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HITTING STRAIGHT LINES BY COMPOUND POISSON PROCESS PATHS[+]

WOLFGANG J. BÜHLER[1*], PREM S. PURI[2**] AND HANS-J. SCHUH[1]

[1]*Fachbereich 17-Mathematik, Johannes Gutenberg-Universität Mainz, Postfach 3980, Saarstrasse 21, D-6500 Mainz, F. R. Germany*
[2]*Department of Statistics, Purdue University, West Lafayette, Indiana 47907, U.S.A.*

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Abstract. In a recent article Mallows and Nair (1989, *Ann. Inst. Statist. Math.*, **41**, 1–8) determined the probability of intersection $P\{X(t) = \alpha t$ for some $t \geq 0\}$ between a compound Poisson process $\{X(t), t \geq 0\}$ and a straight line through the origin. Using four different approaches (direct probabilistic, via differential equations and via Laplace transforms) we extend their results to obtain the probability of intersection between $\{X(t), t \geq 0\}$ and arbitrary lines. Also, we display a relationship with the theory of Galton-Watson processes. Additional results concern the intersections with two (or more) parallel lines.

Key words and phrases: Compound Poisson processes, intersection with lines, transition probabilities, Laplace transforms, Galton-Watson processes.

1. Introduction

This paper was inspired by a recent article by Mallows and Nair (1989). Mallows and Nair considered the probability $\theta := P\{X(t) = t$ for some $t, 0 < t < \infty\}$ where $\{X(t), t \geq 0\}$ is a compound Poisson process allowing positive jumps only. Let λ be the intensity of the Poisson process and H be the common distribution function of the jump sizes. Further let $\varphi(u) = \int_0^\infty \exp(-ux)dH(x)$, $A := \int_0^\infty xdH(x)$ and $\phi(u) = \lambda[1 - \varphi(u)]$. Mallows and Nair (1987) show that $\theta = \phi'(\omega)$ where ω is the largest nonnegative root of the equation $\phi(u) = u$.

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[+]Our coauthor and friend Prem Singh Puri died on August 12, 1989. We dedicate our contribution to this paper to his memory.

Remark. Zero is always a root of $\phi(u) = u$ and $\phi'(0) = \rho$ where $\rho := EX(1)$. (Here and in the sequel we use the notation " $:=$ " for a defining equation.) If $\rho \leq 1$, then it is the only nonnegative root and $\theta = \rho$, a well known result (see Takacs (1967)). In the case $\rho > 1$ there is an additional positive root ω and $\theta = \phi'(\omega) < 1$.

Consider the case with integer jumps only. Then the quantity $q = e^{-\omega}$ is the smallest fixed point in $[0, 1]$ of the probability generating function $f(s) = \exp\{-\phi(-\log s)\}$. Thus q can be viewed as the probability of extinction of a Galton-Watson process governed by the probability generating function f. This interpretation led us to the search for a method of proof displaying and using a relationship with the theory of Galton-Watson processes. This search led us to investigate three different approaches to the problem, one probabilistic, one using differential equalities and finally a method using Laplace transforms. All of these approaches embed the problem in the more general setting of considering the intersection with lines not necessarily passing through the origin, and in fact they are useful in extending Mallows and Nair's (1989) result to this case. However, neither of these methods solves the original problem nor displays a connection with Galton-Watson processes. What we consider the final success of our attempts is contained in Sections 5 and 6. It uses generating functions to solve the original problem together with its extension to lines not passing through the origin. We originally found it via the Galton-Watson processes displayed in Section 7. It should be pointed out that this approach is not restricted to compound Poisson processes but proves the result for processes with stationary increments whose paths are nondecreasing step functions starting at 0. Mallows and Nair had conjectured the result to be true for this more general class of stochastic processes. We shall find it convenient to change the time scale in such a way that the Poisson intensity changes from λ to 1. Thus a line of slope α in our setup would have slope $\alpha\lambda$ in the Mallows and Nair version. Since their result is stated only for intersections with the diagonal, we should note that a different rescaling would show that their result remains true for the line $y = \alpha t$ if in the definition of ω and θ the function ϕ is replaced by $\Psi(u) = \phi(u)/\alpha\lambda$. We should thus keep in mind *that for a given process the value of θ in our case also depends on α.*

The form of our results and some of the techniques needed to obtain them will depend on the particular lines considered, the most important distinction being between "steep" lines $y = \alpha t + \beta$ with $\alpha > A$ and "flat" lines, $0 < \alpha < A$; this is not surprising since it corresponds to the difference between the above mentioned cases $\rho < 1$ having an elegant probabilistic proof and $\rho > 1$ where Mallows and Nair had to use analytical methods. In terms of Galton-Watson processes this is the distinction between subcriticality and supercriticality.

Consider α as fixed and let $h(\beta) = P(X(t) = \alpha t + \beta$ for some $t > 0)$. In Section 2 we shall use probabilistic methods to determine $h(\beta)$ explicitely for $\beta < 0$ and for $\alpha > 1$, $\beta > 0$. In the following section we shall establish that h satisfies a differential equation with retarded argument and give its solution. In these two sections, for the sake of clarity of exposition attention is restricted to the simple Poisson process without compounding. In Section 4 we find the Laplace transform $\hat{h}(s) = \int_0^\infty e^{-\beta s} h(\beta) d\beta$ for a Poisson process with compounding. Section 5 studies a discrete time version of our problem. Its solution is then applied to the continuous time case in the following section. This final solution was originally found via the interpretation in terms of Galton-Watson processes presented in Section 7. Finally, we investigated the intersections of the process with two parallel lines. The main idea of our approaches is to use the renewal property of the process under study. In parts of the paper we consider integer valued processes only. The extension to the nonlattice case is possible, as in the last section of Mallows and Nair (1989).

Remark. Mallows and Nair's result has an alternative elegant probabilistic proof valid for the general lattice case using a random walk associated with the process (cf. Remark 3 of Mallows and Nair (1989)).

2. A probabilistic approach

We are presenting this approach only for a Poisson process $\{N(t), t \geq 0\}$ without compounding, i.e. with $P_n(t) := P(N(t) = n) = (t^n/n!)e^{-t}$.

Excluding trivial cases consider first the probability $h(\beta)$ of intersection with the line $y = \alpha t + \beta$ with $\alpha < 0$ and $\beta > 0$. This line is intersected at the level n if and only if $t_n := (n - \beta)/\alpha > 0$ and $N(t_n) = n$. Since the values $\alpha t_n + \beta$ are decreasing and $N(t_n)$ nondecreasing the events $\{N(t_n) = n\}$ for different n are disjoint and

$$h(\beta) = \sum_{n=0}^{[\beta]} P_n(t_n) = \sum_{n=0}^{[\beta]} \frac{[(n-\beta)/\alpha]^n}{n!} e^{-(n-\beta)/\alpha} . \tag{2.1}$$

Next consider the cases with $\alpha > 0$ and $\beta < 0$. Taking two values $\beta_2 < \beta_1 < 0$ we see that line 2 can only be intersected after line 1. The renewal property of the process then yields $h(\beta_2) = h(\beta_1)h(\beta_2 - \beta_1)$ allowing $h(\beta) = \exp(b\beta)$ as the only measurable solutions. As h is a probability we must have $b \geq 0$. Conditioning on $N(-\beta/\alpha)$ leads to

$$h(\beta) = \sum_{r=0}^{\infty} \frac{(-\beta/\alpha)^r}{r!} e^{\beta/\alpha} h(-r) .$$

For $\beta = -1$, using $h(-r) = (h(-1))^r$, we obtain the equation

$$c := h(-1) = \exp(-1/\alpha) \sum_{r=0}^{\infty} (h(-1)/\alpha)^r / r! = \exp[(c-1)/\alpha] .$$

Thus $h(\beta) = \exp[(1-c)\beta/\alpha]$. The equation for c has the solution 1 and for $\alpha \neq 1$ an additional solution. In the case $0 < \alpha < 1$ it is clear that $h(\beta) < 1$ for $\beta < 0$ and thus that c must be the only solution less than unity. In the cases $\alpha \geq 1$, it is clear from the strong law of large numbers that $h(\beta) = 1$; also in these cases 1 is the only solution that is ≤ 1.

The measurability of h on $]-\infty, 0]$ which was needed to establish its exponential form follows from Lemma 2.1 which will again be used in the following section.

LEMMA 2.1. *Except possibly for a jump at $\beta = 0$ the function h is continuous.*

PROOF. Let $0 < \beta < \beta + \Delta$ or $\beta < \beta + \Delta < 0$. Then the event of intersecting one of the two lines (corresponding to β and to $\beta + \Delta$ resp.) but not the other can only occur when for some n the process has its n-th jump in the interval $[t_n - \Delta/\alpha, t_n]$ (jumping into the space between the two lines) or if it has its $(n+1)$-th jump in this time interval (jumping out of that space). As $\Delta \to 0$ the events A_Δ just described shrink to the event $A :=$ {the n-th or $(n+1)$-th jump occurs at time t_n for some n}. This latter event has probability zero and thus $|h(\beta + \Delta) - h(\beta)| \leq P(A_\Delta) \to 0$ as $\Delta \to 0$. Continuity from the left is established in the same manner.

In order to determine $h(\beta)$ for $\alpha > 1$ and $\beta > 0$ we note that the line $y = \alpha t + \beta$ will be intersected if and only if $N(t)$ ever reaches the level $\alpha t + \beta$. Therefore, the event of no intersection can be described by the validity of the inequalities:

$$N(t_j) \leq j, \qquad \text{for all } j \text{ for which} \quad t_j := (j - \beta)/\alpha > 0 .$$

Define in general for any $0 < y_1 < y_2 < \cdots$

$$\phi_j(y_1, y_2, \ldots) = P(N(y_1) \leq j, N(y_2) \leq j+1, \ldots) .$$

Then conditioning on $N(y_1)$ and using the independence and stationarity of increments of the process

$$\phi_j(y_1, y_2, \ldots) = \sum_{i=0}^{j} P(N(y_1) = i)\phi_{j-i+1}(y_2 - y_1, y_3 - y_1, \ldots) .$$

Denoting the generating function of the sequence $\phi_j, j = 0, 1, \ldots$ by H

$$\begin{aligned}
H(s \mid y_1, y_2, \ldots) &= \sum_{j=0}^{\infty} s^j \phi_j(y_1, y_2, \ldots) \\
&= \sum_{i=0}^{\infty} P_i(y_1) s^i \sum_{j=i}^{\infty} \phi_{j-i+1}(y_2 - y_1, y_3 - y_1, \ldots) s^{j-i} \\
&= \sum_{i=0}^{\infty} P_i(y_1) s^i \left\{ \sum_{l=0}^{\infty} \phi_l(y_2 - y_1, y_3 - y_1, \ldots) s^l \right. \\
&\qquad \left. - \phi_0(y_2 - y_1, y_3 - y_1, \ldots) \right\} \frac{1}{s} \\
&= \frac{1}{s} e^{y_1(s-1)} \{ H(s \mid y_2 - y_1, y_3 - y_1, \ldots) \\
&\qquad - \phi_0(y_2 - y_1, y_3 - y_1, \ldots) \} .
\end{aligned}$$

We know $\phi_0(1/\alpha, 2/\alpha, \ldots)$ to be equal to the probability $1 - \theta$ of never intersecting the line $y = \alpha t$. Thus for the above sequence of t-values denoting the fractional part of β by $\langle \beta \rangle := \beta - [\beta]$ we obtain

$$\begin{aligned}
&H\left(s \mid \frac{1 - \langle \beta \rangle}{\alpha}, \frac{2 - \langle \beta \rangle}{\alpha}, \ldots \right) \\
&\quad = \frac{1}{s} \exp\left(\frac{1 - \langle \beta \rangle}{\alpha} (s - 1) \right) \left\{ H\left(s \mid \frac{1}{\alpha}, \frac{2}{\alpha}, \ldots \right) - (1 - \theta) \right\} .
\end{aligned}$$

In particular for integer β this reduces to an equation for $H(s \mid 1/\alpha, 2/\alpha, \ldots)$ with solution $H(s \mid 1/\alpha, 2/\alpha, \ldots) = (1 - \theta)\{1 - s \exp((1 - s)/\alpha)\}^{-1}$. Consequently,

$$H\left(s \mid \frac{1 - \langle \beta \rangle}{\alpha}, \frac{2 - \langle \beta \rangle}{\alpha}, \ldots \right) = \frac{(1 - \theta) \exp\{ -(1 - \langle \beta \rangle)(1 - s)/\alpha \}}{\exp\{ -(1 - s)/\alpha \} - s} .$$

The following result is established by evaluating

$$\begin{aligned}
1 - h(\beta) &= \phi_{[\beta]}\left(\frac{1 - \langle \beta \rangle}{\alpha}, \frac{2 - \langle \beta \rangle}{\alpha}, \ldots \right) \\
&= \frac{\partial^{[\beta]}}{\partial s^{[\beta]}} H\left(s \mid \frac{1 - \langle \beta \rangle}{\alpha}, \ldots \right) \Bigg|_{s=0} .
\end{aligned}$$

THEOREM 2.1. *For $\alpha > 1$ we have $h(\beta) = 1$ if $\beta < 0$ and if $\beta > 0$*

$$(2.2)\qquad h(\beta)=1-(1-\theta)e^{\beta/\alpha}\sum_{j=0}^{[\beta]}\frac{(-1)^j}{j!}\left\{\frac{(\beta-j)}{\alpha\exp(1/\alpha)}\right\}^j .$$

Unfortunately for $0<\alpha<1$ *the opportunity for the process to never intersect the line* $\alpha t+\beta$ *by jumping above it and staying above forever seems to introduce sufficient complications to make it intractible by probabilistic methods similar to the above.*

3. Using differential equations

In this section we still restrict our attention to the simple Poisson process. The reader may convince himself that at least the extension to integer valued compound Poisson processes is an easy exercise.

We condition on the time of the first jump and thus obtain for arbitrary $\alpha>0$ and $\beta\neq 0$

$$(3.1)\qquad h(\beta)=\begin{cases}\int_0^\infty e^{-t}h(\alpha t+\beta-1)dt & \text{for}\quad \beta>0\\[2ex] e^{\beta/\alpha}+\int_0^{-\beta/\alpha}e^{-t}h(\alpha t+\beta-1)dt & \text{for}\quad \beta<0\,.\end{cases}$$

By substituting $\alpha t+\beta-1=u$ this equation is transformed into

$$(3.2)\qquad h(\beta)=\begin{cases}\frac{1}{\alpha}\int_{\beta-1}^\infty e^{-(u-\beta+1)/\alpha}h(u)du & \text{for}\quad \beta>0\\[2ex] \frac{1}{\alpha}\int_{\beta-1}^\infty e^{-(u-\beta+1)/\alpha}h(u)du+e^{\beta/\alpha} & \text{for}\quad \beta<0\,.\end{cases}$$

Lemma 2.1 allows us to differentiate (3.2) with respect to the limits of integration for all $\beta\neq 0$ which after some manipulation leads to

$$(3.3)\qquad h'(\beta)=\frac{1}{\alpha}\{h(\beta)-h(\beta-1)\},\qquad \text{for}\quad \beta\neq 0\,.$$

Let us first consider (3.3) on $]-\infty,0[$. From this equation it is immediately clear that $g(\beta):=h(\beta-1)$ also defines a solution if h is a solution. For $\alpha<1$ it is easy to verify that $h(\beta)$ has to vanish at $-\infty$ and that except for a multiplicative constant (3.3) allows at most one solution with this property. Hence $h(\beta-1)=g(\beta)=ch(\beta)$ which changes (3.3) into $h'(\beta)=(1-c)/\alpha\cdot h(\beta)$ for $\beta<0$ whose solution with $h(0-)=1$ is $h(\beta)=\exp\{(1-c)/\alpha\cdot\beta\}$. The relation $h(\beta-1)=ch(\beta)$ yields for $\beta\nearrow 0$ the value of c as the solution less than one of $c=\exp\{(c-1)/\alpha\}$. For $\alpha>1$, of course, $c = 1$ is the only solution and $h(\beta) \equiv 1$ for $\beta < 0$. This confirms the corresponding solution found in Section 2 by probabilistic methods.

For positive β we can now solve (3.3) recursively through intervals of length one. Using that for $n \leq \beta \leq n+1$ the function $g(\beta) = h(\beta - 1)$ is already known and that we know the initial value $h(n)$ (for $n = 0$ it is the value θ and for $n = 1, 2, \ldots$ it is determined in the previous step) we have thus shown.

THEOREM 3.1. *For negative β we have $h(\beta) = 1$ if $\alpha > 1$ and $h(\beta) = \exp\{(c-1)\beta/\alpha\}$ if $0 < \alpha < 1$. Furthermore for all $\alpha \neq 1$, $h(0) = \theta$ and for $0 \leq n \leq \beta \leq n+1$*

$$h(\beta) = \exp(\beta/\alpha)\left\{ h(n)\exp(-n/\alpha) - (1/\alpha)\int_n^\beta \exp(-x/\alpha)h(x-1)dx \right\}. \tag{3.4}$$

For $\alpha > 1$ solving this recursion reproves Theorem 2.1. For $0 < \alpha < 1$ the expressions for $h(\beta)$ can also be explicitly calculated from (3.4), but even for small values of n the calculations become rather involved. Therefore we postpone presenting the explicit forms.

4. An approach via Laplace transforms

In this section we allow a general positive jump size distribution for a compound Poisson process $\{X(t), t \geq 0\}$ with finite expectation A. Denote the distribution function of jump sizes by H and keep the time scale adjusted such that the underlying Poisson process has intensity 1 and therefore $EX(1) = A$. The equations (3.1) then are modified into

$$h(\beta) = \begin{cases} \int_0^\infty \exp(-t)\int_0^\infty h(\alpha t + \beta - x)dH(x)dt & \text{for} \quad \beta > 0 \\ \exp(\beta/\alpha) + \int_0^{-\beta/\alpha} \exp(-t)\int_0^\infty h(\alpha t + \beta - x)dH(x)dt & \\ & \text{for} \quad \beta < 0\,. \end{cases} \tag{4.1}$$

Proceeding as in Section 3 we are led to the equations

$$h'(\beta) = \frac{1}{\alpha}\left[h(\beta) - \int_0^\infty h(\beta - x)dH(x) \right], \tag{4.2}$$

valid for all $\beta \neq 0$.

Again, the argumentation of Section 3, appropriately modified, shows that for $\beta < 0$ the solution is given by $h(\beta) = \exp((1-c)\beta/\alpha)$, where

$$c = \int_0^\infty \exp\{(c-1)x/\alpha\} dH(x) \tag{4.3}$$

with $c = 1$ in the case $\alpha > A$ and $c < 1$ for $\alpha < A$. Denoting the Laplace transform of a function f on $]0, \infty[$ by $\hat{f}(\sigma) = \int_0^\infty \exp(-\sigma\beta) f(\beta) d\beta$ with $\operatorname{Re}(\sigma) \geq 0$ we obtain from (4.2).

$$\begin{aligned} -\theta + \sigma\hat{h}(\sigma) &= -h(0+) + \sigma\hat{h}(\sigma) = h'(\sigma) \\ &= \frac{1}{\alpha}\left[\hat{h}(\sigma) - \int_0^\infty e^{-\beta\sigma} d\sigma \int_0^\infty h(\beta - x) dH(x)\right]. \end{aligned} \tag{4.4}$$

After interchanging the order of integration in the last expression we can make use of our knowledge of h for negative values of its argument. More precisely, recalling that $\varphi(\sigma) = \int_0^\infty \exp(-\sigma x) dH(x)$ we can write

$$\begin{aligned} &\int_0^\infty \exp(-\beta\sigma) d\beta \int_0^\infty h(\beta - x) dH(x) \\ &= \int_0^\infty dH(x) \left\{ \int_0^x \exp(-\beta\sigma) h(\beta - x) d\beta \right. \\ &\qquad \left. + \int_x^\infty \exp(-\beta\sigma) h(\beta - x) d\beta \right\} \\ &= \int_0^\infty dH(x) \int_0^x \exp(-\beta\sigma) \exp\{(1-c)(\beta - x)/\alpha\} d\beta \\ &\quad + \int_0^\infty dH(x) \exp(-\sigma x) \int_0^\infty \exp(-\sigma u) h(u) du \\ &= \int_0^\infty dH(x) \{\exp(\sigma x) - \exp(-(1-c)x/\alpha)\}\{(1-c)/\alpha - \sigma\}^{-1} \\ &\quad + \varphi(\sigma)\hat{h}(\sigma) \\ &= \left\{ \int_0^\infty \exp\{-(1-c)x/\alpha\} dH(x) - \varphi(\sigma) \right\} \alpha/(\alpha\sigma - 1 + c) \\ &\quad + \varphi(\sigma) h(\sigma). \end{aligned}$$

Combining this with (4.4) and using the defining equation for c we have shown:

THEOREM 4.1. *The probability $h(\beta)$ that the sample path of a compound Poisson process with jump size distribution H and with $EX(1)$*

$= A$ *intersects the line* $y = \alpha t + \beta$ *is given by* $h(\beta) = \exp\{(1-c)\beta/\alpha\}$ *for* $\beta < 0$, $h(0) = \theta$, *and its restriction to* $]0, \infty[$ *has the Laplace transform*

$$\hat{h}(\sigma) = \frac{\alpha}{\sigma\alpha - 1 + \varphi(\sigma)} \left\{ \theta - \frac{c - \varphi(\sigma)}{\sigma\alpha - 1 + c} \right\} \tag{4.5}$$

where c *is the smallest solution in* $[0, 1]$ *of* $c = \phi((1-c)/\alpha)$.

In the Poisson case H is concentrated at 1, i.e. $\phi(\sigma) = e^{-\sigma}$ and thus

$$\hat{h}(\sigma) = \frac{\alpha}{\sigma\alpha - 1 + \exp(-\sigma)} \left\{ \theta - \frac{c - \exp(-\sigma)}{\sigma\alpha - 1 + c} \right\} \tag{4.5a}$$

which for $\alpha > 1$ further simplifies to

$$\hat{h}(\sigma) = \{\theta\alpha - (1 - e^{-\sigma})/\sigma\}/(\sigma\alpha - 1 + e^{-\sigma}) . \tag{4.5b}$$

This last expression can be expanded and manipulated into the form

$$\hat{h}(\sigma) = \frac{1}{\sigma} - (1-\theta) \sum_{r=0}^{\infty} \left(\frac{1}{\alpha}\right)^r \left(\frac{1}{\sigma}\right)^r \sum_{j=0}^{\infty} \binom{r}{j} (-1)^j (e^{-j\sigma}/\sigma) . \tag{4.6}$$

COROLLARY 4.1. *In the Poisson case without compounding, when* $\alpha > 1$, h *is given by* (2.1). *To prove this we have to identify* σ^{-r} *as the Laplace transform of* $f_r(\beta) := \beta^{(r-1)}/(r-1)!$ *and* $e^{-j\sigma}/\sigma$ *as that of the indicator function of the half line* $[j, \infty[$. *The convolution of* f_r *with this indicator function shifts the argument from* β *to* $\beta - j$ *as long as* $\beta - j > 0$ *and lets those terms vanish in which* $j > \beta$. *Thus interchanging the order of summation the factor of* $(1-\theta)$ *in* (4.6) *is the Laplace transform of*

$$\sum_{j=0}^{[\beta]} \frac{(-1)^j}{j!} \sum_{r=j}^{\infty} \frac{(\beta - j)^{r-j}}{(r-j)!} \left(\frac{1}{\alpha}\right)^{rj} \left(\frac{\beta - j}{\alpha}\right)^j = \sum_{j=0}^{[\beta]} \frac{(-1)^j}{j!} e^{\beta/\alpha} \left(\frac{\beta - j}{\alpha \exp(1/\alpha)}\right)^j ,$$

as asserted.

5. A random walk version

Implicitely we have already used in the previous sections that the behaviour of $X(t)$ only matters at the discrete time points $t_1, t_2, \ldots$ when $\alpha t_n + \beta$ is integer. This fact is behind the development of the present section and its application to the original problem.

Let now $\{X_n, n \in \mathbb{N}_0\}$ be a random walk with $X_0 = 0$ and independent nonnegative integer increments $Y_j = X_j - X_{j-1}$. Further let $f(s) = \sum_{i=0}^{\infty} p_i s^i$ be the common probability generating function of the Y_j and $m = E(Y_j) = \sum_{i=0}^{\infty} ip_i$ the common expected value. We shall be interested in the probabilities

$$r_n := P(-n + X_l = l \text{ for some } l \in \mathbb{N}), \qquad n = 0, 1, 2, \ldots .$$

As an intermediate step we determine first

$$u_n := P(n + X_l = l \text{ for some } l \in \mathbb{N}_0), \qquad n = 0, 1, 2, \ldots .$$

LEMMA 5.1. *If q ist the smallest nonnegative root of $f(s) = s$ then $u_n = q^n$ for all $n \in \mathbb{N}_0$.*

PROOF. As the paths of X_n are integer valued and nondecreasing, the event $E_n =$ "$n + X_l = l$ for some $l \in \mathbb{N}_0$" can only occur when first E_1 has occurred and the strong Markov property implies $P(E_n | E_1) = PE_{n-1}$. This shows $u_n = u_1^n$, $n = 1, 2, \ldots$. On the other hand,

$$\begin{aligned} u_1 &= \sum_{k=0}^{\infty} P(E_1 | Y_1 = k) p_k \\ &= \sum_{k=0}^{\infty} P(k + Y_2 + \cdots + Y_l = l - 1 \text{ for some } l \in \mathbb{N}) p_k \\ &= \sum_{k=0}^{\infty} u_k p_k = \sum_{k=0}^{\infty} u_1^k p_k = f(u_1) . \end{aligned}$$

Thus u_1 is a fixed point of f. If $m \leq 1$, then $q = 1$ is the only fixed point. Otherwise $(X_l/l) - 1 \xrightarrow[l \to \infty]{\text{a.s.}} m - 1 \geq 0$. Hence for n large enough we see that $1 - u_k = P(n + X_l > l$ for all $l \in \mathbb{N}_0) > 0$ which shows that $u_n = u_1^n < 1$. This identifies u_1 as the unique nonnegative fixed point q which is less than 1.

LEMMA 5.2. $r_\infty := \lim_{n \to \infty} r_n = 0$ *if* $m < 1$.

PROOF. As above: $(X_l/l) - 1 \xrightarrow[l \to \infty]{\text{a.s.}} m - 1 < 0$ and hence

$$1 - r_n = P((-n) + X_l < l \text{ for all } l \in \mathbb{N}_0) \xrightarrow[n \to \infty]{} 1 .$$

The following theorem is the discrete time version of the Mallows and Nair (1989) result.

THEOREM 5.1. $r_0 = f'(q)$. *In particular* $r_0 = m$ *if* $q = 1$ *and* $r_0 = p_1$ *if* $q = 0$.

PROOF. We first derive a recursion formula for the r_n and then use it to determine their generating function $R(s) = \sum_{n=0}^{\infty} r_n s^n$ from which r_0 can be read off.

$$\begin{aligned} r_n &= \sum_{k=0}^{\infty} P(-n + X_l = l \text{ for some } l \in \mathbb{N} \mid Y_1 = k) p_k \\ &= \sum_{k=0}^{\infty} P(-n + k - 1 + Y_2 + \cdots + Y_l = l - 1 \text{ for some } l \in \mathbb{N}) p_k \\ &= \sum_{k=0}^{n} r_{n-k+1} p_k + \sum_{k=n+1}^{\infty} u_{k-n-1} p_k = \sum_{k=0}^{n} r_{n-k+1} p_k + \sum_{k=n+1}^{\infty} q^{k-n-1} p_k \,. \end{aligned}$$

The cases $q = 0$, $0 < q < 1$ and $q = 1$ have to be treated separately. Let first $q = 0$. In this case the last sum in the recursion for r_n reduces to p_{n+1}, in particular $r_0 = r_1 p_0 + p_1 = p_1 = f'(q)$. For $q > 0$ use the recursion to see that

$$\begin{aligned} R(s) &= \frac{1}{s} \sum_{n=0}^{\infty} \sum_{k=0}^{n} p_k s^k r_{n-k+1} s^{n+k-1} + \sum_{n=0}^{\infty} s^n \sum_{k=n+1}^{\infty} p_k q^{k-n-1} \\ &= \frac{1}{s} \sum_{k=0}^{\infty} p_k s^k \sum_{n=k}^{\infty} r_{n-k+1} s^{n-k+1} + \sum_{k=1}^{\infty} p_k q^{k-1} \sum_{n=0}^{k-1} (s/q)^n \\ &= \frac{f(s)}{s} (R(s) - r_0) + \sum_{k=1}^{\infty} p_k q^{k-1} \frac{(s/q)^k - 1}{(s/q) - 1} \\ &= \frac{f(s)}{s} (R(s) - r_0) + \frac{f(s) - f(q)}{s - q} \,, \quad \text{if} \quad s \neq q \,. \end{aligned}$$

In the limit for $s \to q$ we have $R(q) = R(q) - r_0 + f'(q)$ which implies the desired result for $0 < q < 1$. If $q = 1$ we conclude from the above equation that

$$\begin{aligned} r_0 &= \left(\frac{f(s)}{s} - 1 \right) (R(s) - r_0) + \frac{1 - f(s)}{1 - s} \\ &= \frac{f(s) - s}{1 - s} (1 - s) \frac{R(s) - r_0}{s} + \frac{1 - f(s)}{1 - s} \,, \quad s \neq 1 \,. \end{aligned}$$

The rest of Theorem 5.1 follows now by taking the limit for $s \to 1$, as

$$\lim_{s \to 1} \frac{f(s) - s}{1 - s} = 1 - f'(1) = 1 - m = 0 \quad \text{for} \quad m = 1, \qquad \text{and}$$

$$\lim_{s \to \infty} (1 - s) \frac{R(s) - r_0}{s} = r_1 \lim_{s \to 1} \sum_{j=1}^{\infty} (r_j - r_{j+1}) s^j = r_1 - \sum_{j=1}^{\infty} (r_j - r_{j+1}) = r_\infty = 0$$

for $m < 1$ by Abel's lemma and by Lemma 5.2.

The generating function R contains the information about all probabilities r_n. These can in fact be explicitly determined. While we do not want to actually give the somewhat involved expressions we want to point out that for $q > 0$ it is not hard to derive them by first using the above equation for R to determine the generating function $\boldsymbol{P}$ of $\rho_n := r_n - q^{-n}$, $n = 0, 1, 2, \ldots$, i.e. $\boldsymbol{P}(s) = R(s) - q/(q-s)$. We obtain $\boldsymbol{P}(s) = \rho_0(1 + s/(f(s) - s))$ for $0 \le s < q$ if $q > 0$. Both for the case $q \neq 0$ and for $q = 0$ where we use R directly, the expressions for r_n can then be found by expanding $1/(f(s) - s)$.

6. Application to the continuous time case

In this section we extend the results of the previous section to the case of an integer valued stochastic process $\{X(t), t \ge 0\}$ with stationary independent increments for which almost all sample paths are nondecreasing step functions vanishing at 0. The question whether $X(t) = \alpha t + \beta$ for some $t > 0$ can be reformulated as "$X((s-\beta)/\alpha) = s$ for some s?" Let therefore $Y_s := X(s/\alpha)$. We are thus interested in the probabilities

$$\begin{aligned} r_{n+\beta} &:= P(-(n+\beta) + Y_s = s \text{ for some } s > 0) \\ &= P(-n + Y_{j-\beta} = j \text{ for some } j \in \mathbb{N}), \\ u_{n+\beta} &:= P(n + \beta + Y_s = s \text{ for some } s \ge 0) \\ &= P(n + Y_{j+\beta} = j \text{ for some } j \in \mathbb{N}_0), \end{aligned}$$

for $n = 0, 1, 2, \ldots$, $0 \le \beta < 1$, the second equalities being true since Y_s is integer valued. In particular we want to know $h(n+\beta) = r_{n+\beta}$.

Obviously $\{Y_n, n \in \mathbb{N}_0\}$ is one of the processes of the previous section. The distribution of Y_t has a p.g.f. of the form $f_t(s) = \sum_{k=0}^{\infty} P_k(t)s^k = \exp\{-\lambda \cdot t\phi(-\log s)\}$. Here $\phi(s) = \int_{0^+}^{\infty}(1 - e^{-sx})dN(x)$ where $N(x)$ is nondecreasing on $\mathbb{R}^+$ with $\lim_{x\to\infty} N(x) = \infty$ and $\int_{0^+}^{\infty} x dN(x) < \infty$. Let $f(s) = f_1(s)$ and q be the smallest nonnegative root of $f(s) = s$. Then clearly $f_t(q) = (f(q))^t$ for $t \ge 0$.

LEMMA 6.1. $u_{n+\beta} = q^{n+\beta}$, $n \in \mathbb{N}_0$, $0 \le \beta < 1$.

PROOF. As in the previous section

$$\begin{aligned} u_{n+\beta} &= \sum_{k=0}^{\infty} P(n + k + (Y_{j+\beta} - Y_\beta) = j \text{ for some } j \in \mathbb{N}_0 \mid X_\beta = k) p_k(\beta) \\ &= \sum_{k=0}^{\infty} u_{n+k} p_k(\beta) = q^n f_\beta(q) = q^{n+\beta}. \end{aligned}$$

Let now $\boldsymbol{P}(s)$ be as in the previous section. Obviously $p_0 > 0$, thus the case $q = 0$ does not occur and we have $\boldsymbol{P}(s) = \rho_0 f(s)/(f(s) - s)$ for $0 \le s < q$ with $\rho_0 = r_0 - 1$. Define generally $\rho_{n+\beta} = r_{n+\beta} - q^{-(n+\beta)}$ and $\boldsymbol{P}_{n+\beta}(s) = \sum_{n=0}^{\infty} \rho_{n+\beta} s^n$, $0 \le s < q$, $0 \le \beta < 1$.

THEOREM 6.1. $\boldsymbol{P}_\beta(s) = f_\beta(s)\boldsymbol{P}(s)$, $0 \le s < q$, $0 \le \beta < 1$.

COROLLARY 6.1. $r_0 = f'(q)$.

COROLLARY 6.2. *In the Poisson case*

$$h(n+\beta) = r_{n+\beta} = q^{-(n+\beta)} - (1 - r_0) \sum_{j=0}^{n} \frac{(-1)^j}{j!} e^{\lambda(n+\beta)} \{(n - \beta - j)e^{-\lambda}\}^j .$$

Up to a rescaling this is the expression given in earlier sections.

PROOF OF THE THEOREM 6.1.

$$\begin{aligned} r_{n+\beta} &= \sum_{k=0}^{\infty} P(-n + k - 1 + (X_{j-\beta} - X_{1-\beta}) = j - 1 \\ &\qquad \text{for some } j \in \mathbb{N} \mid X_{1-\beta} = k) p_k(1-\beta) \\ &= \sum_{k=0}^{n} p_k(1-\beta) r_{n-k+1} + \sum_{k=n+1}^{\infty} p_k(1-\beta) q^k \\ &= \sum_{k=0}^{n} p_k(1-\beta)\rho_{n-k+1} + q^{-(n+1)} \sum_{j=0}^{\infty} p_j(1-\beta) q^j \\ &= \sum_{k=0}^{n} p_k(1-\beta)\rho_{n-k+1} + q^{-(n+\beta)} . \end{aligned}$$

Thus, $\rho_{n+\beta} = r_{n+\beta} - q^{-(n+\beta)} = \sum_{k=0}^{n} p_k(1-\beta)\rho_{n-k+1}$. Using this recursion we see that

$$\begin{aligned} s\boldsymbol{P}_\beta(s) &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} p_k(1-\beta) s^k \rho_{n-k+1} s^{n-k+1} \\ &= \sum_{k=0}^{\infty} p_k(1-\beta) s^k \sum_{n=k}^{\infty} \rho_{n-k+1} s^{n-k+1} \\ &= f_{1-\beta}(s)(\boldsymbol{P}(s) - \rho_0) = f_{1-\beta}(s) \left\{ \rho_0 \left(1 + \frac{s}{f(s) - s} \right) - \rho_0 \right\} \\ &= f_{1-\beta}(s)\rho_0 s/(f(s) - s) = f_{1-\beta}(s) \cdot \boldsymbol{P}(s)/f(s) = f_\beta(s)\boldsymbol{P}(s) . \end{aligned}$$

The Corollaries 6.1 and 6.2 are seen immediately by expansion of $\boldsymbol{P}_\beta$ in a power series, the expression for $\boldsymbol{P}$ taken from the previous section specializing to $\boldsymbol{P}(s) = \rho_0(1 - se^{\lambda(1-s)})^{-1}$.

7. A Galton-Watson process

We return to the situation of Section 5. Remember that $\{X_n, n \geq 0\}$ is a random walk with $X_0 = 0$ and i.i.d. nonnegative increments $Y_j = X_j - X_{j-1}$, having probability generating function $f(s) = \sum_{i=0}^{\infty} p_i s^i$. Let $m = EY_j$ and denote by q the smallest nonnegative root of $f(s) = s$ and $r_0 = f'(q)$.

Let now Z_0 be a positive random variable. Define $V_0 := Z_0$, $V_{n+1} := Z_0 + X_{V_n}$, $Z_{n+1} := V_{n+1} - V_n$ for $n \geq 0$. Then,

LEMMA 7.1. *The sequence $\{Z_n, n \geq 0\}$ constitutes a Galton-Watson process with offspring probability generating function f.*

PROOF. Use the definition of the variables V_j and Z_j to write Z_{n+1} (conditioned on $Z_0, Z_1, \ldots, Z_n$) as $Y_{V_{n-1}+1} + Y_{V_{n-1}+2} + \cdots + Y_{V_{n-1}+Z_n}$, where the variables $Y_{V_{n-1}+j}$ are i.i.d. with probability generating function f and independent of $Z_0, \ldots, Z_n$.

LEMMA 7.2. *$Z_0 + X_l > l$ for all $l < V_n$*

$$V_{n+1} = Z_0 + X_{V_n} \begin{cases} > V_n & \text{if} \quad Z_{n+1} > 0 \\ = V_n & \text{if} \quad Z_{n+1} = 0 \,. \end{cases}$$

PROOF. The second assertion follows immediately from $V_{n+1} - V_n = Z_{n+1}$. We show that $Z_0 + X_l > l$ even for $l \leq V_n$ if $Z_n > 0$ by induction. As $Z_n > 0$ implies $Z_{n-1} > 0$ we assume $Z_0 + X_l > l$ for $l \leq V_{n-1}$. Now, if $V_{n-1} < l \leq V_n = V_{n-1} + Z_n$ we write $l = V_{n-1} + j$ with $1 \leq j \leq Z_n$. Thus, $Z_0 + X_l \geq Z_0 + X_{V_{n-1}} = V_n = V_{n-1} + Z_n > V_{n-1} + j = l$. If $Z_n = 0$, then there is a $k < n$ with $Z_k > 0$, $Z_{k+1} = Z_{k+2} = \cdots = Z_n = 0$. But then $Z_0 + X_l > l$ for $l < V_k = V_n$.

COROLLARY 7.1. *If $Z_0 + X_l > l$ for $l < k$ and $Z_0 + X_k = k$, then*
(a) *there is an n with $Z_n > 0$, $Z_{n+1} = 0$,*
(b) *$V_n = k$.*

PROOF. $k < V_j$ would imply $Z_0 + X_k > k$. Therefore, $V_j \leq k$ for all j. This is only possible when the nonnegative differences $Z_{j+1} = V_{j+1} - V_j$ become 0 at a certain $j = n$, i.e. when (a) is true. But then $Z_0 + X_l > l$ for all $l < V_n$ and $Z_0 + X_{V_n} = V_n$ which displays V_n as having the value k.

Lemma 5.1 is an immediate consequence: Let $Z_0 = k$, then

$$u_k = P(Z_0 + X_l = l \text{ for some } l \geq 0) = P(Z_n = 0 \text{ for some } n \leq 0) = q^k . \tag{7.1}$$

We now want to determine $r_0 = P(X_l = l \text{ for some } l > 0)$. The increasing sequence $X_0, X_1, \ldots$ can cross the diagonal from below only. Define the stopping time $T = \inf\{n > 0, X_n \geq n\}$. Then,

$$\begin{aligned}
D &= \{X_l = l \text{ for some } l > 0\} \\
&= \{X_l = l \text{ for some } l > 0,\, T < \infty\} = \bigcup_{j=1}^{\infty} \{Y_T = j\} \cap D \\
&= \bigcup_{j=1}^{\infty} \{Y_1 = j,\, X_{1+l} = 1 + l \text{ for some } l \geq 0\} \\
&\quad \cup \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{j-1} \bigcup_{t=k+1}^{\infty} \{T = t;\, X_{t-1} = t - 1 - k, \\
&\qquad\qquad Y_t = j,\, X_{t+l} = t + l \text{ for some } l \geq 0\} .
\end{aligned}$$

If we define

$$\begin{aligned}
A_t(k,j) &:= \{X_{t-1} = t - 1 - k,\ Y_t = j,\ T = t\} , \\
B_t(j) &:= \{j + X_{t+l} - X_t = l \text{ for some } l \geq 0\} ,
\end{aligned}$$

this decomposition of D can be written as

$$D = \bigcup_{j=1}^{\infty} \left\{ (A_1(0,j) \cap B_1(j-1)) \cup \bigcup_{k=1}^{j-1} \bigcup_{t=k+1}^{\infty} (A_t(k,j) \cap B_t(j-k-1)) \right\} .$$

From Lemma 7.1 we see that $B_t(j)$ is the extinction of a Galton-Watson process with $Z_0 = j$ i.e. $PB_t(j) = q^j$. Using the independence between A_t and B_t events we calculate

$$\begin{aligned}
r_0 = PD &= \sum_{j=1}^{\infty} \left\{ q^{j-1} P(A_1(0,j)) + \sum_{k=1}^{j-1} q^{j-k-1} \sum_{t=k+1}^{\infty} P(A_t(k,j)) \right\} \\
&= \sum_{j=1}^{\infty} \sum_{k=0}^{j-1} q^{j-k-1} P(A(k,k)), \qquad \text{where}
\end{aligned}$$

$$A(k,j) = \bigcup_{t=k+1}^{\infty} A_t(k,j) \quad \text{for} \quad k > 1, \quad A(0,j) = A_1(0,j) .$$

Next we remark that $\tilde{X}_l := \sum_{i=t-l}^{t-1} Y_i = X_{t-1} - X_{t-l-1}$, $l = 0, \ldots, t-1$ is stochastically the same as X_l, $l = 0, 1, \ldots, t-1$.

LEMMA 7.3. *$A_t(k,j) = \{k + \tilde{X}_l > l$ for $1 \le l \le t-2, k + \tilde{X}_{t-1} = t-1,$ $Y_t = Y_t = j\}$.*

PROOF. $A_t(k,j) = \{X_l < l \text{ for } 1 \le l \le t-2, X_{t-1} = t-1-k, Y_t = j\}$. On $A_t(k,j)$ we have therefore

$$\begin{aligned}
&X_{t-1} = \tilde{X}_{t-1} \\
&\quad \Leftrightarrow \tilde{X}_{t-1} = t-1-k \quad \text{and} \quad X_l < l && \text{for} \quad 1 \le l \le t-2 \\
&\quad \Leftrightarrow \tilde{X}_l = X_{t-1} - X_{t-l-1} \\
&\qquad\quad > (t-1-k) - (t-l-1) = l-k && \text{for} \quad 1 \le l \le t-2 \\
&\quad \Leftrightarrow k + \tilde{X}_l > l && \text{for} \quad 1 \le l \le t-2 \,.
\end{aligned}$$

Now Y_t is independent of $\tilde{X}_l$, $0 \le l \le t-1$. Thus $\{\tilde{X}_l, 0 \le l \le t-1\}$ has on $A_t(k,j)$ the same distribution as $\{X_l, 0 \le l \le t-1\}$ has on $C_t(k) :=$ $\{k + X_l > l, 1 \le l \le t-2, k + X_{t-1} = t-1\}$.

The $A_t(k,j)$, $t = k+1, k+2, \dots$ being disjoint we conclude that $\{\tilde{X}_l,$ $0 \le l \le T-1\}$ is defined on $A(k,j)$ and has the same finite dimensional distributions as $\{X_l, 0 \le l \le L\}$ on $\bigcup_{t=k+1}^{\infty} C_t(k) = \{k + X_l \text{ for some } l > 0\}$ where $L = \min\{l \mid k + X_l = 1\}$.

Using the Galton-Watson process $\tilde{Z}_n$ corresponding to $\tilde{X}_l$ with $\tilde{Z}_0 = k$ we see that $A(k,j) = \{\text{extinction of } \tilde{Z}_n\} \cap \{Y_T = j\}$. As Y_T is independent of $\{\tilde{X}_l, 0 \le l \le T-1\}$ and thus of $\{\tilde{Z}_n, n \ge 0\}$ we have

$$P(A(k,j)) = q^k \cdot p_j \quad \text{for} \quad k \ge 1 \qquad \text{and} \qquad PA(0,j) = P(Y_1 = j) = p_j \,.$$

This leads to

$$r_0 = PD = \sum_{j=1}^{\infty} \sum_{k=0}^{j-1} q^{j-k-1} q^k p_j = \sum_{j=1}^{\infty} j q^{j-1} p_j = f'(q) \,.$$

Remark 7.1. The above derivation of r_0 allows the following interpretation. When at time T the process jumps from k below the diagonal to $j-k-1$ above the diagonal $k = 1, 2, \dots, j-1$ then two Galton-Watson processes bound for extinction are triggered, namely $\{Z_n, n \ge 0\}$ with $Z_0 =$ $j-k-1$ on $B(j-k-1)$ and $\{\tilde{Z}_n, n \ge 0\}$ with $\tilde{Z}_0 = k$ on $A(k,j)$. These processes are independent of Y_T. Hence the probability of D being realized with $Y_T = j$ and $\tilde{Z}_0 = k$ is $p_j q^{j-k-1} q^k = p_j q^{j-1}$. Since j different values of k are possible

$$PD = \sum_{j=1}^{\infty} j p_j q^{j-1} = f'(q) \,.$$

8. The case of two parallel lines

In this section we consider a compound Poisson process, again with $EX(1) = A$, and determine its intersection behaviour with two parallel lines $y = \alpha t + \beta$ (line 1) and $y = \alpha t + \beta + \gamma$ (line 2). Assume $\gamma > 0$ and $\alpha > 0$, $\alpha \neq A$. Our aim is to determine the probabilities defined below:

σ_i is the probability that the process (starting with $X(0) = 0$) intersects line i without intersecting the other line first ($i = 1, 2$),

ε_i is the probability that the process, given its path intersects line i, returns to line i without first intersecting the other line ($i = 1, 2$),

δ_i is the probability that the process, given its path intersects line i, at some later time intersects the other line without an intermediate return to line i ($i = 1, 2$).

Finally, $\rho_0 = 1 - \sigma_1 - \sigma_2$, $\rho_1 = 1 - \varepsilon_1 - \delta_1$ and $\rho_2 = 1 - \varepsilon_2 - \delta_2$ are the probabilities of not intersecting either of the lines, starting from 0, from line 1, or from line 2, respectively.

THEOREM 8.1. *If $\{X(t), t \geq 0\}$ is a compound Poisson process with positive jumps such that $EX(1) = A$ then*

$$\sigma_1 = \frac{h(\beta) - h(\beta + \gamma)h(-\gamma)}{1 - h(\gamma)h(-\gamma)}, \qquad \sigma_2 = \frac{h(\beta + \gamma) - h(\beta)h(\gamma)}{1 - h(\gamma)h(-\gamma)},$$

$$\rho_0 = 1 - \sigma_1 - \sigma_2, \qquad \varepsilon_1 = \varepsilon_2 = \frac{\theta - h(\gamma)h(-\gamma)}{1 - h(\gamma)h(-\gamma)},$$

$$\rho_{1,2} = \frac{(1 - h(\pm\gamma))(1 - \theta)}{1 - h(\gamma)h(-\gamma)}, \qquad \delta_i = 1 - \varepsilon_i - \rho_i .$$

The proof proceeds by determining the joint probability generating functions (p.g.f.) of the numbers of visits to the two lines. Denote these p.g.f. by g_0 if we start the process at 0 and by g_i if the process is started on line i. Then conditioning on the first visit after time 0 to either of the lines (if any), it is easy to establish the equations

$$\begin{aligned} g_0(s_1, s_2) &= \rho_0 + \sigma_1 s_1 g_1(s_1, s_2) + \sigma_2 s_2 g_2(s_1, s_2) , \\ g_1(s_1, s_2) &= \rho_1 + \varepsilon_1 s_1 g_1(s_1, s_2) + \delta_1 s_2 g_2(s_1, s_2) , \\ g_2(s_1, s_2) &= \rho_2 + \delta_2 s_1 g_1(s_1, s_2) + \varepsilon_2 s_2 g_2(s_1, s_2) . \end{aligned} \tag{8.1}$$

It is an easy exercise to solve the last two of these equations for the p.g.f.'s g_1 and g_2 and to compare the expressions thus obtained for the marginals $g_1(s, 1)$, $g_1(1, s)$, $g_2(s, 1)$ and $g_2(1, s)$ with what we know them to be namely $g_1(s, 1) = (1 - \theta)/(1 - \theta s) = g_2(1, s)$, $g_1(1, s) = 1 - h(\gamma) + h(\gamma)(1 - \theta)s/(1 - \theta s)$

and $g_2(s, 1) = 1 - h(-\gamma) + h(-\gamma)(1-\theta)s/(1-\theta s)$. This comparison yields the values of ε_i, δ_i and ρ_i $(i = 1, 2)$ as given in Theorem 8.1. Turning to the first one of the equations (8.1) and again evaluating the marginals completes the proof.

9. Concluding remarks

Remark 9.1. For a Poisson process without compounding and the line $y = \alpha t + \beta$ with $\alpha < 1$, Nair *et al.* (1986) have determined the asymptotic value for $h(\beta)$ as $\beta \to \infty$.

The explicit form of the Laplace transform $\hat{h}$ determined in Section 4 now allows us to generalize their Lemma 4.5 to the general case of a compound Poisson process whose jump sizes are positive and have an expectation A. In fact, applying a Tauberian theorem (e.g., Widder (1941)) and keeping in mind that $\phi'(0) = -A$ one obtains

$$\lim_{\beta\to\infty} h(\beta) = \frac{\alpha(1-\theta)}{A-\alpha} \tag{9.1}$$

for the case $\alpha < A$. Naturally, if $\alpha > A$, then $h(\beta) \to 0$, as $\beta \to \infty$.

Remark 9.2. The most remarkable feature of Theorem 8.1 is the fact that $\varepsilon_1 = \varepsilon_2$. The proof implies even the following slightly stronger statement.

COROLLARY 9.1 TO THEOREM 8.1. *Consider a time homogeneous Markov chain in discrete time with three states* 1, 2 *and* ∞, *say. Let* ∞ *be absorbing. Then the probabilities of never returning to state i given the process starts in i* $(i = 1, 2)$ *are the same if and only if the one step transition probabilities* P_{ii} *for* $i = 1$ *and* 2 *are equal.*

Remark 9.3. In Section 8 if $\alpha > A$ then $h(-\gamma) = 1$ and $\rho_2 = 0$. We could have used this in our analysis there. It would not, however, have led to much simplification. On the other hand, we should note that the expressions given in Theorem 8.1 simplify if we put $h(-\gamma) = 1$.

Remark 9.4. The probabilities determined in Theorem 8.1 could in principle also be obtained by the methods presented in the previous sections. In this connection we wish to point out that the probability $q(y)$ of reaching line 2 (the upper line) without first touching the lower line if the process is started from $X(0) = \beta + y$ satisfies the equation

$$q'(y) = -\frac{1}{\alpha}\{q(y) - q(y+1)\} \tag{9.2}$$

for $y \neq \gamma$, which can be considered dual to (3.3).

The probability $q(y)$ can be read off Theorem 8.1 as σ_2 if we put $\beta = -y$, that is

$$q(y) = [h(\gamma - y) - h(-y)h(\gamma)]/[1 - h(\gamma)h(-\gamma)]\,. \tag{9.3}$$

Remark 9.5. Further insight could be gained by considering intersections with more than two lines. It can be left as an exercise for the reader to pursue the approach of Section 5 further in this direction. The problems connected with two or more non-parallel lines appear far more difficult and additional research in this direction is needed.

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