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ON THE ADMISSIBILITY OF AN ESTIMATOR OF A NORMAL MEAN VECTOR UNDER A LINEX LOSS FUNCTION

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Abstract. For a p-variate normal mean with known variances, the model proposed by Zellner (1986, *J. Amer. Statist. Assoc.*, **81**, 446–451) is discussed in a slightly different framework. A generalized Bayes estimate is derived from a three-stage Bayes point of view under the asymmetric loss function, and the admissibility of such estimators is proved.

Key words and phrases: Admissible estimators, empirical Bayes, hierarchical Bayes analysis, generalized Bayes, LINEX loss, three-stage Bayes estimators.

1. Introduction

In some estimation problems it is appropriate to use asymmetric loss functions. Several authors have considered asymmetric linear loss functions (e.g., Ferguson (1967), Aitchison and Dunsmore (1975), Berger (1985)). Varian (1975) introduced a very useful asymmetric LINEX loss function, which is not linear and is defined as $\boldsymbol{L}(\theta, \hat{\theta}) = v(\hat{\theta} - \theta)$ where $v(x) = b\{\exp(ax) - ax - 1\}$ and $a \neq 0$, $b > 0$ are constants. A full discussion of the properties of the LINEX loss function may be found in Zellner (1986).

For $X_1, \dots, X_n$ as a random sample of size n from $\boldsymbol{N}(\theta, \sigma^2)$, when σ^2 is known, Zellner (1986) showed that the sample mean, $\bar{X}$, fails to be admissible using the LINEX loss function and it is dominated by $\bar{X} - a\sigma^2/2n$. Rojo (1987) showed that estimators of the form $c\bar{X} + d$ are admissible for θ using the LINEX loss whenever $0 \leq c < 1$ or $c = 1$ and $d = -a\sigma^2/2n$. Parsian (1989) showed that $\bar{X} - a\sigma^2/2n$ is the only minimax and admissible estimator of θ in the class of all estimators of the form $c\bar{X} + d$. An important consideration is when σ^2 is unknown. Zellner (1986)

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suggested replacement of σ^2 by $S^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2/(n-1)$ in $\bar{X} - a\sigma^2/2n$ and Parsian (1989) obtained a unique Bayes, hence admissible, estimate of θ using the LINEX loss function.

Zellner (1986) extended the LINEX loss for multiparameter estimation problems as $\boldsymbol{L}(\boldsymbol{\theta}, \hat{\boldsymbol{\theta}}) = \sum_{i=1}^{p} v_i(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})$ where $v_i(\boldsymbol{x}) = b_i\{\exp(a_i x_i) - a_i x_i - 1\}$ and $a_i \neq 0$, $b_i > 0$, $i = 1,\ldots,p$ are constants, and considered the model, which was introduced by Lindley (1962), to discuss the multiparameter estimation problem for the normal case and provided some estimators of the parameters of interest (see Zellner (1986), (4.6), p. 450). However, he did not mention any optimal property of the estimator. The model proposed by Zellner can be discussed in a slightly different framework: namely, a three-stage Bayes point of view (or hierarchical Bayes analysis) under the LINEX loss function, in the terminology of Lindley and Smith (1972) (see also Lindley (1971*a*, 1971*b*). In the simplest situation of p-independent normal variables with unit variances, the Lindley and Smith approach can be described as follows.

Let $X_1,\ldots,X_p$ be independent, $X_i \sim \boldsymbol{N}(\theta_i, 1)$, $i = 1,\ldots,p$. Suppose that conditional on μ, θ_i's are i.i.d. $\boldsymbol{N}(\mu, 1)$, while marginally μ has an improper distribution uniform over the entire real line, i.e., $d\boldsymbol{G}(\mu) = d\mu$. It is often the case to choose the second stage prior as a suitable noninformative prior (see Berger (1985), p. 180). Then the improper prior distribution of $\boldsymbol{\theta} = (\theta_1,\ldots,\theta_p)'$ is given by (see Berger (1985), p. 108)

$$(1.1) \qquad \pi(\boldsymbol{\theta}) \propto \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\sum_{i=1}^{p}(\theta_i - \mu)^2\right] d\mu = \exp\left[-\frac{1}{2}\sum_{i=1}^{p}(\theta_i - \bar{\theta})^2\right]$$
$$= \exp\left[-\frac{1}{2}\boldsymbol{\theta}'\left(\boldsymbol{I}_p - \frac{1}{p}\mathbf{1}_p\mathbf{1}_p'\right)\boldsymbol{\theta}\right]$$

where $\bar{\theta} = (1/p)\sum_{i=1}^{p}\theta_i$ and $\mathbf{1}_p$ is a p-component column vector with all elements equal to 1, and the posterior distribution of $\boldsymbol{\theta}$ given $\boldsymbol{X} = \boldsymbol{x}$ is $\boldsymbol{N}_p(\boldsymbol{Dx}, \boldsymbol{D})$ where $\boldsymbol{D} = (\boldsymbol{I}_p + \mathbf{1}_p\mathbf{1}_p'/p)/2$. Hence, using these results and the extended LINEX loss function, the generalized Bayes estimate of θ_i, $i = 1,\ldots,p$, is

$$(1.2) \qquad \delta_{\mathrm{GB}}^{i}(\boldsymbol{x}) = \frac{1}{2}x_i + \frac{1}{2}\bar{x} - \frac{1}{2}a_i\tau^2 \qquad i = 1,\ldots,p\,.$$

Note that the arguments leading to $\delta_{\mathrm{GB}}^{i}(\boldsymbol{x})$ are essentially the same as those given in Zellner (1986).

In Section 2, under the same assumptions as above, which are the same as those given in Zellner (1986), we will derive the estimate $\delta_{GB}(\boldsymbol{x})$ from a three-stage Bayes point of view under the extended LINEX loss and give an empirical Bayes interpretation of it. In Section 3, we will use Blyth's (1951) method (limiting Bayes argument) to prove admissibility of the obtained estimator under the extended LINEX loss function.

2. Three stage Bayes estimator

Suppose $\boldsymbol{X} \sim N_p(\boldsymbol{\theta}, \boldsymbol{\Sigma})$ where $\boldsymbol{\theta}$ is unknown, but $\boldsymbol{\Sigma}$ is known positive definite. For the sake of simplicity, we consider the case $\boldsymbol{\Sigma} = \boldsymbol{I}_p$. Suppose that conditional on μ, $\boldsymbol{\theta}$ has the prior $N_p(\mu \mathbf{1}_p, \boldsymbol{I}_p)$ whereas μ has the improper uniform distribution over the real line, i.e., $dG(\mu) = d\mu$. It is assumed that the loss function is the extended LINEX loss introduced in Section 1.

To derive the generalized Bayes estimate of $\boldsymbol{\theta}$, first we need to derive the posterior distribution of $\boldsymbol{\theta}$ using the following lemma, which is the key lemma in the article of Lindley and Smith (1972) and will be used in deriving the posterior distribution and marginal distribution of $\boldsymbol{X}$ frequently in this note.

LEMMA 2.1. *Suppose that, given* $\boldsymbol{\theta}_1$, $\boldsymbol{Y} \sim N_p(\boldsymbol{A}_1\boldsymbol{\theta}_1, \boldsymbol{C}_1)$, *given* $\boldsymbol{\theta}_2$, $\boldsymbol{\theta}_1 \sim N_p(\boldsymbol{A}_2\boldsymbol{\theta}_2, \boldsymbol{C}_2)$. *Then* (i) *the marginal distribution of* $\boldsymbol{Y}$ *is* $N_p(\boldsymbol{A}_1\boldsymbol{A}_2\boldsymbol{\theta}_2, \boldsymbol{C}_1 + \boldsymbol{A}_1\boldsymbol{C}_2\boldsymbol{A}_1')$, *and* (ii) *the distribution of* $\boldsymbol{\theta}_1$, *given* $\boldsymbol{Y}$ *is* $N_p(\boldsymbol{D}\boldsymbol{b}, \boldsymbol{D})$ *with* $\boldsymbol{D}^{-1} = \boldsymbol{A}_1'\boldsymbol{C}_1^{-1}\boldsymbol{A}_1 + \boldsymbol{C}_2^{-1}$ *and* $\boldsymbol{b} = \boldsymbol{A}_1'\boldsymbol{C}_1^{-1}\boldsymbol{Y} + \boldsymbol{C}_2^{-1}\boldsymbol{A}_2\boldsymbol{\theta}_2$.

Now, using Lemma 2.1 with prior as in (1.1), the posterior distribution of $\boldsymbol{\theta}$ given $\boldsymbol{X} = \boldsymbol{x}$ is $N_p(\boldsymbol{D}\boldsymbol{x}, \boldsymbol{D})$, where

$$\boldsymbol{D} = \left(2\boldsymbol{I}_p - \frac{1}{p}\mathbf{1}_p\mathbf{1}_p'\right)^{-1} = \frac{1}{2}\left(\boldsymbol{I}_p + \frac{1}{p}\mathbf{1}_p\mathbf{1}_p'\right) = (\boldsymbol{d}_1', \ldots, \boldsymbol{d}_p')' \tag{2.1}$$

is positive definite.

The identity used in deriving (2.1) (and which will be used repeatedly later) is given by

$$(A + \boldsymbol{u}\boldsymbol{v}')^{-1} = A^{-1} - \frac{(A^{-1}\boldsymbol{u})(\boldsymbol{v}'A^{-1})}{1 + \boldsymbol{v}'A^{-1}\boldsymbol{u}}, \qquad \text{(see Rao (1973), p. 33)} \tag{2.2}$$

where A is a $p \times p$ invertible matrix and $\boldsymbol{u}$, $\boldsymbol{v}$ are p-component column vectors. Thus the generalized Bayes estimate of $\boldsymbol{\theta}$, say $\delta_{GB}(\boldsymbol{x})$, under LINEX loss is

$$\delta_{GB}^i(\boldsymbol{x}) = -\frac{1}{a_i}\log \boldsymbol{M}_{\theta_i|\boldsymbol{x}}(-a_i), \qquad i = 1, \ldots, p \tag{2.3}$$

where $\theta_i|\boldsymbol{x} \sim \boldsymbol{N}(\boldsymbol{d}_i'\boldsymbol{x}, \tau^2)$ with $\tau^2 = 1/2 + 1/2p$ and $\boldsymbol{M}_{\theta_i|\boldsymbol{x}}(\cdot)$ denotes the moment generating function of $\theta_i|\boldsymbol{x}$. Therefore

$$\boldsymbol{M}_{\theta_i|\boldsymbol{x}}(-a_i) = \exp\left[-a_i\boldsymbol{d}_i'\boldsymbol{x} + \frac{1}{2}a_i^2\tau^2\right], \quad i = 1,\ldots,p\,. \tag{2.4}$$

Hence, combining (2.3) and (2.4) we get

$$\begin{aligned}\delta_{\mathbf{GB}}^i(\boldsymbol{x}) &= \boldsymbol{d}_i'\boldsymbol{x} - \frac{1}{2}a_i\tau^2 \\ &= \frac{1}{2}x_i + \frac{1}{2}\bar{x} - \frac{1}{2}a_i\tau^2, \quad i = 1,\ldots,p\,,\end{aligned} \tag{2.5}$$

where $\bar{x} = (1/p)\sum_{i=1}^{p} x_i$.

In the terminology of Lindley and Smith (1972), $\boldsymbol{\delta}_{\mathbf{GB}}(\boldsymbol{x})$ is a three-stage Bayes estimate. The estimate can also be given by an interesting empirical Bayes interpretation. If, for example, μ were known, then the Bayes estimate of $\boldsymbol{\theta}$ under LINEX loss is

$$\hat{\theta}_i(\mu) = \frac{1}{2}x_i + \frac{1}{2}\mu - \frac{1}{2}a_i\tau^2, \quad i = 1,\ldots,p\,. \tag{2.6}$$

If, however, μ is an unknown parameter, it can be estimated from the marginal distribution of $\boldsymbol{X}$, which in this case is $N_p(\mu\mathbf{1}_p, 2\boldsymbol{I}_p)$, using Lemma 2.1. Thus μ is estimated by $\hat{\mu} = \bar{x}$, and the empirical Bayes estimate of $\boldsymbol{\theta}$ under the LINEX loss is

$$\hat{\theta}_i(\hat{\mu}) = \frac{1}{2}x_i + \frac{1}{2}\bar{x} - \frac{1}{2}a_i\tau^2, \quad i = 1,\ldots,p\,, \tag{2.7}$$

which is the same as (2.5). The estimate $\hat{\theta}_i(\hat{\mu})$ is an example of a class of general empirical Bayes estimates proposed in Efron and Morris (1973).

3. Admissibility of $\boldsymbol{\delta}_{\mathbf{GB}}$

In this section we will prove that $\boldsymbol{\delta}_{\mathbf{GB}}$ is an admissible estimator of $\boldsymbol{\theta}$ w.r.t. the extended LINEX loss. The technique we are going to use is one due to Blyth (1951), who views the given estimator as the limit of a sequence of Bayes estimators with the difference of the Bayes risks converging to zero at an appropriate rate.

To do this, consider the sequence of priors $\{\xi_m\colon m \geq 1\}$ for $\boldsymbol{\theta}$, where $\xi_m \sim \boldsymbol{N}_p(\mathbf{0}, \boldsymbol{\Sigma}_m)$ with

$$\boldsymbol{\Sigma}_m^{-1} = \left(1 + \frac{1}{m}\right)\boldsymbol{I}_p - \frac{1}{p}\mathbf{1}_p\mathbf{1}_p' . \tag{3.1}$$

The implicit idea is to approximate the improper prior (1.1) by the proper prior ξ_m with ξ_m converging (as $m \to \infty$) to the improper prior (1.1) at an appropriate rate (see also Stein (1965)).

Now, using Lemma 2.1, it is easy to verify that the posterior distribution of $\boldsymbol{\theta}$ given $\boldsymbol{X} = \boldsymbol{x}$ is $N_p(\boldsymbol{D}_m\boldsymbol{x}, \boldsymbol{D}_m)$ where

$$\boldsymbol{D}_m^{-1} = \boldsymbol{I}_p + \boldsymbol{\Sigma}_m^{-1} = \boldsymbol{D}^{-1} + \frac{1}{m}\boldsymbol{I}_p . \tag{3.2}$$

Therefore, using (2.2)

$$\begin{aligned}\boldsymbol{D}_m &= \frac{m}{2m+1}\boldsymbol{I}_p + \frac{m^2}{p(2m+1)(m+1)}\mathbf{1}_p\mathbf{1}_p' \\ &= (\boldsymbol{d}_{1,m}', \ldots, \boldsymbol{d}_{p,m}')'\end{aligned} \tag{3.3}$$

and the marginal distribution of $\boldsymbol{X}$ is $N_p(\boldsymbol{0}, \boldsymbol{I}_p + \boldsymbol{\Sigma}_m)$. Hence, the Bayes estimate of $\boldsymbol{\theta}$, say $\boldsymbol{\delta}_{\mathbf{B}}^m$, under LINEX loss is

$$\begin{aligned}\delta_{\mathbf{B}}^{i,m}(\boldsymbol{x}) &= \boldsymbol{d}_{i,m}'\boldsymbol{x} - \frac{1}{2}a_i\tau_m^2 \\ &= \frac{m}{2m+1}x_i + \frac{m^2}{(2m+1)(m+1)}\bar{x} - \frac{1}{2}a_i\tau_m^2, \qquad i = 1, \ldots, p\end{aligned} \tag{3.4}$$

where $\bar{x} = (1/p)\sum_{i=1}^{p} x_i$ and $\tau_m^2 = m/(2m+1) + m^2/p(2m+1)(m+1)$.

Let $r_L(\xi, \boldsymbol{\delta})$ denote the Bayes risks of $\boldsymbol{\delta}$ w.r.t. the prior ξ under LINEX loss. Then the corresponding Bayes risks of $\boldsymbol{\delta}_{\mathbf{B}}^m$ and $\boldsymbol{\delta}_{\mathbf{GB}}$, respectively, are:

$$\begin{aligned} & r_L(\xi_m, \boldsymbol{\delta}_{\mathbf{B}}^m) \\ &= (-2\pi)^{-p}|\boldsymbol{D}_m|^{-1/2}|\boldsymbol{I}_p + \boldsymbol{\Sigma}_m|^{-1/2} \\ &\quad \times \int \cdots \int L(\boldsymbol{\Delta}) \exp\left[-\frac{1}{2}\left(\boldsymbol{\Delta} + \frac{1}{2}\tau_m^2\boldsymbol{a}\right)'\boldsymbol{D}_m^{-1}\left(\boldsymbol{\Delta} + \frac{1}{2}\tau_m^2\boldsymbol{a}\right)\right] \\ &\quad \times \exp\left[-\frac{1}{2}\boldsymbol{x}'(\boldsymbol{I}_p + \boldsymbol{\Sigma}_m)^{-1}\boldsymbol{x}\right] d\Delta_1 \cdots d\Delta_p dx_1 \cdots dx_p ,\end{aligned} \tag{3.5}$$

(details are given in the Appendix) and

$$(3.6)\quad r_L(\xi_m, \boldsymbol{\delta}_{\mathbf{GB}}) = (-2\pi)^{-p}|\boldsymbol{D}_m|^{-1/2}|\boldsymbol{I}_p + \boldsymbol{\Sigma}_m|^{-1/2}$$
$$\times \int \cdots \int \boldsymbol{L}(\boldsymbol{\Delta}) \exp\left[-\frac{1}{2}\left(\boldsymbol{\Delta} + \frac{1}{2}\tau^2\boldsymbol{a} + \boldsymbol{D}_m^*\boldsymbol{x}\right)' \boldsymbol{D}_m^{-1}\right.$$
$$\left.\times\left(\boldsymbol{\Delta} + \frac{1}{2}\tau^2\boldsymbol{a} + \boldsymbol{D}_m^*\boldsymbol{x}\right)\right]$$
$$\times \exp\left[-\frac{1}{2}\boldsymbol{x}'(\boldsymbol{I}_p + \boldsymbol{\Sigma}_m)^{-1}\boldsymbol{x}\right] d\Delta_1\cdots d\Delta_p dx_1\cdots dx_p\,.$$

Subtraction of (3.5) from (3.6) gives

$$(3.7)\quad r_L(\xi_m, \boldsymbol{\delta}_{\mathbf{GB}}) - r_L(\xi_m, \boldsymbol{\delta}_{\mathbf{B}}^m)$$
$$= (-2\pi)^{-p}|\boldsymbol{D}_m|^{-1/2}|\boldsymbol{I}_p + \boldsymbol{\Sigma}_m|^{-1/2}$$
$$\times \int \cdots \int \boldsymbol{L}(\boldsymbol{\Delta}) \exp\left[-\frac{1}{2}\boldsymbol{x}'\left(\boldsymbol{I}_p + \boldsymbol{\Sigma}_m\right)^{-1}\boldsymbol{x}\right]$$
$$\times\left\{\exp\left[-\frac{1}{2}\left(\boldsymbol{\Delta} + \frac{1}{2}\tau^2\boldsymbol{a} + \boldsymbol{D}_m^*\boldsymbol{x}\right)' \boldsymbol{D}_m^{-1}\right.\right.$$
$$\left.\times\left(\boldsymbol{\Delta} + \frac{1}{2}\tau^2\boldsymbol{a} + \boldsymbol{D}_m^*\boldsymbol{x}\right)\right]$$
$$\left.- \exp\left[-\frac{1}{2}\left(\boldsymbol{\Delta} + \frac{1}{2}\tau_m^2\boldsymbol{a}\right)' \boldsymbol{D}_m^{-1}\left(\boldsymbol{\Delta} + \frac{1}{2}\tau_m^2\boldsymbol{a}\right)\right]\right\}$$
$$\times\, d\Delta_1\cdots d\Delta_p dx_1\cdots dx_p\,.$$

Finally, to prove admissibility of $\boldsymbol{\delta}_{\mathbf{GB}}$ in estimating $\boldsymbol{\theta}$ under the LINEX loss function, suppose $\boldsymbol{\delta}_{\mathbf{GB}}$ is dominated by some estimate $\boldsymbol{\delta}(\boldsymbol{x})$ of $\boldsymbol{\theta}$. Using the continuity of the risk function in $\boldsymbol{\theta}$ for an estimator $\boldsymbol{\delta}(\boldsymbol{X})$, it follows that there exists some $\boldsymbol{\theta}_0$, $\varepsilon > 0$ and $\zeta > 0$ such that

$$(3.8)\quad r_L(\xi_m, \boldsymbol{\delta}_{\mathbf{GB}}) - r_L(\xi_m, \boldsymbol{\delta}) \geq \varepsilon[\xi_m(\boldsymbol{\theta}_0 + \zeta\mathbf{1}_p) - \xi_m(\boldsymbol{\theta}_0 - \zeta\mathbf{1}_p)]$$
$$= \varepsilon(2\pi)^{-p/2}|\boldsymbol{\Sigma}_m|^{-1/2}$$
$$\times \int_{\boldsymbol{\theta}_0-\zeta\mathbf{1}_p}^{\boldsymbol{\theta}_0+\zeta\mathbf{1}_p} \cdots \int \exp\left[-\frac{1}{2}\boldsymbol{\theta}'\boldsymbol{\Sigma}_m^{-1}\boldsymbol{\theta}\right] d\theta_1\cdots d\theta_p$$
$$\geq \varepsilon|\boldsymbol{\Sigma}_m|^{-1/2}(2\pi)^{-p/2}$$
$$\times \int_{\boldsymbol{\theta}_0-\zeta\mathbf{1}_p}^{\boldsymbol{\theta}_0+\zeta\mathbf{1}_p} \cdots \int \exp\left[-\frac{1}{2}\boldsymbol{\theta}'\boldsymbol{\Sigma}_1^{-1}\boldsymbol{\theta}\right] d\theta_1\cdots d\theta_p$$
$$= \boldsymbol{K}|\boldsymbol{\Sigma}_m|^{-1/2}\,.$$

Since for $m \geq 1$, $\boldsymbol{\theta}'\boldsymbol{\Sigma}_m^{-1}\boldsymbol{\theta} < \boldsymbol{\theta}'\boldsymbol{\Sigma}_1^{-1}\boldsymbol{\theta} = \boldsymbol{\theta}'\boldsymbol{D}^{-1}\boldsymbol{\theta}$ and $\boldsymbol{K}$ is a positive constant not depending on m. Hence from (3.7) and (3.8),

$$(3.9) \qquad \boldsymbol{I}_m = \frac{r_L(\xi_m, \boldsymbol{\delta}_{\mathbf{GB}}) - r_L(\xi_m, \boldsymbol{\delta}_{\mathbf{B}}^m)}{r_L(\xi_m, \boldsymbol{\delta}_{\mathbf{GB}}) - r_L(\xi_m, \boldsymbol{\delta})}$$
$$\leq \boldsymbol{K}^{-1}(-2\pi)^{-p}|\boldsymbol{\Sigma}_m|^{1/2}|\boldsymbol{D}_m|^{-1/2}|\boldsymbol{I}_p + \boldsymbol{\Sigma}_m|^{-1/2}$$
$$\times \int \cdots \int \cdots \, d\Delta_1 \cdots d\Delta_p dx_1 \cdots dx_p \, .$$

Now,

$$|\boldsymbol{\Sigma}_m|^{1/2} = \frac{m^{1/2}}{\left(1 + \frac{1}{m}\right)^{(p-1)/2}} \qquad \text{and}$$

$$|\boldsymbol{D}_m|^{-1/2} = \left(1 + \frac{1}{m}\right)^{1/2}\left(2 + \frac{1}{m}\right)^{(p-1)/2} .$$

Also, using (2.2),

$$\boldsymbol{\Sigma}_m = \frac{m}{m+1}\boldsymbol{I}_p + \frac{m^2}{p(m+1)}\mathbf{1}_p\mathbf{1}_p' \qquad \text{and}$$

$$|\boldsymbol{I}_p + \boldsymbol{\Sigma}_m|^{-1/2} = (m+1)^{-1/2}\left(\frac{m+1}{2m+1}\right)^{(p-1)/2} .$$

Hence, (3.9) becomes

$$(3.10) \quad \boldsymbol{I}_m \leq \boldsymbol{K}_1 \int \cdots \int \boldsymbol{L}(\Delta) \exp\left[-\frac{1}{2}\boldsymbol{x}'(\boldsymbol{I}_p + \boldsymbol{\Sigma}_m)^{-1}\boldsymbol{x}\right] \exp\left[-\frac{1}{2}\boldsymbol{\Delta}'\boldsymbol{D}_m^{-1}\boldsymbol{\Delta}\right]$$
$$\times \left\{ \exp\left[-\boldsymbol{\Delta}'\boldsymbol{D}_m^{-1}\left(\frac{1}{2}\tau^2\boldsymbol{a} + \boldsymbol{D}_m^*\boldsymbol{x}\right)\right.\right.$$
$$\left. - \frac{1}{2}\left(\frac{1}{2}\tau^2\boldsymbol{a} + \boldsymbol{D}_m^*\boldsymbol{x}\right)'\boldsymbol{D}_m^{-1}\left(\frac{1}{2}\tau^2\boldsymbol{a} + \boldsymbol{D}_m^*\boldsymbol{x}\right)\right]$$
$$\left. - \exp\left[-\frac{1}{2}\tau_m^2\boldsymbol{\Delta}'\boldsymbol{D}_m^{-1}\boldsymbol{a} - \frac{1}{8}\tau_m^4\boldsymbol{a}'\boldsymbol{D}_m^{-1}\boldsymbol{a}\right]\right\}$$
$$\times \, d\Delta_1 \cdots d\Delta_p dx_1 \cdots dx_p \, .$$

Note that as $m \to \infty$, $\{\cdots\} \to 0$ and

$$(3.11)\qquad |\{\cdots\}| \le 2 + \sum_{r=1}^{\infty} \left(\frac{2^{r-1}+1}{2^r r!} \right) \tau^{2r} |\boldsymbol{\Delta}' \boldsymbol{D}_m^{-1} \boldsymbol{a}|^r + \sum_{r=1}^{\infty} \frac{2^{r-1}}{r!} \left| \left(\boldsymbol{\Delta} + \frac{\tau^2}{2} \boldsymbol{a} \right)' \boldsymbol{D}_m^{-1} \boldsymbol{D}_m^* \boldsymbol{x} \right|^r$$

(details are given in the Appendix).

Using the notation $\boldsymbol{A} \le \boldsymbol{B}$ if and only if $\boldsymbol{B} - \boldsymbol{A}$ is non-negative definite, and in consideration of (2.1) and (3.2), we conclude that,

$$\boldsymbol{D}_m^{-1} \boldsymbol{D}_m^* = \frac{1}{m} \boldsymbol{D}, \qquad \boldsymbol{D} \le \boldsymbol{I}_p, \qquad \boldsymbol{D}_m^{-1} \le \left(2 + \frac{1}{m} \right) \boldsymbol{I}_p .$$

Hence,

$$(3.12)\qquad \left| \left(\boldsymbol{\Delta} + \frac{\tau^2}{2} \boldsymbol{a} \right)' \boldsymbol{D}_m^{-1} \boldsymbol{D}_m^* \boldsymbol{x} \right|^2 = \frac{1}{m^2} \left| \left(\boldsymbol{\Delta} + \frac{\tau^2}{2} \boldsymbol{a} \right)' \boldsymbol{D} \boldsymbol{x} \right|^2 \le \frac{1}{m^2} \left| \left(\boldsymbol{\Delta} + \frac{\tau^2}{2} \boldsymbol{a} \right)' \boldsymbol{D} \left(\boldsymbol{\Delta} + \frac{\tau^2}{2} \boldsymbol{a} \right) \right|^2 |\boldsymbol{x}' \boldsymbol{D} \boldsymbol{x}|^2$$

(Cauchy-Shwarz inequality)

$$\le \frac{1}{m^2} \left\| \boldsymbol{\Delta} + \frac{\tau^2}{2} \boldsymbol{a} \right\|^2 \|\boldsymbol{x}\|^2 \le \frac{1}{m_0^2} \left\| \boldsymbol{\Delta} + \frac{\tau^2}{2} \boldsymbol{a} \right\|^2 \|\boldsymbol{x}\|^2 \quad \forall m \ge m_0$$

and

$$(3.13)\qquad |\boldsymbol{\Delta}' \boldsymbol{D}_m^{-1} \boldsymbol{a}|^2 \le |\boldsymbol{\Delta}' \boldsymbol{D}_m^{-1} \boldsymbol{\Delta}|^2 |\boldsymbol{a}' \boldsymbol{D}_m^{-1} \boldsymbol{a}|^2 \le \left(2 + \frac{1}{m} \right)^4 \|\boldsymbol{\Delta}\|^2 \|\boldsymbol{a}\|^2 \le \left(2 + \frac{1}{m_0} \right)^4 \|\boldsymbol{\Delta}\|^2 \|\boldsymbol{a}\|^2 \qquad \forall\, m \ge m_0 .$$

Also,

$$(3.14)\qquad \exp\left[-\frac{1}{2} \boldsymbol{\Delta}' \boldsymbol{D}_m^{-1} \boldsymbol{\Delta} \right] \le \exp\left(-\frac{1}{2} \boldsymbol{\Delta}' \boldsymbol{\Delta} \right)$$

and

$$\exp\left[-\frac{1}{2}\boldsymbol{x}'(\boldsymbol{I}_p+\boldsymbol{\Sigma}_m)^{-1}\boldsymbol{x}\right]\le\exp\left(-\frac{1}{2}\boldsymbol{x}'\boldsymbol{x}\right). \tag{3.15}$$

Therefore, by consideration of (3.11)–(3.15) and using the dominated convergence theorem we conclude that

$$\boldsymbol{I}_m\to 0 \quad \text{as} \quad m\to\infty .$$

Hence, there exists an m_0 such that $\boldsymbol{I}_{m_0}<1$, i.e.,

$$r(\xi_{m_0},\boldsymbol{\delta})<r(\xi_{m_0},\boldsymbol{\delta}_{\mathbf{B}}^{m_0})$$

which contradicts the Bayesness of $\boldsymbol{\delta}_{\mathbf{B}}^{m_0}$ w.r.t. ξ_{m_0}. This proves the admissibility of $\boldsymbol{\delta}_{\mathbf{GB}}$ w.r.t. the extended LINEX loss function.

Remark 1. If instead, we start with $\boldsymbol{X}\sim N_p(\boldsymbol{\theta},\sigma^2\boldsymbol{I}_p)$ and conditional on μ, $\boldsymbol{\theta}\sim N_p(\mu\mathbf{1}_p,r^2\boldsymbol{I}_p)$ and μ had the uniform (improper) prior distribution where σ^2, $r^2>0$ are both known, then the (improper) Bayes estimate (three-stage Bayes estimate) of θ_i, $1\le i\le p$, is given by

$$\frac{x_i+\lambda\bar{x}}{1+\lambda}-\frac{a_i}{1+\lambda}\sigma^2\tau^2 \qquad i=1,\dots,p$$

with $\lambda=\sigma^2/r^2$, and the admissibility of such an estimator can be proved in the same way as in the special case $\sigma^2=r^2=1$.

Remark 2. One can generalize the model in Remark 1 to $\boldsymbol{X}\sim N_p(\boldsymbol{\theta},\boldsymbol{\Sigma}_1)$ where conditional on μ, $\boldsymbol{\theta}\sim N_p(\mu\mathbf{1}_p,\boldsymbol{\Sigma}_2)$ and μ has the improper uniform prior distribution over the entire real line, and find the three-stage (or empirical Bayes) estimate of $\boldsymbol{\theta}$ as described in Section 2 when $\boldsymbol{\Sigma}_1$ and $\boldsymbol{\Sigma}_2$ are known.

Remark 3. Instead of considering the model in Remark 2, one can consider the model given $\boldsymbol{\theta}$, $\boldsymbol{X}\sim N_p(\boldsymbol{\theta},\boldsymbol{\Sigma}_1)$, given $\boldsymbol{\mu}$, $\boldsymbol{\theta}\sim N_p(\boldsymbol{\mu},\boldsymbol{\Sigma}_2)$ and given $\boldsymbol{\eta}$, $\boldsymbol{\mu}\sim N_p(\boldsymbol{\eta},\boldsymbol{\Sigma}_3)$, where the variance-convariance matrices $\boldsymbol{\Sigma}_1$, $\boldsymbol{\Sigma}_2$ and $\boldsymbol{\Sigma}_3$ are known and are positive definite. Then using Lemma 2.1 and the extended LINEX loss function, one can obtain the unique proper Bayes estimates of $\boldsymbol{\theta}$ and such estimates are, therefore, admissible.

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Appendix

To see (3.5) and (3.6), note that

$$
\begin{aligned}
& r_L(\xi_m, \boldsymbol{\delta}_{\mathbf{B}}^m) \\
&\quad = \int \cdots \int \boldsymbol{L}(\boldsymbol{\theta}, \boldsymbol{\delta}_{\mathbf{B}}^m)(2\pi)^{-p/2}|\boldsymbol{D}_m|^{-1/2} \exp\left[-\frac{1}{2}(\boldsymbol{\theta} - \boldsymbol{D}_m\boldsymbol{x})'\boldsymbol{D}_m^{-1}(\boldsymbol{\theta} - \boldsymbol{D}_m\boldsymbol{x})\right] \\
&\qquad\qquad \times (2\pi)^{-p/2}|\boldsymbol{I}_p + \boldsymbol{\Sigma}_m|^{-1/2} \\
&\qquad\qquad \times \exp\left[-\frac{1}{2}\boldsymbol{x}'(\boldsymbol{I}_p + \boldsymbol{\Sigma}_m)^{-1}\boldsymbol{x}\right] d\theta_1\cdots d\theta_p dx_1\cdots dx_p \\
&\quad = (2\pi)^{-p}|\boldsymbol{D}_m|^{-1/2}|\boldsymbol{I}_p + \boldsymbol{\Sigma}_m|^{-1/2} \\
&\qquad \times \int \cdots \int \boldsymbol{L}(\boldsymbol{\theta}, \boldsymbol{\delta}_{\mathbf{B}}^m) \exp\left[-\frac{1}{2}(\boldsymbol{\theta} - \boldsymbol{D}_m\boldsymbol{x})'\boldsymbol{D}_m^{-1}(\boldsymbol{\theta} - \boldsymbol{D}_m\boldsymbol{x})\right] \\
&\qquad\qquad \times \exp\left[-\frac{1}{2}\boldsymbol{x}'(\boldsymbol{I}_p + \boldsymbol{\Sigma}_m)^{-1}\boldsymbol{x}\right] d\theta_1\cdots d\theta_p dx_1\cdots dx_p .
\end{aligned}
$$

Let $\boldsymbol{\Delta} = \boldsymbol{\delta}_{\mathbf{B}}^m - \boldsymbol{\theta} = (\Delta_1,\ldots,\Delta_p)'$ where $\Delta_i = \delta_{\mathbf{B}}^{i,m}(\boldsymbol{x}) - \theta_i$. Then

$$\Delta_i + \frac{1}{2} a_i\tau_m^2 = \boldsymbol{d}_{i,m}'\boldsymbol{x} - \theta_i, \qquad i = 1,\ldots,p$$

or

$$\boldsymbol{\Delta} + \frac{1}{2}\tau_m^2\boldsymbol{a} = \boldsymbol{D}_m\boldsymbol{x} - \boldsymbol{\theta}$$

with $\boldsymbol{a} = (a_1,\ldots,a_p)'$. Hence, (3.5).

$$\begin{aligned}
&r_L(\xi_m, \boldsymbol{\delta}_{\mathrm{GB}}) \\
&= \int \cdots \int L(\boldsymbol{\theta}, \boldsymbol{\delta}_{\mathrm{GB}})(2\pi)^{-p/2}|\boldsymbol{D}_m|^{-1/2} \exp\left[-\frac{1}{2} (\boldsymbol{\theta} - \boldsymbol{D}_m\boldsymbol{x})'\boldsymbol{D}_m^{-1}(\boldsymbol{\theta} - \boldsymbol{D}_m\boldsymbol{x}) \right] \\
&\quad \times (2\pi)^{-p/2}|\boldsymbol{I}_p + \boldsymbol{\Sigma}_m|^{-1/2} \\
&\quad \times \exp\left[-\frac{1}{2} \boldsymbol{x}'(\boldsymbol{I}_p + \boldsymbol{\Sigma}_m)^{-1}\boldsymbol{x} \right] d\theta_1 \cdots d\theta_p dx_1 \cdots dx_p \\
&= (2\pi)^{-p}|\boldsymbol{D}_m|^{-1/2}|\boldsymbol{I}_p + \boldsymbol{\Sigma}_m|^{-1/2} \\
&\quad \times \int \cdots \int L(\boldsymbol{\theta}, \boldsymbol{\delta}_{\mathrm{GB}}) \\
&\quad \times \exp\left[-\frac{1}{2} (\boldsymbol{\theta} - \boldsymbol{D}_m\boldsymbol{x})'\boldsymbol{D}_m^{-1}(\boldsymbol{\theta} - \boldsymbol{D}_m\boldsymbol{x}) \right] \\
&\quad \times \exp\left[-\frac{1}{2} \boldsymbol{x}'(\boldsymbol{I}_p + \boldsymbol{\Sigma}_m)^{-1}\boldsymbol{x} \right] d\theta_1 \cdots d\theta_p dx_1 \cdots dx_p \,.
\end{aligned}$$

Let $\boldsymbol{\Delta} = \boldsymbol{\delta}_{\mathrm{GB}} - \boldsymbol{\theta} = (\Delta_1, \ldots, \Delta_p)'$ with $\Delta_i = \delta^i_{\mathrm{GB}}(\boldsymbol{x}) - \theta_i$. Then

$$\Delta_i = \boldsymbol{d}'_{i,m}\boldsymbol{x} - \theta_i + \boldsymbol{d}'_i\boldsymbol{x} - \boldsymbol{d}'_{i,m}\boldsymbol{x} - \frac{1}{2} a_i\tau^2$$

or

$$\boldsymbol{d}'_{i,m}\boldsymbol{x} - \theta_i = \Delta_i + \frac{1}{2} a_i\tau^2 + \boldsymbol{d}^{*\prime}_{i,m}\boldsymbol{x}, \qquad i = 1, \ldots, p$$

with $\boldsymbol{d}^*_{i,m} = \boldsymbol{d}_{i,m} - \boldsymbol{d}_i$. Taking $\boldsymbol{D}^*_m = \boldsymbol{D}_m - \boldsymbol{D}$, we conclude (3.6).

To see (3.11), note that

$$\begin{aligned}
|\{\cdots\}| &\le \exp\Bigg[-\frac{1}{2} \tau^2\boldsymbol{\Delta}'\boldsymbol{D}_m^{-1}\boldsymbol{a} - \boldsymbol{\Delta}'\boldsymbol{D}_m^{-1}\boldsymbol{D}^*_m\boldsymbol{x} \\
&\qquad - \frac{\tau^2}{2} \boldsymbol{a}'\boldsymbol{D}_m^{-1}\boldsymbol{D}^*_m\boldsymbol{x} - \frac{1}{2} \boldsymbol{x}'\boldsymbol{D}^*_m\boldsymbol{D}_m^{-1}\boldsymbol{D}^*_m\boldsymbol{x} - \frac{\tau^4}{8} \boldsymbol{a}'\boldsymbol{D}_m^{-1}\boldsymbol{a} \Bigg] \\
&\quad + \exp\left[-\frac{1}{2} \tau_m^2\boldsymbol{\Delta}'\boldsymbol{D}_m^{-1}\boldsymbol{a} - \frac{1}{8} \tau_m^4\boldsymbol{a}'\boldsymbol{D}_m^{-1}\boldsymbol{a} \right] \\
&\le \exp\left[-\frac{\tau^2}{2} \boldsymbol{\Delta}'\boldsymbol{D}_m^{-1}\boldsymbol{a} - \boldsymbol{\Delta}'\boldsymbol{D}_m^{-1}\boldsymbol{D}^*_m\boldsymbol{x} - \frac{\tau^2}{2} \boldsymbol{a}'\boldsymbol{D}_m^{-1}\boldsymbol{D}^*_m\boldsymbol{x} \right] \\
&\quad + \exp\left[-\frac{\tau_m^2}{2} \boldsymbol{\Delta}'\boldsymbol{D}_m^{-1}\boldsymbol{a} \right]
\end{aligned}$$

$$= \exp\left[-\frac{\tau^2}{2} \boldsymbol{\Delta}' \boldsymbol{D}_m^{-1} \boldsymbol{a} - \left(\boldsymbol{\Delta} + \frac{\tau^2}{2} \boldsymbol{a} \right)' \boldsymbol{D}_m^{-1} \boldsymbol{D}_m^* \boldsymbol{x} \right]$$

$$+ \exp\left[-\frac{\tau_m^2}{2} \boldsymbol{\Delta}' \boldsymbol{D}_m^{-1} \boldsymbol{a} \right]$$

$$= \sum_{r=0}^{\infty} \left[-\frac{\tau^2}{2} \boldsymbol{\Delta}' \boldsymbol{D}_m^{-1} \boldsymbol{a} - \left(\boldsymbol{\Delta} + \frac{\tau^2}{2} \boldsymbol{a} \right)' \boldsymbol{D}_m^{-1} \boldsymbol{D}_m^* \boldsymbol{x} \right]^r \Big/ r!$$

$$+ \sum_{r=0}^{\infty} \left(-\frac{\tau_m^2}{2} \boldsymbol{\Delta}' \boldsymbol{D}_m^{-1} \boldsymbol{a} \right)^r \Big/ r!$$

$$\leq 2 + \sum_{r=1}^{\infty} 2^{r-1} \left[\frac{\tau^{2r}}{2^r} |\boldsymbol{\Delta}' \boldsymbol{D}_m^{-1} \boldsymbol{a}|^r + \left| \left(\boldsymbol{\Delta} + \frac{\tau^2}{2} \boldsymbol{a} \right)' \boldsymbol{D}_m^{-1} \boldsymbol{D}_m^* \boldsymbol{x} \right|^r \right] \Big/ r!$$

$$+ \sum_{r=1}^{\infty} \frac{\tau_m^{2r}}{2^r} |\boldsymbol{\Delta}' \boldsymbol{D}_m^{-1} \boldsymbol{a}|^r / r!$$

$$= 2 + \sum_{r=1}^{\infty} \left(\frac{\tau^{2r}}{2r!} + \frac{\tau_m^{2r}}{2^r r!} \right) |\boldsymbol{\Delta}' \boldsymbol{D}_m^{-1} \boldsymbol{a}|^r$$

$$+ \sum_{r=1}^{\infty} \frac{2^{r-1}}{r!} \left| \left(\boldsymbol{\Delta} + \frac{\tau^2}{2} \boldsymbol{a} \right)' \boldsymbol{D}_m^{-1} \boldsymbol{D}_m^* \boldsymbol{x} \right|^r$$

$$\leq 2 + \sum_{r=1}^{\infty} \left(\frac{2^{r-1} + 1}{2^r r!} \right) \tau^{2r} |\boldsymbol{\Delta}' \boldsymbol{D}_m^{-1} \boldsymbol{a}|^r$$

$$+ \sum_{r=1}^{\infty} \frac{2^{r-1}}{r!} \left| \left(\boldsymbol{\Delta} + \frac{\tau^2}{2} \boldsymbol{a} \right)' \boldsymbol{D}_m^{-1} \boldsymbol{D}_m^* \boldsymbol{x} \right|^r .$$

Using the fact that $\tau_m^2 \leq \tau^2$, $\boldsymbol{x}' \boldsymbol{D}_m^* \boldsymbol{D}_m^{-1} \boldsymbol{D}_m^* \boldsymbol{x} \geq 0$ and $\boldsymbol{a}' \boldsymbol{D}_m^{-1} \boldsymbol{a} \geq 0 \ \forall m \geq 1$.

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