$$(3.57)\qquad [\boldsymbol{p}'_{h_10}\times\boldsymbol{p}'_{h_3q}\times(\boldsymbol{p}'_{h_5r}-\boldsymbol{p}'_{h_50})]\boldsymbol{\alpha}_1=0\,.$$

For any fixed r,k $(1\le r<k\le s-1)$, summing the s equations that (3.55) and (3.56), and then subtracting from (3.55), one gets

$$(3.58)\qquad [\boldsymbol{p}'_{h_1k}\times\boldsymbol{p}'_{h_30}\times(\boldsymbol{p}'_{h_5r}-\boldsymbol{p}'_{h_50})]\boldsymbol{\alpha}_1=0\,.$$

Equations (3.56) to (3.58) implies that $[P_{h_1}\times P_{h_3}\times P^*_{h_5}]\boldsymbol{\alpha}_1=\mathbf{0}$, where $P^*_{h_5}$ is an $(s-1)\times(s-1)$ matrix with rows $(\boldsymbol{p}'_{h_5r}-\boldsymbol{p}'_{h_50})$, $1\le r\le s-1$. Since P_{h_1}, P_{h_3} and $P^*_{h_5}$ are of full column rank, one obtains $\boldsymbol{\alpha}_1=\mathbf{0}$. Proceeding in a similar way, one can obtain $\boldsymbol{\theta}_{h_1}=\mathbf{0}$, $\boldsymbol{\theta}_{h_3}=\mathbf{0}$, $\boldsymbol{\alpha}_2=\mathbf{0}$, $\boldsymbol{\theta}_{h_6}=\mathbf{0}$ and $\theta_0=0$. Hence $\boldsymbol{\theta}=\mathbf{0}$.

The proofs of other cases follow in a similar manner through considerable algebra. The details of such cases are omitted here to save space but may be obtained from the author on request.

4. Main result in the asymmetric case

The following notation will be helpful in describing the search designs considered here. Admittedly, the notation is somewhat involved, but it is expected that the examples stated later in this section would be helpful in illustrating the plan under consideration. Let $s_0=0$.

For $1\le l\le m-1$, if $s_{l-1}<s_l$, then define the $m\times 1$ vectors $\boldsymbol{a}_j^{(1,l)}$ $(s_{l-1}\le j\le s_l-1)$ such that in $\boldsymbol{a}_j^{(1,l)}$, the first $(l-1)$ elements equal 0 and the remaining elements are j; let $A^{(1,l)}$ be an $m\times(s_l-s_{l-1})$ array whose

columns are given by $\boldsymbol{a}_j^{(1,l)}$ for all possible j. Note that $A^{(1,l)}$ is defined only when $s_{l-1} < s_l$.

For $1 \le l \le m-2$, if $s_l < s_{l+1}$, then define the $m \times 1$ vectors $\boldsymbol{a}_{j,u,i_u}^{(2,l)}$ ($s_l \le j \le s_{l+1}-1$, $1 \le u \le l$, $1 \le i_u \le s_u - 1$) such that in $\boldsymbol{a}_{j,u,i_u}^{(2,l)}$, the u-th element is i_u, the last $(m-l)$ elements equal j and the remaining elements equal 0; let $A^{(2,l)}$ be an $m \times \sum_{u=1}^{l} (s_u - 1)(s_{l+1} - s_l)$ array whose columns are $\boldsymbol{a}_{j,u,i_u}^{(2,l)}$ for all possible j, u and i_u. Note that $A^{(2,l)}$ is defined only when $s_l < s_{l+1}$.

For $1 \le l \le m-2$, if $s_{l-1} < s_l$, then define the $m \times 1$ vectors $\boldsymbol{a}_{j,u,i_u}^{(3,l)}$ ($s_{l-1} \le j \le s_l - 1$, $l \le u \le m$, $0 \le i_u \le s_u - 1$, $i_u \ne j$) such that in $\boldsymbol{a}_{j,u,i_u}^{(3,l)}$, the first $(l-1)$ elements equal 0, the u-th element is i_u and the remaining elements equal j; let $A^{(3,l)}$ be an $m \times \sum_{u=l}^{m} (s_l - s_{l-1})(s_u - 1)$ array whose columns are $\boldsymbol{a}_{j,u,i_u}^{(3,l)}$ for all possible j, u and i_u. Note that $A^{(3,l)}$ is defined only when $s_{l-1} < s_l$.

For $1 \le l \le m-3$, if $s_l < s_{l+1}$, define the $m \times 1$ vectors $\boldsymbol{a}_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(4,l)}$ ($s_l \le j \le s_{l+1} - 1$, $1 \le u_1 \le l$, $1 \le i_{u_1} \le s_{u_1} - 1$, $l+1 \le u_2 \le m$, $j+1 \le i_{u_2} \le s_{u_2} - 1$) such that in $\boldsymbol{a}_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(4,l)}$, the u-th element is i_{u_1}, the remaining of the first l elements equal 0, the u_2-th element is i_{u_2} and the remaining elements equal j; let $A^{(4,l)}$ be an $m \times \left[\sum_{u_1=1}^{l} (s_{u_1} - 1)\right]\left[\sum_{j=s_l}^{s_{l+1}-1} \sum_{u_2=l+1}^{m} (s_{u_2} - j - 1)\right]$ array whose columns are $\boldsymbol{a}_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(4,l)}$ for all possible j, u_1, u_2, i_{u_1} and i_{u_2}. Note that $A^{(4,l)}$ is defined only when $s_l < s_{l+1}$.

For $1 \le l \le m-3$, if $s_{l-1} < s_l$, then define the $m \times 1$ vectors $\boldsymbol{a}_{j,u_1,u_2}^{(5,l)}$ ($s_{l-1} \le j \le s_l - 1$, $j \ne 0$, $l \le u_1 < u_2 \le m$) such that in $\boldsymbol{a}_{j,u_1,u_2}^{(5,l)}$, the first $(l-1)$ elements equal 0, the u_1-th and u_2-th elements equal 0 and the remaining elements equal j; let $A^{(5,l)}$ be an $m \times [((m-l+1)(m-l)/2)(s_l - s_{l-1} - \delta_{1l})]$ array whose columns are $\boldsymbol{a}_{j,u_1,u_2}^{(5,l)}$ for all possible j, u_1 and u_2. Note that $A^{(5,l)}$ is defined only when $s_{l-1} < s_l$.

For $1 \le l \le m-3$, if $s_{l-1} < s_l$, then define the $m \times 1$ vectors $\boldsymbol{a}_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(6,l)}$ ($s_{l-1} \le j \le s_l - 1$, $j \ne 0$, $l \le u_1 < u_2 \le m$, $j+1 \le i_{u_1} \le s_{u_1} - 1$, $j+1 \le i_{u_2} \le s_{u_2} - 1$) such that in $\boldsymbol{a}_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(6,l)}$, the first $(l-1)$ elements equal 0, the u_1-th element is i_{u_1}, the u_2-th element is i_{u_2} and the remaining elements equal j; let $A^{(6,l)}$ be an $m \times \left[\sum_{\substack{j=s_{l-1}\\ j\ne 0}}^{s_l-1} \sum_{u_1=l}^{m} \sum_{\substack{u_2=l\\ u_1<u_2}}^{m} (s_{u_1} - j - 1)(s_{u_2} - j - 1)\right]$ array whose columns are $\boldsymbol{a}_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(6,l)}$ for all possible j, u_1, u_2, i_{u_1} and i_{u_2}. Note that $A^{(6,l)}$ is defined only when $s_{l-1} < s_l$.

For $1 \le l \le m-3$, if $s_{l-1} < s_l$, then define the $m \times 1$ vectors $\boldsymbol{a}_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(7,l)}$ ($s_{l-1} \le j \le s_l - 1$, $j \ne 0$, $l \le u_1 < u_2 \le m$, $0 \le i_{u_1} \le s_{u_1} - 1$, $0 \le i_{u_2} \le s_{u_2} - 1$, $\min(i_{u_1}, i_{u_2}) < j$ and $\max(i_{u_1}, i_{u_2}) > j$) such that in $\boldsymbol{a}_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(7,l)}$, the first $(l-1)$ elements equal 0, the u_1-th element is i_{u_1}, the u_2-th element is i_{u_2} and the remaining elements equal j; let $A^{(7,l)}$ be an $m \times \left[\sum_{\substack{j=s_{l-1}\\ j\ne 0}}^{s_l-1} \sum_{u_1=l}^{m} \sum_{\substack{u_2=l\\ u_1<u_2}}^{m} j(s_{u_1} + s_{u_2} -\right.$

$2j-2)\Big]$ array whose columns are $\boldsymbol{a}^{(7,l)}_{j,u_1,u_2,i_{u_1},i_{u_2}}$ for all possible j, u_1, u_2, i_{u_1} and i_{u_2}. Note that $A^{(7,l)}$ is defined only when $s_{l-1} < s_l$.

If $s_{m-3} < s_{m-2}$, then define the $m \times 1$ vectors $\boldsymbol{a}^{(8)}_{j,u_1,u_2,i_{u_1},i_{u_2}}$ ($s_{m-3} \le j \le s_{m-2} - 1$, $m-2 \le u_1 < u_2 \le m$, $0 \le i_{u_1} \le s_{u_1} - 1$, $0 \le i_{u_2} \le s_{u_2} - 1$, $\min(i_{u_1}, i_{u_2}) = 0$, $\max(i_{u_1}, i_{u_2}) > j$) such that in $\boldsymbol{a}^{(8)}_{j,u_1,u_2,i_{u_1},i_{u_2}}$, the first $(m-3)$ elements equal 0, the u_1-th element is i_{u_1}, the u_2-th element is i_{u_2} and the remaining elements are j; let $A^{(8)}$ be an $m \times \left[\sum_{j=s_{m-3}}^{s_{m-2}-1} \sum_{u_1=m-2}^{m} \sum_{\substack{u_2=m-2\\u_1<u_2}}^{m} (s_{u_1} + s_{u_2} - 2j - 2)\right]$ array whose columns are $\boldsymbol{a}^{(8)}_{j,u_1,u_2,i_{u_1},i_{u_2}}$ for all possible j, u_1, u_2, i_{u_1} and i_{u_2}. Note that $A^{(8)}$ is defined only when $s_{m-3} < s_{m-2}$.

If $s_{m-2} < s_{m-1}$, then define the $m \times 1$ vectors $\boldsymbol{a}^{(9)}_{j,k,u,i_u}$ ($s_{m-2} \le j \le s_{m-1} - 1$, $s_{m-2} \le k \le s_m - 1$, $j \ne k$, $1 \le u \le m-2$, $1 \le i_u \le s_u - 1$) such that in $\boldsymbol{a}^{(9)}_{j,k,u,i_u}$, the u-th element is i_u, the last two elements are j and k and the remaining elements equal 0; let $A^{(9)}$ be an $m \times \left[\left(\sum_{u=1}^{m-2} (s_u - 1)\right)(s_{m-1} - s_{m-2})(s_m - s_{m-2} - 1)\right]$ array whose columns are $\boldsymbol{a}^{(9)}_{j,k,u,i_u}$ for all possible j, k, u and i_u. Note that $A^{(9)}$ is defined only when $s_{m-2} < s_{m-1}$.

If $s_{m-2} < s_{m-1} - 1$, then define the $m \times 1$ vectors $\boldsymbol{a}^{(10)}_{jk}$ ($s_{m-2} \le j < s_{m-1} - 1$, $s_{m-2} \le k \le s_m - 1$, $j \ne k$) such that in $\boldsymbol{a}^{(10)}_{jk}$, the last two elements are j and k and all other elements equal 0; let $A^{(10)}$ be an $m \times [(s_{m-1} - s_{m-2} - 1)(s_m - s_{m-2} - 1)]$ array whose columns are $\boldsymbol{a}^{(10)}_{jk}$ for all possible j and k. Note that $A^{(10)}$ is defined only when $s_{m-2} < s_{m-1} - 1$.

Finally, if $s_{m-2} < s_m$, then define the $m \times 1$ vectors $\boldsymbol{a}^{(11)}_k$ ($s_{m-2} \le k \le s_m - 1$, $k \ne s_{m-1} - 1$) such that in $\boldsymbol{a}^{(11)}_k$, the first $(m-2)$ elements equal 0, the $(m-1)$-th element is $s_{m-1} - 1$ and the m-th element is k; let $A^{(11)}$ be an $m \times (s_m - s_{m-2} - 1 + \delta_{s_{m-1}s_{m-2}})$ array whose columns are $\boldsymbol{a}^{(11)}_k$ for all possible k, where $\delta_{s_{m-1}s_{m-2}}$ is the Kronecker delta. Note that $A^{(11)}$ is defined only when $s_{m-2} < s_m$.

Let

$$
\begin{aligned}
n &= \text{total number of runs} \\
&= \sum_{l=1}^{m-1} (s_l - s_{l-1}) + \sum_{l=1}^{m-2} \sum_{u=1}^{l} (s_u - 1)(s_{l+1} - s_l) + \sum_{l=1}^{m-2} \sum_{u=l}^{m} (s_l - s_{l-1})(s_u - 1) \\
&\quad + \sum_{l=1}^{m-3} \left[\sum_{u_1=1}^{l} (s_{u_1} - 1) \left(\sum_{j=s_l}^{s_{l+1}-1} \sum_{u_2=l+1}^{m} (s_{u_2} - j - 1) \right)\right] \\
&\quad + \sum_{l=1}^{m-3} \sum_{\substack{j=s_{l-1}\\j\ne 0}}^{s_l-1} \sum_{u_1=l}^{m} \sum_{\substack{u_2=l\\u_1<u_2}}^{m} (s_{u_1} - j - 1)(s_{u_2} - j - 1) \\
&\quad + \sum_{l=1}^{m-3} [((m-l-1)(m-l)(s_l - s_{l-1} - \delta_{1l})/2)]
\end{aligned}
$$

$$+\sum_{l=1}^{m-3}\sum_{\substack{j=s_{l-1}\\ j\neq 0}}^{s_l-1}\sum_{\substack{u_1=l\\ u_1<u_2}}^{m}\sum_{u_2=l}^{m}[j(s_{u_1}+s_{u_2}-2j-2)]$$

$$+\sum_{j=s_{m-3}}^{s_{m-2}-1}\sum_{\substack{u_1=m-2\\ u_1<u_2}}^{m}\sum_{u_2=m-2}^{m}[s_{u_1}+s_{u_2}-2j-2]$$

$$+(s_{m-1}-s_{m-2})(s_m-s_{m-2}-1)\sum_{u=1}^{m-2}(s_u-1)$$

$$+(s_{m-1}-s_{m-2}-1)(s_m-s_{m-2}-1)+(s_m-s_{m-2}-1+\delta_{s_{m-1}s_{m-2}})$$

and define the following $m\times n$ array:

$$Q=[A^{(1,1)},\ldots,A^{(1,m-1)},A^{(2,1)},\ldots,A^{(2,m-2)},A^{(3,1)},\ldots,A^{(3,m-2)},$$
$$A^{(4,1)},\ldots,A^{(4,m-3)},A^{(5,1)},\ldots,A^{(5,m-3)},A^{(6,1)},\ldots,A^{(6,m-3)},$$
$$A^{(7,1)},\ldots,A^{(7,m-3)},A^{(8)},A^{(9)},A^{(10)},A^{(11)}] \tag{4.1}$$

where in Q, any subarray of the form $A^{(i,j)}$ or $A^{(i)}$ is included provided it is defined

Example 4.1. Let $m=6$, $s_1=2$, $s_2=3$, $s_3=4$, $s_4=5$, $s_5=6$, $s_6=7$.

$$A^{(1,1)}:\begin{matrix}01\\01\\01\\01\\01\\01\end{matrix};\quad A^{(1,2)}:\begin{matrix}0\\2\\2\\2\\2\\2\end{matrix};\quad A^{(1,3)}:\begin{matrix}0\\0\\3\\3\\3\\3\end{matrix};\quad A^{(1,4)}:\begin{matrix}0\\0\\0\\4\\4\\4\end{matrix};\quad A^{(1,5)}:\begin{matrix}0\\0\\0\\5\\5\\5\end{matrix};\quad A^{(2,1)}:\begin{matrix}1\\2\\2\\2\\2\\2\end{matrix};$$

$$A^{(2,2)}:\begin{matrix}100\\012\\333\\333\\333\\333\end{matrix};\quad A^{(2,3)}:\begin{matrix}100000\\012000\\000123\\444444\\444444\\444444\end{matrix};\quad A^{(2,4)}:\begin{matrix}1000000000\\0120000000\\0001230000\\0000001234\\5555555555\\5555555555\end{matrix};$$

$$A^{(3,1)}:\begin{matrix}10000000000000000000011111111111111111111\\01200000000000000000010211111111111111111\\00012300000000000000011023111111111111111\\00000012340000000000011111023411111111111\\00000000001234500000011111111102345111111\\00000000000000012345611111111111111023456\end{matrix};\quad A^{(3,2)}:\begin{matrix}00000000000000000000\\01222222222222222222\\22013222222222222222\\22222013422222222222\\22222222201345222222\\22222222222222013456\end{matrix};$$

$$A^{(3,3)}:\begin{matrix}000000000000000000\\000000000000000000\\012333333333333333\\333012433333333333\\333333301245333333\\333333333333012456\end{matrix};\quad A^{(3,4)}:\begin{matrix}000000000000000\\000000000000000\\000000000000000\\012344444444444\\444401235444444\\444444444012356\end{matrix};\quad A^{(4,1)}:\begin{matrix}1111111111\\2222222222\\3222222222\\2342222222\\2223452222\\2222223456\end{matrix};$$

$A^{(4,2)}$:
111111000000000000
000000121212121212
333333333333333333
433333443333333333
345333334455333333
333456333333445566 ;

$A^{(4,3)}$:
111000000000000000
000121212000000000
000000000123123123
444444444444444444
544554444555444444
456445566444555666 ;

$A^{(5,1)}$:
000001111111111
011110000111111
101110111000111
110111011011001
111011101101010
111101110110100 ;

$A^{(5,2)}$:
0000000000
0000222222
0222000222
2022022002
2202202020
2220220200 ;

$A^{(5,3)}$:
000000
000000
000333
033003
303030
330300 ;

$A^{(6,1)}$:
1111111111111111111111111111111111111111111111111111111111111111111111111111111111
2222222222222211111111111111111111111111111111111111111111111111111111111111111111
2311111111111222333222233332222233333111111111111111111111111111111111111111111111
1123411111111123423411111111111111111222233334444222223333344444111111111111111111
1111123451111111111123452345111111111234523452345111111111111122222333334444455555
1111111123456111111111111123456234561111111111234562345623456234562345623456234562 3456 ;

$A^{(6,2)}$:
00000000000000000000000000000000000
22222222222222222222222222222222222
33333333322222222222222222222222222
34222222223334443333444422222222222
22345222234534522222222233334444555 5
22222345622222234563456345634563456 ;

$A^{(6,3)}$:
00000000000
00000000000
33333333333
44444333333
45333444555
33456456456 ;

$A^{(7,1)}$:
0000000000000011111111111111111111111111111111111111111111111111111111111111
2111111111111100200020000200000211111111111111111111111111111111111111111111
1231111111111123011111111111111100023000023000002311111111111111111111111111
1112341111111111112340111111111112340011111111111110000234000002341111111111
1111123451111111111112345011111111111234500111111234500011111111000002345
1111111112345611111111111123456011111111111234560011111123456000234560000 ;

$A^{(7,2)}$:
0000000000000000000000000000000000000000000000000000000000000000000000000000
0100110001110000111122222222222222222222222222222222222222222222222222222222
3322222222222222222200113300011133000011113322222222222222222222222222222222
2234342222222222222234340122222222222222222000111334400001111334422222222222222
2222223453452222222222222345345012222222222345345010122222222222220000111334455
2222222222223456345622222222222222234563456012222222222345634560101345634560 10101 ;

$A^{(7,3)}$:
000000000000000000000000000000000000000000000000000000
000000000000000000000000000000000000000000000000000000
012001122000111222333333333333333333333333333333333333
444333333333333333300112244400011122244433333333333333
333454545333333333345454501233333333333330001112224445 55
333333333456456456333333333456456456012456456456012012 ;

$A^{(8)}$:
000000
000000
000000
000444
544005
456560 ;

$A^{(9)}$:
1000000000
0120000000
0001230000
0000001234
5555555555
6666666666 ;

$A^{(11)}$:
0
0
0
0
5
6 ;

and $A^{(10)}$ is not defined.

THEOREM 4.1. *The columns of the matrix Q, interpreted as the level combinations of $F_1, F_2, \ldots, F_m$, give the required search design in the general* *asymmetric factorial set-up in n runs.*

The proof of the above theorem which follows essentially along the same line of the symmetric case but involves more complicated algebraic manipulations, is omitted here to save space. The details of the proof may be obtained from the author on request.

5. Discussion

The paper presented here describes a method for constructing search designs for general asymmetric factorials that allow the estimation of the general mean and main-effects, and search for and estimate at most one among two- and three-factor interactions. The proposed design is worthwhile when $m \geq 5$, where m is the number of factors. This design also usually requires a small number of runs compared to the corresponding regular fractions. The search designs presented in the following examples require considerably smaller numbers of runs compared to the corresponding regular plans (see Table 1). These regular plans are, indeed, capable of estimating many more parameters than those considered in this paper. Anyway, if prior knowledge is available that at most one among two- and three-factor interactions is present, then such elaborate estimation is not required and, in such situations, the designs presented in this paper appear to be economical.

Table 1. Some examples illustrating the number of runs in our plan and in the regular plan.

Serial No.	Set-up	Number of runs	
		Our plan	Regular plan
1	3^6	114	233
2	3^7	150	379
3	3^8	191	577
4	$3^3 \times 4^4$	295	741
5	$3^7 \times 4$	268	788
6	$3^5 \times 4^3$	310	914
7	4^5	224	376
8	4^7	424	1,156
9	$2^3 \times 3^3 \times 4^2$	187	454
10	$2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9$	2,171	5,119

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