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E-OPTIMALITY OF SOME ROW AND COLUMN DESIGNS

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Abstract. In this paper we consider experimental settings where v treatments are being tested in b_1 rows and b_2 columns of sizes k_{1i} and k_{2j}, respectively, $i = 1, 2, \ldots, b_1$, $j = 1, 2, \ldots, b_2$. Some sufficient conditions for designs to be E-optimal in these classes are derived and some necessary and sufficient conditions for the E-optimality of some special classes of row and column designs are presented. Examples are also given to illustrate this theory.

Key words and phrases: E-optimality, orthogonality, balancing, connectedness, block design, row and column design.

1. Introduction

Let us consider the row and column designs with the following model of observations:

$$y = [\mathbf{1}_n, \boldsymbol{D}'_1, \boldsymbol{D}'_2, \boldsymbol{\Delta}'] \begin{pmatrix} \mu \\ \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \\ \gamma \end{pmatrix} + \boldsymbol{e} ,$$

where y is a $n \times 1$ dimensional vector of random observations, $\mathbf{1}_n$ is the $n \times 1$ vector of ones, $\boldsymbol{D}'_1$, $\boldsymbol{D}'_2$, $\boldsymbol{\Delta}'$ are $n \times b_1$, $n \times b_2$ and $n \times v$ dimensional design matrices for rows, columns and treatments, respectively, μ is an overall mean parameter and $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$ and γ are $b_1 \times 1$, $b_2 \times 1$ and $v \times 1$ vectors of unknown row, column and treatment parameters, respectively. The vector $\boldsymbol{e}$ contains n uncorrelated random variables having expectation zero and variance σ^2 each.

For a given design let $\boldsymbol{N}_1 = \boldsymbol{\Delta D}'_1$ denote the $v \times b_1$ treatment vs. row incidence matrix, let $\boldsymbol{N}_2 = \boldsymbol{\Delta D}'_2$ be the $v \times b_2$ treatment vs. column incidence matrix and let $\boldsymbol{N}_3 = \boldsymbol{D}_1\boldsymbol{D}'_2$ be the $b_1 \times b_2$ row vs. column incidence matrix.

The properties of these row and column designs can be considered by examining the patterns of the matrices

$$C = C_1 - (N_2 - N_1 K_1^{-1} N_3) C_3^- (N_2 - N_1 K_1^{-1} N_3)' , \tag{1.1}$$

where

$$C_1 = R - N_1 K_1^{-1} N_1' ,$$

C_3^- is a generalized inverse of

$$C_3 = K_2 - N_3' K_1^{-1} N_3 ,$$

and K_1, K_2 and R are the diagonal matrices with elements equal to the row sizes, k_{1i}, the column sizes, k_{2j} and the treatment replications, r_l, respectively, $i = 1, 2, \ldots, b_1, j = 1, 2, \ldots, b_2$, $l = 1, 2, \ldots, v$. An equivalent formula for C is

$$C = C_2 - (N_1 - N_2 K_2^{-1} N_3') C_4^- (N_1 - N_2 K_2^{-1} N_3')' , \tag{1.2}$$

where

$$C_2 = R - N_2 K_2^{-1} N_2'$$

and C_4^- is a generalized inverse of

$$C_4 = K_1 - N_3 K_2^{-1} N_3' .$$

The matrix C and its eigenvalues indicate some important properties of a given design, e.g., connectedness, orthogonality, balance, C-property or optimality with respect to some criterion. The E-optimality criterion was introduced by Ehrenfeld (1955). The design d belonging to any class of designs is E-optimal in this class when the smallest nonzero eigenvalue of the matrix C of d is not less than the smallest nonzero eigenvalue of the matrix C of each other design from the class. The E-optimality of row and column designs was considered by Eccleston and Kiefer (1981) and Jacroux (1982, 1985). Any row and column design whose row vs. column incidence matrix is $N_3 = \mathbf{1}_{b_1}\mathbf{1}_{b_2}'$ is said to be ordinary (see e.g. Raghavarao and Federer (1975)).

2. Main results

We will consider the problem of choosing an E-optimal design in classes of connected row and column designs without assumption of ordinarity.

Henceforth, we will denote by $\mathscr{D}(v, \boldsymbol{k}_1, \boldsymbol{k}_2)$ the class of all connected row and column designs having v treatments arranged in rows and columns having sizes as defined by the elements of the vectors $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$, respectively. With each row and column design, we associate the two block designs d_1 and d_2 defined by the matrices $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$. The class of connected block designs having v treatments tested in blocks having sizes as defined by the elements of the vector $\boldsymbol{k}_i$, will be denoted by $\mathscr{D}(v, \boldsymbol{k}_i)$, $i = 1, 2$.

For $d \in \mathscr{D}(v, \boldsymbol{k}_1, \boldsymbol{k}_2)$ let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{v-1}$ denote the eigenvalues of the matrix $\boldsymbol{C}$, $0 = \lambda_{10} < \lambda_{11} \leq \lambda_{12} \leq \cdots \leq \lambda_{1,(v-1)}$ the eigenvalues of the matrix $\boldsymbol{C}_1$ and $0 = \lambda_{20} < \lambda_{21} \leq \lambda_{22} \leq \cdots \leq \lambda_{2,(v-1)}$ the eigenvalues of $\boldsymbol{C}_2$. Notice that the matrices $(\boldsymbol{N}_2 - \boldsymbol{N}_1\boldsymbol{K}_1^{-1}\boldsymbol{N}_3)\boldsymbol{C}_3^{-}(\boldsymbol{N}_2 - \boldsymbol{N}_1\boldsymbol{K}_1^{-1}\boldsymbol{N}_3)'$ and $(\boldsymbol{N}_1 - \boldsymbol{N}_2\boldsymbol{K}_2^{-1}\boldsymbol{N}_3')\boldsymbol{C}_4^{-}(\boldsymbol{N}_1 - \boldsymbol{N}_2\boldsymbol{K}_2^{-1}\boldsymbol{N}_3')'$ are positive semidefinite. From this it follows from equations (1.1) and (1.2) that

$$\lambda_j \leq \lambda_{ij} \tag{2.1}$$

for $i = 1, 2, j = 1, 2, \ldots, v - 1$ (see Seber (1984)).

LEMMA 2.1. *For $i = 1$ or $i = 2$ let the design d_i be E-optimal in $\mathscr{D}(v, \boldsymbol{k}_i)$ and let it be associated with the row and column design $d \in \mathscr{D}(v, \boldsymbol{k}_1, \boldsymbol{k}_2)$. Then there does not exist any row and column design $d^0 \in \mathscr{D}(v, \boldsymbol{k}_1, \boldsymbol{k}_2)$ such that for associated with it the block design $d_{i'}^0 \in \mathscr{D}(v, \boldsymbol{k}_{i'})$*

$$\lambda_{i'1}^0 > \lambda_{i1} \quad \textit{and} \quad \lambda_{i'1}^0 = \lambda_1^0, \qquad i' \neq i, \quad i' = 1, 2\,. \tag{2.2}$$

PROOF. From the relations (2.2) and from the E-optimality of d_i, we have $\lambda_1^0 = \lambda_{i'1}^0 > \lambda_{i1} \geq \lambda_{i1}^0$. This is a contradiction with (2.1).

THEOREM 2.1. *Let $d \in \mathscr{D}(v, \boldsymbol{k}_1, \boldsymbol{k}_2)$ be such that the design d_i for $i = 1$ or $i = 2$ is E-optimal in the class $\mathscr{D}(v, \boldsymbol{k}_i)$. If $\lambda_1 = \lambda_{i1}$, then d is E-optimal in $\mathscr{D}(v, \boldsymbol{k}_1, \boldsymbol{k}_2)$.*

PROOF. From (2.1) and from the assumptions of this theorem, we have for $i = 1$ or $i = 2$, $\lambda_1^* \leq \lambda_{i1}^* \leq \lambda_{i1} = \lambda_1$, where λ_1^* and λ_{i1}^* are the eigenvalues of the matrices $\boldsymbol{C}$ and $\boldsymbol{C}_i$ of any design d^* belonging to the class $\mathscr{D}(v, \boldsymbol{k}_1, \boldsymbol{k}_2)$. It now follows from Lemma 2.1 that d is E-optimal in $\mathscr{D}(v, \boldsymbol{k}_1, \boldsymbol{k}_2)$.

Some special cases of row and column designs are now considered having row vs. column incidence matrix $\boldsymbol{N}_3 = \boldsymbol{k}_1\boldsymbol{k}_2'/n$. These designs were studied by Pal (1977).

COROLLARY 2.1. *Let $d \in \mathscr{D}(v, \boldsymbol{k}_1, \boldsymbol{k}_2)$ be such that its incidence matrix $\boldsymbol{N}_3$ is of the form $\boldsymbol{N}_3 = \boldsymbol{k}_1\boldsymbol{k}_2'/n$. If for $i = 1$ or $i = 2$*

$$\boldsymbol{N}_i = \boldsymbol{r}\boldsymbol{k}_i'/n \tag{2.3}$$

and if $d_{i'}$ ($i' \neq i$, $i' = 1, 2$) is E-optimal in the class $\mathscr{D}(v, \boldsymbol{k}_{i'})$, then d is E-

optimal in $\mathscr{D}(v, \boldsymbol{k}_1, \boldsymbol{k}_2)$.

PROOF. When for $i = 1$ or $i = 2$ relation (2.3) is satisfied, then for the row and column design with $\boldsymbol{N}_3 = \boldsymbol{k}_1\boldsymbol{k}_2'/n$ the matrix $\boldsymbol{C}$ is of the form $\boldsymbol{C} = \boldsymbol{C}_{i'}$ $(i' \neq i,\ i' = 1, 2)$. In this case $\lambda_1 = \lambda_{i'1}$. Hence from Theorem 2.1 the result follows.

Example 2.1. Let us consider the block design described by Lee and Jacroux (1987). This design is E-optimal in the class $\mathscr{D}(10, [2 * \mathbf{1}_{40}', 4 * \mathbf{1}_{25}']')$. The incidence matrix of this design is $\boldsymbol{N} = [\boldsymbol{N}_{R36}, \boldsymbol{N}_{R108}]$ where $\boldsymbol{N}_{R36}$ and $\boldsymbol{N}_{R108}$ are the incidence matrices of the designs R36 and R108 given in Clatworthy (1973). Now consider the row and column design $d \in \mathscr{D}(10, 90 * \mathbf{1}_2, [2 * \mathbf{1}_{40}', 4 * \mathbf{1}_{25}']')$ having matrices $\boldsymbol{N}_2 = \boldsymbol{N}$, $\boldsymbol{N}_1 = 9 * \mathbf{1}_{10}\mathbf{1}_2'$ and

$$\boldsymbol{N}_3 = \begin{pmatrix} \mathbf{1}_{40}' & 2 * \mathbf{1}_{25}' \\ \mathbf{1}_{40}' & 2 * \mathbf{1}_{25}' \end{pmatrix}.$$

Since the assumptions of Corollary 2.1 hold, the design d is E-optimal in the class $\mathscr{D}(10, 90 * \mathbf{1}_2, [2 * \mathbf{1}_{40}', 4 * \mathbf{1}_{25}']')$.

Let us now consider the special case of row and column designs having

$$\text{(2.4)} \qquad \boldsymbol{N}_3 = \boldsymbol{k}_1\boldsymbol{k}_2'/n, \qquad \boldsymbol{N}_1'\boldsymbol{N}_2 = \boldsymbol{k}_1\boldsymbol{k}_2'/v \qquad \text{and} \qquad \boldsymbol{r} = r * \mathbf{1}_v .$$

These designs are related to those which were studied by Eccleston and Kiefer (1981). They considered ordinary row and column designs having $\boldsymbol{N}_1'\boldsymbol{N}_2 = r * \mathbf{1}_{b_1}\mathbf{1}_{b_2}'$ and $r = b_1b_2/v$.

THEOREM 2.2. *The row and column design* $d \in \mathscr{D}(v, \boldsymbol{k}_1, \boldsymbol{k}_2)$ *satisfying* (2.4) *is E-optimal in this class if for* $i = 1$ *or* $i = 2$ *the design* d_i *is E-optimal in the class* $\mathscr{D}(v, \boldsymbol{k}_i)$ *and* $\lambda_{i1} \leq \lambda_{i'1}$ $(i' \neq i,\ i' = 1, 2)$.

PROOF. If $\boldsymbol{N}_3 = \boldsymbol{k}_1\boldsymbol{k}_2'/n$, then the matrix $\boldsymbol{C}$ is of the form $\boldsymbol{C} = \boldsymbol{C}_1 + \boldsymbol{C}_2 - \boldsymbol{C}_0$ where $\boldsymbol{C}_0 = r(\boldsymbol{I} - \mathbf{1}_v\mathbf{1}_v'/v)$. Since $\boldsymbol{N}_1'\boldsymbol{N}_2 = \boldsymbol{k}_1\boldsymbol{k}_2'/v$ the matrices $\boldsymbol{C}$, $\boldsymbol{C}_1$, $\boldsymbol{C}_2$ and $\boldsymbol{C}_0$ have a common set of eigenvectors. The eigenvalues of these matrices satisfy for $j = 1, 2, \ldots, v-1$ the following relation:

$$\text{(2.5)} \qquad \lambda_j = \lambda_{1j} + \lambda_{2j} - r .$$

Let us write $\boldsymbol{C} = \boldsymbol{C}_0 - (\boldsymbol{C}_0 - \boldsymbol{C}_1) - (\boldsymbol{C}_0 - \boldsymbol{C}_2)$. The matrices $\boldsymbol{C}_0 - \boldsymbol{C}_1$ and $\boldsymbol{C}_0 - \boldsymbol{C}_2$ are orthogonal; thus $(\boldsymbol{C}_0 - \boldsymbol{C}_1)(\boldsymbol{C}_0 - \boldsymbol{C}_2) = \mathbf{0}$ and from (2.5), if $0 < \lambda_{ij} < r$, then $\lambda_{i'j} = r$ and $\lambda_j = \lambda_{ij}$ $(i' \neq i,\ i, i' = 1, 2)$. So $d_i \in \mathscr{D}(v, \boldsymbol{k}_i)$ is E-optimal and $\lambda_{i1} \leq \lambda_{i'1}$; it follows from Theorem 2.1 that d is E-optimal.

Example 2.2. Consider the design $d \in \mathscr{D}(11, [11\ \ 22\ \ 22]', 5 * \mathbf{1}_{11})$ having

$$N_1' = N_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix}$$

and having N_2 which is the incidence matrix of the BIB design $d_2 \in \mathscr{D}(11, 5 * \mathbf{1}_{11})$. The above matrices satisfying the relation (2.4) and the block design d_2 is E-optimal in $\mathscr{D}(11, 5 * \mathbf{1}_{11})$. Hence d is E-optimal in $\mathscr{D}(11, [11\ \ 22\ \ 22]', 5 * \mathbf{1}_{11})$.

Eccleston and Russel (1975) studied row and column designs having incidence matrices satisfying for $i = 1$ or $i = 2$ the relation

$$N_i = N_{i'} K_{i'}^{-1} N_{3i} \tag{2.6}$$

where $i' \neq i$, $i' = 1, 2$, $N_{31} = N_3'$ and $N_{32} = N_3$. Let us denote the classes of these connected designs by $\mathscr{D}_i(v, \mathbf{k}_1, \mathbf{k}_2)$, $i = 1, 2$.

THEOREM 2.3. *A row and column design $d \in \mathscr{D}_i(v, \mathbf{k}_1, \mathbf{k}_2)$ where $i = 1$ or $i = 2$ is E-optimal in this class if and only if $d_{i'}$ is E-optimal in the class $\mathscr{D}(v, \mathbf{k}_{i'})$, $i' \neq i$, $i' = 1, 2$.*

PROOF. Since $\mathscr{D}_i(v, \mathbf{k}_1, \mathbf{k}_2) \subset \mathscr{D}(v, \mathbf{k}_1, \mathbf{k}_2)$, the sufficiency is evident from Theorem 2.1. When d is E-optimal in the class $\mathscr{D}_i(v, \mathbf{k}_1, \mathbf{k}_2)$, then $\lambda_1 \geq \lambda_1^0$ where λ_1^0 is the eigenvalue of the matrix C of any design d^0 from $\mathscr{D}_i(v, \mathbf{k}_1, \mathbf{k}_2)$. From (2.6) we have $C = C_{i'}$, hence the result follows.

Example 2.3. The block design defined by the incidence matrix

$$N' = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

is the partially balanced block design E-optimal in the class $\mathscr{D}(10, [4 * \mathbf{1}_5', \mathbf{1}_2']')$ (see Brzeskwiniewicz (1988)). Let $N_2 = N$,

$$\boldsymbol{N}_1' = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 \end{pmatrix}$$

and

$$\boldsymbol{N}_3' = \begin{pmatrix} 2 & 2 & 2 & 2 & 2 & 0 & 1 \\ 2 & 2 & 2 & 2 & 2 & 1 & 0 \end{pmatrix}$$

are the incidence matrices of the row and column design $d \in \mathscr{D}_1(10, 11 * \mathbf{1}_2, [4 * \mathbf{1}_5', \mathbf{1}_2']')$. The above matrices satisfy the relation (2.6) for $i = 1$. Hence, from Theorem 2.3, this design is E-optimal in the class $\mathscr{D}_1(10, 11 * \mathbf{1}_2, [4 * \mathbf{1}_5', \mathbf{1}_2']')$.

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