Ann. Inst. Statist. Math.
Vol. 42, No. 3, 581–596 (1990)

TESTING LINEAR HYPOTHESES IN ERRORS IN VARIABLES MODEL

NAVEEN K. BANSAL

Department of Mathematics, Statistics and Computer Science, Marquette University, Milwaukee, WI 53233, U.S.A.

(Received December 26, 1988; revised June 16, 1989)

Abstract. A multivariate errors-in-variables model in the matrix form can be written as $X = U + E$, $Y = UA' + WB + F$, where X $(n \times p)$ and Y $(n \times q)$ are observed matrices, E and F are error matrices whose rows are normally distributed, W $(n \times k)$ is a known matrix of rank k, and U, A and B are unknown matrices. We consider the problems of testing linear hypotheses: (i) H_0: $AR = K$ and (ii) H_0: $S'A = L$, where R, K, S and L are known matrices, and we derive the likelihood ratio tests for testing these hypotheses.

Key words and phrases: Eigenvalues and eigenvectors, likelihood function, likelihood ratio test, Jacobian of a transformation, Poincaré separation theorem.

1. Introduction

An errors-in-variables model for the multivariate case is written as

$$x_i = u_i + e_i, \qquad y_i = b + Au_i + f_i; \qquad i = 1, 2, \dots, n\,, \tag{1.1}$$

where x_i $(p \times 1)$ and y_i $(q \times 1)$, $i = 1, 2, \dots, n$ are observed variables, u_i $(p \times 1)$, $i = 1, \dots, n$ are unobserved variables, A $(q \times p)$ and b $(q \times 1)$ are unknown, and e_i $(p \times 1)$, f_i $(q \times 1)$, $i = 1, \dots, n$ are i.i.d. error vectors. We assume that

$$(e_i', f_i')' \sim N\left[0, \sigma^2 \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix}\right], \tag{1.2}$$

where σ^2 is unknown, and Σ_1 $(p \times p)$ and Σ_2 $(q \times q)$ are known p.d. matrices.

In the geophysical surveying problem mentioned in Gleser and Watson

(1973), there is some justification for this assumption. First, prior experience with the error involved in measuring the longitude, latitude and altitude of a stake placed on a glacier can yield the matrix of correlations $P = \Sigma_1 = \Sigma_2$ among the three kinds of measurement error. By symmetry, one can argue that the error variances in measuring the three dimensions are approximately equal to σ^2. Further, since the first and second surveys of any stake are widely separated in time, the errors made in these two surveys should be independent of one another. Putting these arguments together, we get

$$(e_i', f_i')' \sim N\left[0, \sigma^2 \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}\right].$$

In the matrix form (1.1) can be written as

$$X = U + E, \qquad Y = UA' + 1b' + F,$$

where

$$X = [x_1', x_2', \ldots, x_n']', \qquad Y = [y_1', y_2', \ldots, y_n']' ;$$
$$E = [e_1', e_2', \ldots, e_n']', \qquad F = [f_1', f_2', \ldots, f_n']' ;$$
$$U = [u_1', u_2', \ldots, u_n']'$$

and 1 $(n \times 1)$ is a vector of 1's. Here

$$(E\colon F) \sim N\left[0, I \otimes \sigma^2 \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix}\right].$$

In a more general form we write an errors-in-variables model as

$$X = U + E, \qquad Y = UA' + WB + F, \tag{1.3}$$

where X, Y, U and A are as defined before, $W(n \times k)$ is a known matrix of rank k, $B\ (k \times q)$ is an unknown matrix, and

$$(E\colon F) \sim N\left[0, G \otimes \sigma^2 \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix}\right], \tag{1.4}$$

where $G\ (n \times n)$ is a known p.d. matrix.

In this paper, we consider the problem of testing the linear hypotheses: (i) $H_0\colon AR = K$ and (ii) $S'A = L$, where $R\ (p \times r)$ and $S\ (q \times s)$ are known matrices of ranks r and s, respectively, and $K\,(q\times r)$ and $L\,(s\times p)$ are some known matrices.

It should be noted that the assumption (1.4) can be generalized to the following assumption: $(e_i', f_i')' \sim N[0, \sigma^2\Sigma]$, where

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

is a known matrix. Then the model (1.3) can be transformed to

$$X = U + E, \qquad Y^* = UA^{*\prime} + WB + F^* ,$$

where $Y^* = Y - X\Sigma_{11}^{-1}\Sigma_{12}$, $A^* = -\Sigma_{21}\Sigma_{11}^{-1} + A$ and

$$(E\colon F^*) \sim N\left[0, G \otimes \sigma^2 \begin{pmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22\cdot 1} \end{pmatrix}\right],$$

where $\Sigma_{22\cdot 1} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$.

Thus the hypothesis $H_0\colon AR = K$ is equivalent to $H_0\colon A^*R = K^*$, where $K^* = -\Sigma_{21}\Sigma_{11}^{-1}R + K$, and $H_0\colon S'A = L$ is equivalent to $H_0\colon S'A^* = L^*$, where $L^* = -S'\Sigma_{21}\Sigma_{11}^{-1} + L$.

2. Some preliminary results

For our subsequent analysis we need the following two lemmas which we shall prove in this section. We denote $\lambda_1(\mathscr{A}) \geq \lambda_2(\mathscr{A}) \geq \cdots$ as the eigenvalues of the matrix $\mathscr{A}$.

LEMMA 2.1. *Let $\mathscr{X}$ $(m\times l)$, $\mathscr{Y}$ $(m\times c)$ and $\mathscr{H}$ $(l\times l)$ be given matrices $(m > l + c)$, where $\mathscr{H}$ is nonsingular and let Ω_1 $(l\times l)$ and Ω_2 $(c\times c)$ be given p.d. matrices. Then, for any Θ $(m\times l)$ and $\mathscr{C}$ $(l\times c)$,*

$$\operatorname{tr}\Omega_1(\mathscr{X} - \Theta\mathscr{H})'(\mathscr{X} - \Theta\mathscr{H}) + \operatorname{tr}\Omega_2(\mathscr{Y} - \Theta\mathscr{C})'(\mathscr{Y} - \Theta\mathscr{C})$$
$$\geq \sum_{i=l+1}^{l+c} \lambda_i[\operatorname{diag}(\Omega_1, \Omega_2)(\mathscr{X}\colon\mathscr{Y})'(\mathscr{X}\colon\mathscr{Y})]$$

with equality achieved when

$$\Theta = \mathscr{P}\mathscr{P}'\mathscr{X}\mathscr{H}^{-1} \quad \textit{and} \quad \mathscr{C} = (\mathscr{P}'\mathscr{X}\mathscr{H}^{-1})^{-}(\mathscr{P}'\mathscr{Y}),$$

where $\mathscr{P}$ is an $m\times l$ matrix whose columns are the eigenvectors corresponding to the first l eigenvalues of the matrix $\mathscr{X}\Omega_1\mathscr{X}' + \mathscr{Y}\Omega_2\mathscr{Y}'$.

LEMMA 2.2. *Let $\mathscr{X}_1$ $(m \times l_1)$, $\mathscr{X}_2$ $(m \times l_2)$, $\mathscr{Y}$ $(m \times c)$ and $\mathscr{K}$ $(l_1 \times c)$ be given matrices, and Ω_1 $(l_1 \times l_1)$, Ω_2 $(l_2 \times l_2)$ and Ω_3 $(c \times c)$ be known p.d. matrices. Then, for any Θ_1 $(m \times l_1)$, Θ_2 $(m \times l_2)$ and $\mathscr{C}$ $(l_2 \times c)$,*

$$\begin{aligned}
&\operatorname{tr} \Omega_1(\mathscr{X}_1 - \Theta_1)'(\mathscr{X}_1 - \Theta_1) + \operatorname{tr} \Omega_2(\mathscr{X}_2 - \Theta_2)'(\mathscr{X}_2 - \Theta_2) \\
&\quad + \operatorname{tr} \Omega_3(\mathscr{Y} - \Theta_1\mathscr{K} - \Theta_2\mathscr{C})'(\mathscr{Y} - \Theta_1\mathscr{K} - \Theta_2\mathscr{C}) \\
&\quad \geq \sum_{l_2+1}^{l_2+c} \lambda_i[\operatorname{diag}((\Omega_3^{-1} + \mathscr{K}'\Omega_1^{-1}\mathscr{K})^{-1}, \Omega_2) \\
&\qquad \times (\mathscr{Y} - \mathscr{X}_1\mathscr{K} : \mathscr{X}_2)'(\mathscr{Y} - \mathscr{X}_1\mathscr{K} : \mathscr{X}_2)]
\end{aligned}$$

with equality achieved when

$$\Theta_1 = \mathscr{P}_2\mathscr{P}_2'\mathscr{X}_1 + (I - \mathscr{P}_2\mathscr{P}_2')(\mathscr{X}_1\Omega_1 + \mathscr{Y}\Omega_3\mathscr{K}')(\Omega_1 + \mathscr{K}\Omega_3\mathscr{K}')^{-1},$$
$$\Theta_2 = \mathscr{P}_2\mathscr{P}_2'\mathscr{X}_2, \qquad \mathscr{C} = (\mathscr{P}_2'\mathscr{X}_2)^{-}\mathscr{P}_2'(\mathscr{Y} - \Theta_1\mathscr{K}),$$

where $\mathscr{P}_2$ is an $m \times l_2$ matrix whose columns are the eigenvectors corresponding to the first l_2 eigenvalues of the matrix

$$(\mathscr{Y} - \mathscr{X}_1\mathscr{K})(\Omega_3^{-1} + \mathscr{K}'\Omega_1^{-1}\mathscr{K})^{-1}(\mathscr{Y} - \mathscr{X}_1\mathscr{K})' + \mathscr{X}_2\Omega_2\mathscr{X}_2'.$$

Lemma 2.1 was proved before by Gleser and Watson (1973), when $\mathscr{H} = I$. Here we shall give a different proof.

PROOF OF LEMMA 2.1. Let $\Theta = \mathscr{P}\mathscr{D}$ be a decomposition of Θ, where $\mathscr{P}$ $(m \times k)$ is a matrix such that $\mathscr{P}'\mathscr{P} = I_k$ and $\mathscr{D}$ $(k \times l)$ is a matrix of rank k $(k \leq l)$. Then

$$\begin{aligned}
(2.1) \quad &\operatorname{tr} \Omega_1(\mathscr{X} - \Theta\mathscr{H})'(\mathscr{X} - \Theta\mathscr{H}) + \operatorname{tr} \Omega_2(\mathscr{Y} - \Theta\mathscr{C})'(\mathscr{Y} - \Theta\mathscr{C}) \\
&= \operatorname{tr} \Omega_1(\mathscr{X} - \mathscr{P}\mathscr{D}\mathscr{H})'(\mathscr{X} - \mathscr{P}\mathscr{D}\mathscr{H}) \\
&\quad + \operatorname{tr} \Omega_2(\mathscr{Y} - \mathscr{P}\mathscr{D}\mathscr{C})'(\mathscr{Y} - \mathscr{P}\mathscr{D}\mathscr{C}) \\
&\geq \operatorname{tr} \Omega_1\mathscr{X}'(I - \mathscr{P}\mathscr{P}')\mathscr{X} + \operatorname{tr} \Omega_2\mathscr{Y}'(I - \mathscr{P}\mathscr{P}')\mathscr{Y}
\end{aligned}$$

with the equality achieved when

$$\mathscr{D} = \mathscr{P}'\mathscr{X}\mathscr{H}^{-1}, \qquad \mathscr{D}\mathscr{C} = \mathscr{P}'\mathscr{Y}$$

i.e., when

$$(2.2) \qquad \Theta = \mathscr{P}\mathscr{P}'\mathscr{X}\mathscr{H}^{-1}, \qquad \mathscr{C} = (\mathscr{P}'\mathscr{X}\mathscr{H}^{-1})^{-}\mathscr{P}'\mathscr{Y}.$$

Now let $I - \mathscr{P}\mathscr{P}' = \mathscr{B}\mathscr{B}'$, where $\mathscr{B}$ is an $m \times (m-k)$ matrix such that $\mathscr{B}'\mathscr{B} = I_{m-k}$. Then, by the Poincaré separation theorem (see Rao (1973)), and since $k \le l$,

$$\begin{aligned}
(2.3) \qquad & \operatorname{tr} \Omega_1 \mathscr{X}'(I - \mathscr{P}\mathscr{P}')\mathscr{X} + \operatorname{tr} \Omega_2 \mathscr{Y}'(I - \mathscr{P}\mathscr{P}')\mathscr{Y} \\
&= \operatorname{tr} \mathscr{B}'(\mathscr{X}\Omega_1\mathscr{X}' + \mathscr{Y}\Omega_2\mathscr{Y}')\mathscr{B} \\
&\ge \sum_{k+1}^{m} \lambda_i[\mathscr{X}\Omega_1\mathscr{X}' + \mathscr{Y}\Omega_2\mathscr{Y}'] \\
&\ge \sum_{l+1}^{m} \lambda_i[\mathscr{X}\Omega_1\mathscr{X}' + \mathscr{Y}\Omega_2\mathscr{Y}'] \\
&= \sum_{l+1}^{l+c} \lambda_i[\operatorname{diag}(\Omega_1, \Omega_2)(\mathscr{X}:\mathscr{Y})'(\mathscr{X}:\mathscr{Y})]
\end{aligned}$$

with equality achieved when $k = l$ and $\mathscr{B}$ is an $m \times (m-l)$ matrix whose columns are the eigenvectors corresponding to the last $m - l$ eigenvalues of $\mathscr{X}\Omega_1\mathscr{X}' + \mathscr{Y}\Omega_2\mathscr{Y}'$ and thus $\mathscr{P}$ is an $m \times l$ matrix whose columns are the eigenvectors corresponding to the first l eigenvalues of $\mathscr{X}\Omega_1\mathscr{X}' + \mathscr{Y}\Omega_2\mathscr{Y}'$. Thus the lemma is proved from (2.1)–(2.3).

PROOF OF LEMMA 2.2. Let $\Theta_2 = \mathscr{P}_2\mathscr{D}_2$ be a decomposition of Θ_2, where $\mathscr{P}_2$ is an $m \times k_2$ matrix such that $\mathscr{P}_2'\mathscr{P}_2 = I_{k_2}$ and $\mathscr{D}_2$ is a $k_2 \times l_2$ matrix of rank k_2 $(k_2 \le l_2)$. Then

$$\begin{aligned}
(2.4) \quad & \operatorname{tr} \Omega_1(\mathscr{X}_1 - \Theta_1)'(\mathscr{X}_1 - \Theta_1) + \operatorname{tr} \Omega_2(\mathscr{X}_2 - \mathscr{P}_2\mathscr{D}_2)'(\mathscr{X}_2 - \mathscr{P}_2\mathscr{D}_2) \\
&\quad + \operatorname{tr} \Omega_3(\mathscr{Y} - \Theta_1\mathscr{K} - \mathscr{P}_2\mathscr{D}_2\mathscr{C})'(\mathscr{Y} - \Theta_1\mathscr{K} - \mathscr{P}_2\mathscr{D}_2\mathscr{C}) \\
&\qquad \ge \operatorname{tr} \Omega_1(\mathscr{X}_1 - \Theta_1)'(\mathscr{X}_1 - \Theta_1) + \operatorname{tr} \Omega_2\mathscr{X}_2'(I - \mathscr{P}_2\mathscr{P}_2')\mathscr{X}_2 \\
&\qquad\quad + \operatorname{tr} \Omega_3(\mathscr{Y} - \Theta_1\mathscr{K})'(I - \mathscr{P}_2\mathscr{P}_2')(\mathscr{Y} - \Theta_1\mathscr{K})
\end{aligned}$$

with the equality achieved when

$$(2.5) \qquad \Theta_2 = \mathscr{P}_2\mathscr{P}_2'\mathscr{X}_2, \qquad \mathscr{C} = (\mathscr{P}_2'\mathscr{X}_2)^{-}\mathscr{P}_2'(\mathscr{Y} - \Theta_1\mathscr{K}) .$$

Now let $I - \mathscr{P}_2\mathscr{P}_2' = \mathscr{B}_2\mathscr{B}_2'$, where $\mathscr{B}_2$ is an $m \times (m - k_2)$ matrix such that $\mathscr{B}_2'\mathscr{B}_2 = I$, and let

$$\mathscr{V}_1 = \mathscr{P}_2'\Theta_1, \qquad \mathscr{V}_2 = \mathscr{B}_2'\Theta_1 .$$

Then, since $\mathscr{P}_2\mathscr{P}_2' + \mathscr{B}_2\mathscr{B}_2' = I$,

$$
\begin{aligned}
(2.6)\qquad & \operatorname{tr} \Omega_1(\mathscr{X}_1 - \Theta_1)'(\mathscr{X}_1 - \Theta_1) + \operatorname{tr} \Omega_2\mathscr{X}_2'\mathscr{B}_2\mathscr{B}_2'\mathscr{X}_2 \\
& \quad + \operatorname{tr} \Omega_3(\mathscr{Y} - \Theta_1\mathscr{K})'\mathscr{B}_2\mathscr{B}_2'(\mathscr{Y} - \Theta_1\mathscr{K}) \\
& \qquad = \operatorname{tr} \Omega_1(\mathscr{P}_2'\mathscr{X}_1 - \mathscr{V}_1)'(\mathscr{P}_2'\mathscr{X}_1 - \mathscr{V}_1) \\
& \qquad\quad + \operatorname{tr} \Omega_1(\mathscr{B}_2'\mathscr{X}_1 - \mathscr{V}_2)'(\mathscr{B}_2'\mathscr{X}_1 - \mathscr{V}_2) \\
& \qquad\quad + \operatorname{tr} \Omega_3(\mathscr{B}_2'\mathscr{Y} - \mathscr{V}_2\mathscr{K})'(\mathscr{B}_2'\mathscr{Y} - \mathscr{V}_2\mathscr{K}) \\
& \qquad\quad + \operatorname{tr} \Omega_2\mathscr{X}_2'\mathscr{B}_2\mathscr{B}_2'\mathscr{X}_2 \\
& \qquad \geq \operatorname{tr} [(\mathscr{B}_2'\mathscr{X}_1 : \mathscr{B}_2'\mathscr{Y}) - \mathscr{V}_2(I:\mathscr{K})] \\
& \qquad\quad \times \operatorname{diag}(\Omega_1, \Omega_3)[(\mathscr{B}_2'\mathscr{X}_1 : \mathscr{B}_2'\mathscr{Y}) - \mathscr{V}_2(I:\mathscr{K})]' \\
& \qquad\quad + \operatorname{tr} \Omega_2\mathscr{X}_2'\mathscr{B}_2\mathscr{B}_2'\mathscr{X}_2' \\
& \qquad \geq \operatorname{tr} (\mathscr{B}_2'\mathscr{X}_1 : \mathscr{B}_2'\mathscr{Y})\Psi(\mathscr{B}_2'\mathscr{X}_1 : \mathscr{B}_2'\mathscr{Y})' \\
& \qquad\quad + \operatorname{tr} \Omega_2\mathscr{X}_2'\mathscr{B}_2\mathscr{B}_2'\mathscr{X}_2 ,
\end{aligned}
$$

where

$$
\begin{aligned}
\Psi &= \operatorname{diag}(\Omega_1, \Omega_3) \\
& \quad - \operatorname{diag}(\Omega_1, \Omega_3)\begin{pmatrix} I \\ \mathscr{K}' \end{pmatrix}(\Omega_1 + \mathscr{K}\Omega_3\mathscr{K}')^{-1}(I:\mathscr{K}) \operatorname{diag}(\Omega_1, \Omega_3) \\
&= \begin{bmatrix} \Omega_1 - \Omega_1(\Omega_1 + \mathscr{K}\Omega_3\mathscr{K}')^{-1}\Omega_1 & -\Omega_1(\Omega_1 + \mathscr{K}\Omega_3\mathscr{K}')^{-1}\mathscr{K}\Omega_3 \\ -\Omega_3\mathscr{K}'(\Omega_1 + \mathscr{K}\Omega_3\mathscr{K}')^{-1}\Omega_1 & \Omega_3 - \Omega_3\mathscr{K}'(\Omega_1 + \mathscr{K}\Omega_3\mathscr{K}')^{-1}\mathscr{K}\Omega_3 \end{bmatrix}.
\end{aligned}
$$

Note that (see Rao (1973), p. 33),

$$
\begin{aligned}
(\Omega_3^{-1} + \mathscr{K}'\Omega_1^{-1}\mathscr{K})^{-1} &= \Omega_3 - \Omega_3\mathscr{K}'(\Omega_1 + \mathscr{K}\Omega_3\mathscr{K}')^{-1}\mathscr{K}\Omega_3 , \\
(\Omega_1 + \mathscr{K}\Omega_3\mathscr{K}')^{-1} &= \Omega_1^{-1} - \Omega_1^{-1}\mathscr{K}(\Omega_3^{-1} + \mathscr{K}'\Omega_1^{-1}\mathscr{K})^{-1}\mathscr{K}'\Omega_1^{-1} .
\end{aligned}
$$

Thus,

$$
\begin{aligned}
(2.7)\qquad \Psi &= \begin{bmatrix} \mathscr{K}(\Omega_3^{-1} + \mathscr{K}'\Omega_1^{-1}\mathscr{K})^{-1}\mathscr{K}' & -\mathscr{K}(\Omega_3^{-1} + \mathscr{K}'\Omega_1^{-1}\mathscr{K})^{-1} \\ -(\Omega_3^{-1} + \mathscr{K}'\Omega_1^{-1}\mathscr{K})^{-1}\mathscr{K}' & (\Omega_3^{-1} + \mathscr{K}'\Omega_1^{-1}\mathscr{K})^{-1} \end{bmatrix} \\
&= (-\mathscr{K}' : I)'(\Omega_3^{-1} + \mathscr{K}'\Omega_1^{-1}\mathscr{K})^{-1}(-\mathscr{K}' : I) .
\end{aligned}
$$

It can be seen that the equality in (2.6) is achieved when

$$
\begin{aligned}
\mathscr{V}_1 &= \mathscr{P}_2'\Theta_1 = \mathscr{P}_2'\mathscr{X}_1 , \\
\mathscr{V}_2 &= \mathscr{B}_2'\Theta_1 = (\mathscr{B}_2'\mathscr{X}_1\Omega_1 + \mathscr{B}_2'\mathscr{Y}\Omega_3\mathscr{K}')(\Omega_1 + \mathscr{K}\Omega_3\mathscr{K}')^{-1}
\end{aligned}
$$

i.e., when

$$\Theta_1 = \mathscr{P}_2\mathscr{P}_2'\mathscr{X}_1 + \mathscr{B}_2\mathscr{B}_2'(\mathscr{X}_1\Omega_1 + \mathscr{Y}\Omega_3\mathscr{K}')(\Omega_1 + \mathscr{K}\Omega_3\mathscr{K}')^{-1} . \tag{2.8}$$

Now, from (2.7), and by the Poincaré separation theorem, and since $k_2 \le l_2$

$$\begin{aligned}
&\operatorname{tr}(\mathscr{B}_2'\mathscr{X}_1 : \mathscr{B}_2'\mathscr{Y})\Psi(\mathscr{B}_2'\mathscr{X}_1 : \mathscr{B}_2'\mathscr{Y}) + \operatorname{tr}\Omega_2\mathscr{X}_2'\mathscr{B}_2\mathscr{B}_2'\mathscr{X}_2 \\
&\quad = \operatorname{tr}\mathscr{B}_2'[(\mathscr{Y} - \mathscr{X}_1\mathscr{K})(\Omega_3^{-1} + \mathscr{K}'\Omega_1^{-1}\mathscr{K})^{-1} \\
&\qquad\qquad \times (\mathscr{Y} - \mathscr{X}_1\mathscr{K})' + \mathscr{X}_2\Omega_2\mathscr{X}_2']\mathscr{B}_2 \\
&\quad \ge \sum_{k_2+1}^{m} \lambda_i[(\mathscr{Y} - \mathscr{X}_1\mathscr{K})(\Omega_3^{-1} + \mathscr{K}'\Omega_1^{-1}\mathscr{K})^{-1} \\
&\qquad\qquad \times (\mathscr{Y} - \mathscr{X}_1\mathscr{K})' + \mathscr{X}_2\Omega_2\mathscr{X}_2'] \\
&\quad \ge \sum_{l_2+1}^{l_2+c} \lambda_i[\operatorname{diag}((\Omega_3^{-1} + \mathscr{K}'\Omega_1^{-1}\mathscr{K})^{-1}, \Omega_2) \\
&\qquad\qquad \times (\mathscr{Y} - \mathscr{X}_1\mathscr{K} : \mathscr{X}_2)'(\mathscr{Y} - \mathscr{X}_1\mathscr{K} : \mathscr{X}_2)]
\end{aligned} \tag{2.9}$$

with the equality achieved when $k_2 = l_2$ and $\mathscr{B}_2$ is an $m \times (m - l_2)$ matrix whose columns are the eigenvectors corresponding to the last $m - l_2$ eigenvalues of the matrix

$$(\mathscr{Y} - \mathscr{X}_1\mathscr{K})(\Omega_3^{-1} + \mathscr{K}'\Omega_1^{-1}\mathscr{K})^{-1}(\mathscr{Y} - \mathscr{X}_1\mathscr{K})' + \mathscr{X}_2\Omega_2\mathscr{X}_2'$$

i.e., when $\mathscr{P}_2$ is an $m \times l_2$ matrix whose columns are eigenvectors corresponding to the first l_2 eigenvalues of the above matrix.

Thus the lemma is proved from (2.4)–(2.6), (2.8) and (2.9).

3. Likelihood ratio test for testing $AR = K$

We first make the following one-to-one onto transformation

$$\begin{aligned}
\underset{n\times r}{X_1} &= X\Sigma_1^{-1}R(R'\Sigma_1^{-1}R)^{-1}, & \underset{n\times(p-r)}{X_2} &= X\Sigma_1^{-1}Z_R(Z_R'\Sigma_1^{-1}Z_R)^{-1}, \\
\underset{n\times r}{U_1} &= X\Sigma_1^{-1}R(R'\Sigma_1^{-1}R)^{-1}, & \underset{n\times(p-r)}{U_2} &= U\Sigma_1^{-1}Z_R(Z_R'\Sigma_1^{-1}Z_R)^{-1},
\end{aligned} \tag{3.1}$$

where $Z_R\ (p \times (p - r))$ is such that $Z_R'\Sigma_1^{-1}R = 0$ and $Z_R'Z_R = I_{p-r}$. Then

$$(X_1 : X_2) \sim N\left[(U_1 : U_2), G \otimes \sigma^2 \begin{pmatrix} (R'\Sigma_1^{-1}R)^{-1} & 0 \\ 0 & (Z_R'\Sigma_1^{-1}Z_R)^{-1} \end{pmatrix}\right].$$

Now, since $\Sigma_1^{-1}R(R'\Sigma_1^{-1}R)^{-1}R' + \Sigma_1^{-1}Z_R(Z_R'\Sigma_1^{-1}Z_R)^{-1}Z_R' = I$ (see Rao (1973), p. 77),

$$Y = U[\Sigma_1^{-1}R(R'\Sigma_1^{-1}R)^{-1} : \Sigma_1^{-1}Z_R(Z_R'\Sigma_1^{-1}Z_R)^{-1}]\begin{pmatrix} R'A' \\ Z_R'A' \end{pmatrix} + WB + F$$

i.e.,

$$Y = U_1A_1' + U_2A_2' + WB + F ,$$

where $A_1 = AR$ and $A_2 = AZ_R$.

Under the null hypothesis $H_0, A_1 = K$. Thus, under H_0, the likelihood function based on observed matrices X_1, X_2 and Y is given by

$$L = (2\pi\sigma^2)^{-n(p+q)/2} \frac{|R'\Sigma_1^{-1}R|^{n/2}|Z_R'\Sigma_1^{-1}Z_R|^{n/2}}{|G|^{(p+q)/2}|\Sigma_2|^{n/2}} \tag{3.2}$$
$$\times \exp\left\{-\frac{1}{2\sigma^2}[\mathrm{tr}\,(R'\Sigma_1^{-1}R)(X_1 - U_1)'G^{-1}(X_1 - U_1)\right.$$
$$+ \mathrm{tr}\,(Z_R'\Sigma_1^{-1}Z_R)(X_2 - U_2)'G^{-1}(X_2 - U_2)$$
$$+ \mathrm{tr}\,\Sigma_2^{-1}(Y - U_1K' - U_2A_2' - WB)'$$
$$\left.\times G^{-1}(Y - U_1K' - U_2A_2' - WB)]\right\} .$$

Let

$$\begin{aligned} Q_1 &= G^{-1} - G^{-1}W(W'G^{-1}W)^{-1}W'G^{-1} = M'M , \\ Q_2 &= G^{-1}W(W'G^{-1}W)^{-1}W'G^{-1} = N'N , \end{aligned} \tag{3.3}$$

where M $((n-k)\times n)$ is of rank $n-k$ and N $(k\times n)$ is of rank k. Also note that $(M' : N')$ is nonsingular. We denote

$$V_1 = MU_1 \quad \text{and} \quad V_2 = MU_2 . \tag{3.4}$$

Then, using Lemma 2.2, and from (3.3) and (3.4), we get

$$\mathrm{tr}\,(R'\Sigma_1^{-1}R)(X_1 - U_1)'G^{-1}(X_1 - U_1) \tag{3.5}$$
$$+ \mathrm{tr}\,(Z_R'\Sigma_1^{-1}Z_R)(X_2 - U_2)'G^{-1}(X_2 - U_2)$$
$$+ \mathrm{tr}\,\Sigma_1^{-1}(Y - U_1K' - U_2A_2' - WB)'$$
$$\times G^{-1}(Y - U_1K' - U_2A_2' - WB)$$

$$\geq \operatorname{tr}(R'\Sigma_1^{-1}R)(MX_1 - V_1)'(MX_1 - V_1)$$
$$+ \operatorname{tr}(Z_R'\Sigma_1^{-1}Z_R)(MX_2 - V_2)'(MX_2 - V_2)$$
$$+ \operatorname{tr}\Sigma_2^{-1}(MY - V_1K' - V_2A_2')'(MY - V_1K' - V_2A_2')$$
$$\geq \sum_{p-r+1}^{p+q-r} \lambda_i[\operatorname{diag}((\Sigma_2 + K(R'\Sigma_1^{-1}R)^{-1}K')^{-1}, Z_R'\Sigma_1^{-1}Z_R)$$
$$\times (MY - MX_1K' \colon MX_2)'(MY - MX_1K' \colon MX_2)]$$
$$= \sum_{p-r+1}^{p+q-r} \lambda_i[\operatorname{diag}((\Sigma_2 + K(R'\Sigma_1^{-1}R)^{-1}K')^{-1}, Z_R'\Sigma_1^{-1}Z_R)$$
$$\times (Y - X_1K' \colon X_2)'Q_1(Y - X_1K' \colon X_2)]$$

with equality achieved when

$$\begin{aligned}
&B = (W'G^{-1}W)^{-1}W'G^{-1}(Y - U_1K' - U_2A_2'),\\
&A_2' = (P_2'MX_2)^{-}P_2'M(Y - U_1K'),\\
&NU_1 = NX_1, \qquad NU_2 = NX_2,\\
&V_1 = P_2P_2'MX_1\\
&\qquad + (I - P_2P_2')M(X_1R'\Sigma_1^{-1}R + Y\Sigma_2^{-1}K)(R'\Sigma_1^{-1}R + K'\Sigma_2^{-1}K)^{-1},\\
&V_2 = P_2P_2'MX_2,
\end{aligned} \tag{3.6}$$

where P_2 is $(n-k) \times (p-r)$ matrix whose columns are the eigenvectors corresponding to the first $(p-r)$ eigenvalues of the matrix

$$M[(Y - X_1K')(\Sigma_2 + K(R'\Sigma_1^{-1}R)^{-1}K')^{-1}(Y - X_1K')' + X_2Z_R'\Sigma_1^{-1}Z_RX_2']M'.$$

Note that, from (3.4) and (3.6), and since $M'M + N'N = G^{-1}$,

$$U_1 = G[M'V_1 + N'NX_1], \qquad U_2 = G[M'V_2 + N'NX_2].$$

Now from (3.2) and (3.5), and by maximizing the likelihood function with respect to σ^2, we get

$$\sup_{H_0} L = (2\pi)^{-n(p+q)/2} \frac{|R'\Sigma_1^{-1}R|^{n/2}|Z_R'\Sigma_1^{-1}Z_R|^{n/2}}{|G|^{(p+q)/2}|\Sigma_2|^{n/2}} \times \left[\frac{1}{n(p+q)} \sum_{p-r+1}^{p+q-r} \lambda_i(H_0)\right]^{-n(p+q)/2} e^{-n(p+q)/2}, \tag{3.7}$$

where

$$H_0 = \operatorname{diag}\left((\Sigma_2 + K(R'\Sigma_1^{-1}R)^{-1}K')^{-1}, Z_R'\Sigma_1^{-1}Z_R\right) \times (Y - X_1K' : X_2)'Q_1(Y - X_1K' : X_2)\,. \tag{3.8}$$

Now to maximize the likelihood function under full parameter space, note that K is also unknown in (3.2). We denote

$$V = M(U_1 : U_2), \qquad A_3' = (K : A_2)'\,. \tag{3.9}$$

Then

$$L = (2\pi\sigma^2)^{-n(p+q)/2} \frac{|R'\Sigma_1^{-1}R|^{n/2}|Z_R'\Sigma_1^{-1}Z_R|^{n/2}}{|G|^{(p+q)/2}|\Sigma_2|^{n/2}}$$
$$\times \exp\left\{-\frac{1}{2\sigma^2}[\operatorname{tr}\operatorname{diag}(R'\Sigma_1^{-1}R, Z_R'\Sigma_1^{-1}Z_R)[(X_1 : X_2) - (U_1 : U_2)]' \times G^{-1}[(X_1 : X_2) - (U_1 : U_2)] + \operatorname{tr}\Sigma_2^{-1}[Y - (U_1 : U_2)A_3' - WB]'G^{-1}[Y - (U_1 : U_2)A_3' - WB]]\right\}.$$

From (3.3), (3.9), and using Lemma 2.1, we get

$$\begin{aligned}
&\operatorname{tr}\operatorname{diag}(R'\Sigma_1^{-1}R, Z_R'\Sigma_1^{-1}Z_R)[(X_1 : X_2) - (U_1 : U_2)]'G^{-1}[(X_1 : X_2) - (U_1 : U_2)]\\
&\quad + \operatorname{tr}\Sigma_2^{-1}[Y - (U_1 : U_2)A_3' - WB]'G^{-1}[Y - (U_1 : U_2)A_3' - WB]\\
&\qquad \geq \operatorname{tr}\operatorname{diag}(R'\Sigma_1^{-1}R, Z_R'\Sigma_1^{-1}Z_R)[M(X_1 : X_2) - V]'[M(X_1 : X_2) - V]\\
&\qquad\quad + \operatorname{tr}\Sigma_1^{-2}[MY - VA_3']'[MY - VA_3']\\
&\qquad \geq \sum_{p+1}^{p+q} \lambda_i[\operatorname{diag}(R'\Sigma_1^{-1}R, Z_R'\Sigma_1^{-1}Z_R, \Sigma_2^{-1})(X_1 : X_2 : Y)'Q_1(X_1 : X_2 : Y)]
\end{aligned}$$

with the equality achieved when

$$\begin{aligned}
B &= (W'G^{-1}W)^{-1}W'G^{-1}[Y - (U_1 : U_2)A_3']\,,\\
A_3' &= [P'M(X_1 : X_2)]^{-}P'MY\,,\\
N(U_1 : U_2) &= N(X_1 : X_2)\,,\\
V &= PP'M(X_1 : X_2)\,,
\end{aligned}$$

where P is an $(n-k)\times p$ matrix whose columns are the eigenvectors corresponding to the first p eigenvalues of the matrix

$$M[X_1R'\Sigma_1^{-1}RX_1' + X_2Z_R'\Sigma_1^{-1}Z_RX_2' + Y\Sigma_2^{-1}Y']M' .$$

Thus, after maximizing the likelihood with respect to σ^2, we get

$$\text{(3.10)} \qquad \sup L = (2\pi)^{-n(p+q)/2} \frac{|R'\Sigma_1^{-1}R|^{n/2}|Z_R'\Sigma_1^{-1}Z_R|^{n/2}}{|G|^{(p+q)/2}|\Sigma_2|^{n/2}} \times\left[\frac{1}{n(p+q)}\sum_{p+1}^{p+q}\lambda_i(H_*)\right]^{-n(p+q)/2} e^{-n(p+q)/2} ,$$

where

$$H_* = \text{diag}\,(R'\Sigma_1^{-1}R, Z_R'\Sigma_1^{-1}Z_R, \Sigma_2^{-1})(X_1\colon X_2\colon Y)'Q_1(X_1\colon X_2\colon Y) .$$

It can be seen, since $R(R'\Sigma_1^{-1}R)^{-1}R'\Sigma_1^{-1} + Z_R(Z_R'\Sigma_1^{-1}Z_R)^{-1}Z_R'\Sigma_1^{-1} = I$, that

$$\lambda_i(H_*) = \lambda_i(H)$$

where

$$\text{(3.11)} \qquad H = \text{diag}\,(\Sigma_2^{-1}, \Sigma_1^{-1})(Y\colon X)'Q_1(Y\colon X) .$$

Thus, from (3.7) and (3.10), the likelihood ratio test is given by

$$\lambda^{-2/n(p+q)} = \frac{\sum_{p-r+1}^{p+q-r}\lambda_i(H_0)}{\sum_{p+1}^{p+q}\lambda_i(H)} ,$$

where H_0 and H are given by (3.8) and (3.11).

Remark 1. If Σ_1 and Σ_2 are unknown, they can be estimated under repeated measurements. The model, under repeated measurements, is

$$x_{ij} = u_i + \varepsilon_{ij} ,$$
$$y_{ij} = b + Au_i + \eta_{ij}; \quad j = 1, 2, \ldots, m_i, \quad i = 1, \ldots, n .$$

Let

$$X = [\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n]', \qquad Y = [\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n]' ,$$
$$W = (1, 1, \ldots, 1)' \quad \text{and} \quad G = \text{diag}\,(m_1^{-1}, m_2^{-1}, \ldots, m_n^{-1}) .$$

Denote

$$S_1 = \frac{1}{\sum m_i - n} \sum_{i=1}^{n} \sum_{j=1}^{m_i} (x_{ij} - \bar{x}_i)(x_{ij} - \bar{x}_i)' ,$$

$$S_2 = \frac{1}{\sum m_i - n} \sum_{i=1}^{n} \sum_{j=1}^{m_i} (y_{ij} - \bar{y}_i)(y_{ij} - \bar{y}_i)' ,$$

$$T = \sum_1^n m_i \begin{pmatrix} \bar{x}_i - \bar{x} \\ \bar{y}_i - \bar{y} \end{pmatrix} \begin{pmatrix} \bar{x}_i - \bar{x} \\ \bar{y}_i - \bar{y} \end{pmatrix}' ,$$

$$T_0 = \sum_1^n m_i \begin{bmatrix} \bar{y}_i - \bar{y} - K(R'S_1^{-1}R)^{-1}R'S_1^{-1}(\bar{x}_i - \bar{x}) \\ (Z_R'S_1^{-1}Z_R)^{-1}Z_R'S_1^{-1}(\bar{x}_i - \bar{x}) \end{bmatrix} \times \begin{bmatrix} \bar{y}_i - \bar{y} - K(R'S_1^{-1}R)^{-1}R'S_1^{-1}(\bar{x}_i - \bar{x}) \\ (Z_R'S_1^{-1}Z_R)^{-1}Z_R'S_1^{-1}(\bar{x}_i - \bar{x}) \end{bmatrix}' ,$$

where $\bar{x} = (\sum m_i)^{-1} \sum_1^n m_i \bar{x}_i$ and $\bar{y} = (\sum m_i)^{-1} \sum_1^n m_i \bar{y}_i$. Then a test can be propose based on

$$-2 \ln \hat{\lambda} = \sum_{p-r+1}^{p-r+q} \lambda_i[\operatorname{diag}((S_2 + K(R'S_1^{-1}R)^{-1}K')^{-1}, Z_R'S_1^{-1}Z_R)T_0] - \sum_{p+1}^{p+q} \lambda_i[\operatorname{diag}(S_1^{-1}, S_2^{-1})T] .$$

4. Likelihood ratio test for testing $S'A = L$

We, first, make the following one-to-one onto transformation

$$Y_M = MY, \quad Y_N = NY, \quad X_M = MX, \quad X_N = NX , \tag{4.1}$$

where M and N are as defined in (3.3). Let $V_1^* = MU$ and $V_2^* = NU$. Then, since $MW = 0$ (note that $M'MW = 0$),

$$\begin{aligned} Y_M &= V_1^* A' + F_M , \\ Y_N &= V_2^* A' + NWB + F_N , \\ X_M &= V_1^* + E_M, \quad X_N = V_2^* + E_N ; \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} \begin{pmatrix} F_M \\ F_N \end{pmatrix} &= \begin{pmatrix} M \\ N \end{pmatrix} F \sim N\left[0, \begin{pmatrix} M \\ N \end{pmatrix} G(M' \colon N') \otimes \sigma^2 \Sigma_2 \right], \\ \begin{pmatrix} E_M \\ E_N \end{pmatrix} &= \begin{pmatrix} M \\ N \end{pmatrix} E \sim N\left[0, \begin{pmatrix} M \\ N \end{pmatrix} G(M' \colon N') \otimes \sigma^2 \Sigma_1 \right]. \end{aligned} \tag{4.3}$$

Note that NW is nonsingular. From (3.3), and since $MW=0$, we get

$$MGM' = (MGM')(MGM'), \qquad (MGN')(NGM') = 0 ,$$

thus, since MGM' is nonsingular, we get

$$MGM' = I, \qquad MGN' = 0 . \tag{4.4}$$

Now, since $G^{-1} = M'M + N'N$, from (4.4),

$$NGN' = I .$$

From (4.3),

$$\begin{pmatrix} F_M \\ F_N \end{pmatrix} \sim N[0, I \otimes \sigma^2 \Sigma_2], \qquad \begin{pmatrix} E_M \\ E_N \end{pmatrix} \sim N[0, I \otimes \sigma^2 \Sigma_1] . \tag{4.5}$$

Now let

$$\begin{aligned} Z_1 &= Y_M S - X_M L' , \\ Z_2 &= Y_M S(S'\Sigma_2 S)^{-1} L + X_M \Sigma_1^{-1} , \\ Z_3 &= Y_M Z_S , \end{aligned} \tag{4.6}$$

where Z_S $(q \times (q-s))$ is such that $Z_S'\Sigma_2 S = 0$ and $Z_S' Z_S = I_{q-s}$. Note that $(X_M, Y_M) \to (Z_1, Z_2, Z_3)$ is a one-to-one and onto transformation with the Jacobian

$$J = \begin{vmatrix} S' & -L \\ L'(S'\Sigma_2 S)^{-1} S' & \Sigma_1^{-1} \\ Z_S' & 0 \end{vmatrix}^{n-k} . \tag{4.7}$$

From (4.2) and (4.6)

$$\begin{aligned} Z_1 &= V_1^*(A'S - L') + F_M S - E_M L' , \\ Z_2 &= V_1^*(A'S(S'\Sigma_2 S)^{-1} L + \Sigma_1^{-1}) + F_M S(S'\Sigma_2 S)^{-1} L + E_M \Sigma_1^{-1} , \\ Z_3 &= V_1^* A' Z_S + F_M Z_S . \end{aligned}$$

Thus under H_0: $A'S = L'$, from (4.5), we get

$$
\begin{aligned}
&Z_1 \sim N[0, I \otimes (S'\Sigma_2 S + L\Sigma_1 L')]\,, \\
(4.8) \qquad &Z_2 \sim N[V_1^* \Gamma, I \otimes \Gamma]\,, \\
&Z_3 \sim N[V_1^* A_*', I \otimes Z_S' \Sigma_2 Z_S]\,,
\end{aligned}
$$

where

$$\Gamma = [L'(S'\Sigma_2 S)^{-1} L + \Sigma_1^{-1}], \qquad A_* = Z_S' A\,.$$

It can be seen from (4.7), since $(S' :\, - L)\operatorname{diag}(\Sigma_2, \Sigma_1)(L'(S'\Sigma_2 S)^{-1} S' : \Sigma_1^{-1})' = 0$ and $Z_S'\Sigma_2 S = 0$ that

$$J = \left[\frac{|S'\Sigma_2 S + L\Sigma_1 L'|\,|L'(S'\Sigma_2 S)^{-1} L + \Sigma_1^{-1}|\,|Z_S'\Sigma_2 Z_S|}{|\Sigma_1|\,|\Sigma_2|} \right]^{(n-k)/2}.$$

Thus, from (4.2), (4.5), (4.6) and (4.8), the likelihood function based on (X_M, X_N, Y_M, Y_N), under H_0, is given by

$$
\begin{aligned}
L = {} & (2\pi\sigma^2)^{-n(p+q)/2} |\Sigma_1|^{-n/2} |\Sigma_2|^{-n/2} \\
& \times \exp\Bigg\{ -\frac{1}{2\sigma^2} [\operatorname{tr}(S'\Sigma_2 S + L\Sigma_1 L')^{-1} (Y_M S - X_M L')'(Y_M S - X_M L') \\
& \quad + \operatorname{tr} \Gamma^{-1} (Y_M S (S'\Sigma_2 S)^{-1} L + X_M \Sigma_1^{-1} - V_1^* \Gamma)' \\
& \qquad \times (Y_M S (S'\Sigma_2 S)^{-1} L + X_M \Sigma_1^{-1} - V_1^* \Gamma) \\
& \quad + \operatorname{tr}(Z_S'\Sigma_2 Z_S)^{-1} (Y_M Z_S - V_1^* A_*')'(Y_M Z_S - V_1^* A_*') \\
& \quad + \operatorname{tr} \Sigma_2^{-1} (Y_N - V_2^* A' - NWB)'(Y_N - V_2^* A' - NWB) \\
& \quad + \operatorname{tr} \Sigma_1^{-1} (X_N - V_2^*)'(X_N - V_2^*)] \Bigg\}.
\end{aligned}
$$

Now, using Lemma 2.1 and since NW is nonsingular, it can be seen, after maximizing the likelihood function with respect to σ^2, that

$$
\begin{aligned}
(4.9) \qquad \sup_{H_0} L = {} & (2\pi)^{-n(p+q)/2} |\Sigma_1|^{-n/2} |\Sigma_2|^{-n/2} \\
& \times \Bigg[\frac{1}{n(p+q)} (\operatorname{tr}(S'\Sigma_2 S + L\Sigma_1 L')^{-1} (YS - XL')' \\
& \qquad \times Q_1 (YS - XL') + \sum_{p+1}^{p+q-s} \lambda_i(\mathscr{M}_0)) \Bigg]^{-n(p+q)/2},
\end{aligned}
$$

where

$$(4.10)\quad \mathscr{M}_0 = \text{diag}\,(\Gamma^{-1}, (Z_S'\Sigma_2 Z_S)^{-1})(YS(S'\Sigma_2 S)^{-1}L + X\Sigma_1^{-1} : YZ_S)' \times Q_1(YS(S'\Sigma_2 S)^{-1}L + X\Sigma_1^{-1} : YZ_S)\,.$$

The supremum above is attained when

$$V_2^* = X_N, \qquad B = (NW)^{-1}(Y_N - X_N A')\,,$$
$$V_1^* = P_* P_*'(Y_M S(S'\Sigma_2 S)^{-1}L + X_M\Sigma_1^{-1})\Gamma^{-1}\,,$$
$$A_*' = [P_*'(Y_M S(S'\Sigma_2 S)^{-1}L + X_M\Sigma_1^{-1})\Gamma^{-1}]^{-}[P_*' Y_M Z_S]\,,$$

where P_* $((n-k)\times p)$ is a matrix whose columns are the eigenvectors corresponding to the first p eigenvalues of the matrix

$$M[(YS(S'\Sigma_2 S)^{-1}L + X\Sigma_1^{-1})\Gamma^{-1}(YS(S'\Sigma_2 S)^{-1}L + X\Sigma_1^{-1})' + YZ_S(Z_S'\Sigma_2 Z_S)^{-1}Z_S'Y']M'\,.$$

Note that $S'A = L$, $A_* = Z_S'A$ implies that $A = \Sigma_2[S(S'\Sigma_2 S)^{-1}L + Z_S(Z_S'\Sigma_2 Z_S)^{-1}A_*]$.

Now to maximize the likelihood over the full parameter space, note that the likelihood function based on (Y_M, Y_N, X_M, X_N) is given by

$$L = (2\pi\sigma^2)^{-n(p+q)/2}|\Sigma_1|^{-n/2}|\Sigma_2|^{-n/2} \times \exp\Big\{-\frac{1}{2\sigma^2}(\text{tr}\,\Sigma_1^{-1}(X_M - V_1^*)'(X_M - V_1^*) + \text{tr}\,\Sigma_1^{-1}(X_N - V_2^*)'(X_N - V_2^*) + \text{tr}\,\Sigma_2^{-1}(Y_N - V_2^*A' - NWB)'(Y_N - V_2^*A' - NWB) + \text{tr}\,\Sigma_2^{-1}(Y_M - V_1^*A')'(Y_M - V_1^*A'))\Big\}\,.$$

Thus using Lemma 2.1, and by maximizing the likelihood function with respect to σ^2, we get

$$(4.11)\quad \sup L = (2\pi)^{-n(p+q)/2}|\Sigma_1|^{-n/2}|\Sigma_2|^{-n/2} \times \left[\frac{1}{n(p+q)}\sum_{p+1}^{p+q}\lambda_i(\mathscr{M})\right]^{-n(p+q)/2} e^{-n(p+q)/2}\,,$$

where

$$(4.12)\quad \mathscr{M} = \text{diag}\,(\Sigma_1^{-1}, \Sigma_2^{-1})(X : Y)'Q_1(X : Y)$$

with the supremum attained when

$$B=(NW)^{-1}(Y_N-X_NA'),\qquad V_2^*=X_N\,,$$
$$V_1^*=Q_*Q_*'X_M,\qquad A'=(Q_*'X_M)^-(Q_*'Y_M)\,,$$

where Q_* is an $(n-k)\times p$ matrix whose columns are the eigenvectors corresponding to the first p eigenvalues of the matrix $M[X\Sigma_1^{-1}X'+Y\Sigma_2^{-1}Y']M'$.

From (4.9) and (4.11), the likelihood ratio test statistics is given by

$$\lambda^{-2/n(p+q)}$$
$$=\frac{\operatorname{tr}(S'\Sigma_2S+L\Sigma_1L')^{-1}(YS-XL')'Q_1(YS-XL')+\sum_{p+1}^{p+q-s}\lambda_i(\mathscr{M}_0)}{\sum_{p+1}^{p+q}\lambda_i(\mathscr{M})},$$

where $\mathscr{M}$ and $\mathscr{M}_0$ are defined by (4.10) and (4.12).

Remark 2. Some exact test can be constructed based on Y_MS and X_M. Note that, from (4.2),

$$Y_MS=V_1^*A'S+F_MS,\qquad X_M=V_1^*+E_M\,.$$

If the covariance matrix of $(Y_MS\colon E_M)$ has an arbitrary structure, then $A'S=L'$ can be tested exactly (see Basu (1969), Villegas (1964) and Bansal (1987)).

Acknowledgements

The author is indebted to the referees for their careful reading of this paper, and for valuable comments which helped me to improve its organization and broaden its content.

REFERENCES

Bansal, N. K. (1987). Some statistical inferences on latent variables, Ph.D. Dissertation, University of Pittsburgh, Pennsylvania.

Basu, A. P. (1969). On some tests for several linear relations, *J. Roy. Statist. Soc. Ser. B*, **31**, 65–71.

Gleser, L. J. and Watson, G. S. (1973). Estimation of a linear transformation, *Biometrika*, **60**, 525–534.

Rao, C. R. (1973). *Linear Statistical Inference and Its Application*, 2nd ed., Wiley, New York.

Villegas, C. (1964). Confidence region for a linear relation, *Ann. Math. Statist.*, **35**, 780–788.