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SOME NEW CONSTRUCTIONS OF BIVARIATE WEIBULL MODELS

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Abstract. In this article, several approaches are advanced towards the construction of bivariate Weibull models from the consideration of failure behaviors of the components of a two-component system. First, a general method of construction of bivariate life models is developed in the setting of random environmental effects. Some new bivariate Weibull models are derived as special cases and added insights are provided for some of the existing ones. In the course of model formulation in terms of the dependence structure, a new bivariate family of life distributions is constructed so as to incorporate both positive and negative quadrant dependence in the same parametric setting, and a bivariate Weibull model is obtained as a special case. Finally, some distributional properties are presented for a bivariate Weibull model derived from the consideration of random hazards.

Key words and phrases: Bivariate Weibull, bivariate exponential, Weibull minimum, random hazards, quadrant dependence, independence representation, moments.

1. Introduction

The Weibull distribution is a versatile family of life distributions in view of its physical interpretation and its flexibility for empirical fit, and has been extensively applied to analysis of life data concerning many types of manufactured items. In practical applications, common handling or a similar environment may lead to induced dependence for the components of a system (cf. Esary and Proschan (1970)). At present, the study of the univariate Weibull model is well documented, however, the investigation of bivariate/multivariate Weibull distribution is rather limited. The usefulness of a bivariate Weibull model can be visualized in many contexts, such as the times to first and second failures of a repairable device, the breakdown times of dual generators in a power plant, or the survival times of the organs in a two-organ system, such as lungs or kidneys, in the human body.

There is extensive literature on the construction of bivariate exponential models, for instance, Gumbel (1960), Freund (1961), Marshall and Olkin (1967), Block and Basu (1974), Clayton (1978) and Sarkar (1987) (see Basu (1988) for a review). Other references on the general construction of bivariate/multivariate lifetime distributions are Marshall and Olkin (1988) and Oakes (1989). One obvious way of generating a bivariate Weibull model is to make a power transformation of the marginals of a bivariate exponential, for instances, Marshall and Olkin (1967) and Lee (1979). Since power transformation is only a mathematical artifact, it would be desirable to motivate a model from physical considerations. The object of this article is to present some physically meaningful approaches for construction of bivariate extensions of the Weibull distribution. These derivations also serve to indicate the conditions under which the distributions are appropriate.

In the context of modeling dependent lifetimes, the bivariate exponential models proposed by Freund (1961) and Marshall and Olkin (1967) are very popular and well-grounded on physical bases. For a two-component system, Freund's model draws from the idea that the failure rate of one component changes upon the failure of the other component. In Marshall and Olkin's fatal shock model, the dependence of the component lifetimes arises from simultaneous failures of both components. Lu and Bhattacharyya (1988*a*) pursue their ideas to construct bivariate extensions of the Weibull model. However, because of a singular part, the Weibull generalization of the exponential shock model is not appropriate for situations where the components are unlikely to fail simultaneously. Although the Weibull generalization of Freund's exponential model is absolutely continuous, its marginals are not Weibull.

In Section 2, stemming from the idea of Hougaard (1986), a general method of construction of bivariate life models is provided on the basis of the idea that random environmental hazards affect both components in a system. This leads to the bivariate Weibull models derived by Hougaard (1986), Clayton (1978) and Oakes (1982) as special cases. One of these models, called BVW, is a transformation of Gumbel's type B bivariate extreme-value distribution proposed by Gumbel (1960). It was extensively studied by Lee (1979) and was used in cancer research (cf. Hougaard (1984, 1986)) and, in analysis of the annual maximum sea levels (cf. Tawn (1988)). Section 2 includes another construction of a bivariate Weibull which shares the basic properties of the BVW, namely, absolute continuity and Weibull marginals and minimum, but not stemming from a random hazards setting. Modeling a bivariate life distribution from the consideration of its dependence structure is approached in Section 3. We construct a new family of bivariate survival functions which not only has a simple form but also can incorporate both positive and negative dependence according to the values of the dependence parameter. It is different from the Farlie-Gumbel-Morgenstern family and is also distinct from several constructions provided by Kimeldorf and Sampson (1975). The choice of Weibull marginals leads to a bivariate Weibull model which incorporates both positive and negative dependence, while most other familiar models are either only positively quadrant dependent or negatively quadrant dependent. Finally, Section 4 explores some distributional properties of the BVW model via a representation in terms of independent random variables.

2. A class of models based on random hazards

Consider a two-component system for which the association between the component lifetimes X and Y arises from the effect of some common environmental factor (stress). Let $h_1(x)$ and $h_2(y)$ be two arbitrary failure rate functions on $[0,\infty)$, and $H_1(x)$ and $H_2(y)$ their corresponding cumulative failure rate (CFR) functions. We assume that conditionally, given the stress $S=s>0$, the failure rates of X and Y are $h_1(x)s$ and $h_2(y)s$ respectively, so their survival functions are $\exp[-H_1(x)s]$ and $\exp[-H_2(y)s]$. Furthermore, conditionally given s, we consider the following joint survival function (SF) of the components

$$\bar{F}(x,y\,|\,s)=\exp\{-[H_1(x)+H_2(y)]^{\gamma}s\}\,, \tag{2.1}$$

where the parameter γ, which measures the conditional association of X and Y, is assumed to be a constant free of s. In particular, when $\gamma=1$, X and Y are conditionally independent. This is the case that Salvia and Bollinger (1984) and Cantor and Knapp (1985) considered in testing equality of the survival distributions based on paired observations. Incidentally, note that if we take $H_i(x)$ to be the Weibull CFR function $\lambda_i x^{\beta}$, $i=1,2$ and $0<\gamma\leq 1$ then (2.1) reduces to Lee's (1979) bivariate Weibull survival function $\exp\{-[\lambda_1^* x^{\beta}+\lambda_2^* y^{\beta}]^{\gamma}\}$ with $\lambda_i^*=\lambda_i s^{1/\gamma}$, $i=1,2$.

Before proceeding further, we must ensure that with the arbitrary CFR functions $H_i(x)$, (2.1) is a proper bivariate survival model. Evidently, the boundary conditions

$$\begin{aligned}&\bar{F}(\infty,y\,|\,s)=\bar{F}(x,\infty\,|\,s)=0, &&\bar{F}(0,0\,|\,s)=1\,,\\ &\bar{F}(x,0\,|\,s)=\bar{F}_X(x\,|\,s), &&\bar{F}(0,y\,|\,s)=\bar{F}_Y(y\,|\,s)\,,\end{aligned} \tag{2.2}$$

all hold. Differentiating (2.1) with respect to x and y, we obtain

$$\begin{aligned}f(x,y\,|\,s)=&\exp\{-[H_1(x)+H_2(y)]^{\gamma}s\}\gamma[H_1(x)+H_2(y)]^{\gamma-1}sh_1(x)h_2(y)\\ &\times\{\gamma s[H_1(x)+H_2(y)]^{\gamma-1}+(1-\gamma)[H_1(x)+H_2(y)]^{-1}\}\,,\end{aligned}$$

which is non-negative for all $x, y > 0$ provided $0 < \gamma \leq 1$. Therefore, when γ is restricted to the interval $(0, 1]$, (2.1) is a valid (conditional) model with arbitrary CFR functions H_1 and H_2.

We now view the stress S to be a positive random variable and let $\phi(s)$ and $\Psi(t) = E[\exp(-tS)]$ denote its probability density function (pdf) and Laplace transform, respectively. In view of (2.1), the unconditional joint distribution of (X, Y) is determined by the distribution assumed for S, and for ease of reference, we call such a joint distribution a "random hazards" (RH) model generated by (2.1). We first address the general question: if X and Y are to have the specified marginal survival functions $\bar{F}_X(x)$ and $\bar{F}_Y(y)$ on $[0, \infty)$, is it possible to choose the CFR functions H_1 and H_2 in (2.1) and a distribution of S so that the joint distribution of (X, Y) is an RH model? The following theorem provides the answer and serves as a general method of constructing bivariate life models with specified marginals.

THEOREM 2.1. *Suppose* (2.1) *represents the conditional survival function of* (X, Y) *given* $S = s\ (>0)$, *and assume that the Laplace transform* $\Psi(t)$ *of* S *exists on* $[0, \infty)$, *is strictly decreasing,* $\Psi(t) \to 0$ *as* $t \to \infty$, *and* $\Psi^{-1}(u)$ *is absolutely continuous on* $(0, 1]$. *Let*

$$(2.3)\qquad \begin{aligned} H_1^*(x) &= \{\Psi^{-1}[\bar{F}_X(x)]\}^{1/\gamma}, & H_2^*(y) &= \{\Psi^{-1}[\bar{F}_Y(y)]\}^{1/\gamma}, \\ q(x, y) &= [H_1^*(x) + H_2^*(y)]^\gamma, & \bar{F}(x, y) &= \Psi[q(x, y)]. \end{aligned}$$

Then $\bar{F}(x, y)$ *is a bivariate survival function with the marginals* $\bar{F}_X$ *and* $\bar{F}_Y$.

PROOF. The proof hinges on the simple fact that any absolutely continuous and nondecreasing function $H(x)$ on $[0, \infty)$ such that $H(0) = 0$ and $H(x) \to \infty$ as $x \to \infty$, is a valid CFR function for a univariate life distribution. Letting $y = 0$ in (2.1) and taking expectation over S, we get the relation

$$\bar{F}_X(x) = \bar{F}(x, 0) = \int_0^\infty \exp[-H_1^\gamma(x)s]\phi(s)ds = \Psi[H_1^\gamma(x)].$$

Solving for H_1 we get $H_1^*(x) = \{\Psi^{-1}[\bar{F}_X(x)]\}^{1/\gamma}$ which is a valid CFR function on $[0, \infty)$ in the light of the assumptions made on $\Psi(t)$. Similarly, $H_2^*(y)$ is also a valid CFR function so (2.1) with H_i, replaced by H_i^*, $i = 1, 2$ is a valid conditional model. The joint distribution of (X, Y) is then

$$\bar{F}(x, y) = E[\exp\{-[H_1^*(x) + H_2^*(y)]^\gamma S\}] = \Psi[q(x, y)].$$

The X-marginal of this joint survival function is $\Psi[\{H_1^*(x)\}^\gamma] = \bar{F}_X(x)$

which was initially targeted and likewise for the Y-marginal. □

Besides γ, additional parameters can be brought into the RH model (2.3) via parametric assumptions for the marginal distributions as well as the function ϕ. For instance, scale parameters can be incorporated by specifying the marginals to be $\bar{F}_X(x/\theta_1)$ and $\bar{F}_Y(y/\theta_2)$. Some interesting special constructs are illustrated in the following examples.

Example 1. Consider the Weibull marginals $\bar{F}_X(x) = \exp[-(x/\theta_1)^{\beta_1}]$, $\bar{F}_Y(y) = \exp[-(y/\theta_2)^{\beta_2}]$, $0 < x, y < \infty$ and let $\Psi(t) = \exp(-t^{\alpha})$, $0 < \alpha \le 1$. It is the Laplace transform of a positive stable distribution (cf. Hougaard (1984, 1986)) and it satisfies the conditions of our theorem. Since $\Psi^{-1}(u) = (-\log u)^{1/\alpha}$, we have from (2.3)

$$H_1^*(x) = (x/\theta_1)^{\beta_1/\alpha\gamma}, \quad H_2^*(y) = (y/\theta_2)^{\beta_2/\alpha\gamma},$$
$$\bar{F}(x,y) = \exp\{-[(x/\theta_1)^{\beta_1/\alpha\gamma} + (y/\theta_2)^{\beta_2/\alpha\gamma}]^{\alpha\gamma}\}, \qquad 0 < \gamma \le 1, \quad 0 < \alpha \le 1 .$$

Obviously, in this end result, the parameters α an γ are not individually identifiable. Combining them into a single parameter $\delta = \alpha\gamma$, $0 < \delta \le 1$, we arrive at the bivariate Weibull model

$$\text{(2.4)} \qquad \bar{F}(x,y) = \exp\{-[(x/\theta_1)^{\beta_1/\delta} + (y/\theta_2)^{\beta_2/\delta}]^{\delta}\}, \qquad 0 < \delta \le 1 .$$

Hougaard (1986) constructed this model in essentially the same way as above except that he assumed $\gamma = 1$ in the conditional distribution (2.1). This assumption entails that conditionally given s, x and y are independent Weibull. Our derivation shows that the same bivariate Weibull model can also arise in the context where, conditionally given s, X and Y are still dependent, perhaps due to the common effect of some other influencing factor. Thus, the assumption of conditional independence is more of a convenience than a necessity in deriving the bivariate model.

The model (2.4) can also be derived by assuming a simpler distribution of S. Assume that $1/S$ has the gamma distribution with the scale and shape parameters equal to 1 and 1/2, respectively. Then $\phi(s) \propto s^{-3/2}\exp(1/s)$ and $\Psi(t) = \exp[-2t^{1/2}]$. Using the Weibull marginals and applying this Laplace transformation to (2.3), we obtain $H_1^*(x) = \{(1/2)(x/\theta_1)^{\beta_1}\}^{2/\gamma}$ and $H_2^*(y) = \{(1/2)(y/\theta_2)^{\beta_2}\}^{2/\gamma}$ which lead to the bivariate Weibull model (2.4) with $\delta = \gamma/2$.

Example 2. Let $\bar{F}_X$ and $\bar{F}_Y$ be arbitrary survival functions which are desired to be the marginals, and take

$$\Psi(t) = [1 + (\delta - 1)t]^{-1/(\delta-1)}, \qquad 0 \le t < \infty, \quad 1 < \delta < \infty ,$$

so S has the gamma distribution with the scale and shape parameters equal to 1 and $(\delta - 1)^{-1}$, respectively. Here also, the conditions of the theorem hold. We have $\Psi^{-1}(t) = (\delta - 1)^{-1}(u^{1-\delta} - 1)$ so (2.3) yields

$$
\begin{aligned}
&H_1^*(x) = (\delta - 1)^{-1/\gamma}[(\bar{F}_X(x))^{1-\delta} - 1]^{1/\gamma}, \\
(2.5)\quad &\bar{F}(x, y) = \{1 + [\{(\bar{F}_X(x))^{1-\delta} - 1\}^{1/\gamma} + \{(\bar{F}_Y(y))^{1-\delta} - 1\}^{1/\gamma}]^{\gamma}\}^{-1/(\delta-1)} .
\end{aligned}
$$

If we take the marginals to be Weibull, (2.5) would readily provide a bivariate Weibull model. Incidentally, a special case of (2.5) namely $\gamma = 1$, is considered by Oakes (1982) for analysis of bivariate survival data where δ is identified as the parameter that governs the association between X and Y. He gives a random hazards interpretation assuming that X and Y are conditionally independent and also relates the model to one due to Clayton (1978). Allowing for the possibility of conditional dependence, we have the more general model (2.5).

Example 3. Take the Weibull marginals as in Example 1, the Laplace transform $\Psi(t)$ as in Example 2, and further take $\delta = 2$, leaving γ as a free parameter. We then obtain

$$
\begin{aligned}
&H_1^*(x) = \{\exp[(x/\theta_1)^{\beta_1}] - 1\}^{1/\gamma}, \\
(2.6)\quad &\bar{F}(x, y) = [1 + [\{\exp[(x/\theta_1)^{\beta_1}] - 1\}^{1/\gamma} + \{\exp[(y/\theta_2)^{\beta_2}] - 1\}^{1/\gamma}]^{\gamma}]^{-1} .
\end{aligned}
$$

The expression (2.6) yields a bivariate Weibull model which is quite different from (2.4) but also has a random hazards interpretation. Here, the dependence parameter γ has its source in the conditional dependence of X and Y rather than in the distribution assumed for S. Despite an interesting structural interpretation, this model has the undesirable feature that no value of γ yields independence of X and Y. To see this, we first note from (2.6) that

$$\{[\bar{F}(x, y)]^{-1} - 1\}^{1/\gamma} = \{[\bar{F}_X(x)]^{-1} - 1\}^{1/\gamma} + \{[\bar{F}_Y(y)]^{-1} - 1\}^{1/\gamma} .$$

Due to convexity of the function $x^{1/\gamma}$ on $[0, \infty)$, it follows that

$$\frac{1}{\bar{F}(x, y)} \le \frac{1}{\bar{F}_X(x)} + \frac{1}{\bar{F}_Y(y)} - 1 ,$$

and a simple algebraic manipulation then yields

$$\bar{F}_X(x)\bar{F}_Y(y)/\bar{F}(x, y) \le 1 - F_X(x)F_Y(y) \le 1 .$$

Here the last equality is not possible for all (x, y), thus ruling out independence.

The bivariate Weibull model (2.4) can be related to Gumbel's type B bivariate extreme-value distribution (cf. Johnson and Kotz (1972), p. 251)

$$F(u,v) = \exp\{-[(-\log F_U(u))^m + (-\log F_V(v))^m]^{1/m}\}, \qquad 1 \le m, \quad -\infty < u, \quad v < \infty .$$

Specifically, if we take the type I extreme-value marginals

$$F_U(u) = \exp\{-\exp[-(u-\mu_1)/\eta_1]\},$$
$$F_V(v) = \exp\{-\exp[-(v-\mu_2)/\eta_2]\},$$

transform to $X/\theta_1 = \exp(\mu_1 - u)$, $Y/\theta_2 = \exp(\mu_2 - v)$, and take $\beta_i = 1/\eta_i$, $i = 1, 2$, $\delta = 1/m$, we obtain the distribution (2.4). Lee (1979) examines the model (2.4) and its Weibull minimum property; namely, if the shape parameters are equal, the minimum of X and Y is a Weibull. Furthermore, he represents (X, Y) in terms of independent random variables U and V, which is quite useful to derive distributional results for the model (2.4). We designate the model BVW for ease of reference.

Example 4. We conclude this section by providing another bivariate Weibull model which shares all the aforementioned properties of BVW except that its construction is not based on the random hazards model. We begin with requiring the bivariate survival function to be of the form

$$\bar{F}(x,y) = \exp\{-(x/\theta_1)^{\beta_1} - (y/\theta_2)^{\beta_2} - \delta h(x,y)\}, \tag{2.7}$$

where h may also depend on the parameters. We note that the Marshall and Olkin (1967) bivariate Weibull model is of the above form with $h(x,y) = \max(x^{\beta_1}, y^{\beta_2})$; it has Weibull marginals as well as a Weibull minimum, but $\partial^2 h/(\partial x \partial y)$ does not exist. To avoid this last difficulty while retaining the other properties, one simple construction is

$$h(x,y) = [(x/\theta_1)^{\beta_1/m} + (y/\theta_2)^{\beta_2/m}]^m,$$

where $m > 0$ is a constant. With this choice (2.7) takes the form

$$\bar{F}(x,y) = \exp\{-(x/\theta_1)^{\beta_1} - (y/\theta_2)^{\beta_2} - \delta[(x/\theta_1)^{\beta_1/m} + (y/\theta_2)^{\beta_2/m}]^m\}. \tag{2.8}$$

To determine the allowable range of the parameter values δ and m, it is convenient to work with the transformed variables $U=(X/\theta_1)^{\beta_1}$ and $V=(Y/\theta_2)^{\beta_2}$ whose joint SF is

$$\bar{F}(u,v)=\exp\left[-u-v-\delta(u^{1/m}+v^{1/m})^m\right].$$

This evidently satisfies the boundary conditions and it can be shown that in order to have $\partial^2\bar{F}(u,v)/(\partial u\partial v)=f(u,v)\geq 0$ for $u,v\in[0,1]$ we require that $\delta\geq 0$ and $0<m\leq 1$. Taking $\beta_1=\beta_2=1$, in particular, we have a new bivariate exponential distribution

$$\bar{F}(x,y)=\exp\{-(x/\theta_1)-(y/\theta_2)-\delta[(x/\theta_1)^{1/m}+(y/\theta_2)^{1/m}]^m\}, \quad 0<m\leq 1,\quad 0<\theta_1,\theta_2,\quad 0\leq\delta, \tag{2.9}$$

which is absolutely continuous, has exponential marginals and minimum.

3. A model motivated from specified dependence structure

Random variables X and Y are said to be positively quadrant dependent (PQD) (Lehmann (1966)) if, for all (x,y),

$$P(X>x,\ Y>y)\geq P(X>x)P(Y>y).$$

Negative quadrant dependence (NQD) is correspondingly defined by reversing the inequality in the middle. For PQD random variables, large values of one variable tend to accompany large values of the other, and likewise for small values. Marshall and Olkin's bivariate Weibull model and the BVW model are both PQD. Specific work environments may justify negative dependence of the component lives. This happens, for instance, with a two-component system if, upon failure of one component, the other has to function in a reduced load condition or under a lesser demand of the output. One example of a bivariate model having the NQD property is Gumbel's (1960) bivariate exponential distribution

$$\bar{F}(x,y)=\exp(-x-y-\delta xy),\qquad \delta>0,\quad x,y>0. \tag{3.1}$$

All aforementioned models are either only PDQ or only NQD. None of them incorporate both PQD and NQD properties within the same functional form. Moreover, all survival functions of these models have the functional form

$$\bar{F}(x,y)=\exp[-h(x,y)], \tag{3.2}$$

for positive $h(x,y)$ which involves the model parameters and has a cumulative failure rate interpretation $h(x,0) = -\log \bar{F}_X(x)$ in the univariate case. Our object here is to construct a bivariate Weibull model which has a survival function of the form (3.2) and contains a dependence parameter δ such that the model is PQD for an interval of δ-values and is NQD for another interval. To this end, we first examine why some of the preceding models, such as model (2.8) and Gumbel's model (3.1), cannot be both PQD and NQD. We observe that both of these models are of the form

$$(3.3) \qquad \bar{F}(x,y) = \bar{F}_X(x)\bar{F}_Y(y)\exp[\delta k(x,y)], \qquad k(x,y) \geq 0 .$$

Note that $k(x,y)$ satisfies $k(x,0) = k(0,y) = 0$. The form (3.3) is convenient for determining PQD and NQD properties. Specifically, (3.3) would include PQD as well as NQD properties if, for all (x,y), $\exp[\delta k(x,y)] > (<) 1$ for $\delta > (<) 0$, and if both positive and negative δ-values could be allowed.

For a further simplification, we consider the probability integral transformation $(U, V) = [F_X(X), F_Y(Y)]$ of the general model (3.3), and get the uniform representation of $\bar{F}(x,y)$ as

$$(3.4) \qquad \bar{G}(u,v) = (1-u)(1-v)\exp[\delta k^*(u,v)], \qquad 0 \leq u, v \leq 1 ,$$

where $k^*(u,v) = k[F_X^{-1}(u), F_Y^{-1}(v)]$, and $F^{-1}(t) = \inf\{x\colon F(x) \geq t\}$. Note that a distribution is PQD (NQD) whenever its uniform representation is PQD (NQD) (cf. Kimeldorf and Sampson (1975)). Therefore, to construct a bivariate distribution with a specified dependence structure, it suffices to focus attention on its uniform representation.

As we will see below, the reason why Gumbel's model (3.1) is only NQD and model (2.8) is only PQD is the violation of the positivity of the pdf for a certain range of δ which is needed to have the opposite dependence property. Instead of writing the pdf's of the uniform representation of (3.1) and (2.8) individually, we write them in a general form and then specify it to the particular model later. Assuming the existence of the second cross partial derivative $\partial^2 k^*(u,v)/(\partial u \partial v)$, we have from (3.4),

$$\begin{aligned} g(u,v) &= \partial^2 \bar{G}(u,v)/(\partial u \partial v) \\ &= \partial\{(1-v)\exp[\delta k^*(u,v)][-1 + \delta(1-u)\partial k^*(u,v)/\partial u]\}/\partial v \\ &= \exp[\delta k^*(u,v)]C(u,v) , \end{aligned}$$

where

$$(3.5) \qquad C(u,v) = 1 + \delta D(u,v) + \delta^2(1-u)(1-v)\frac{\partial k^*}{\partial u}\frac{\partial k^*}{\partial v}$$

and

$$(3.6) \qquad D(u, v) = (1-u)(1-v)\frac{\partial^2 k^*}{\partial u \partial v} - (1-u)\frac{\partial k^*}{\partial u} - (1-v)\frac{\partial k^*}{\partial v},$$

Since $\exp[\delta k^*(u, v)]$ is always positive, let us examine the $C(u, v)$ corresponding to each of the models (3.1) and (2.8).

Example 5. The uniform representation of Gumbel's model (3.1) leads to the following $C(u, v)$ and $D(u, v)$:

$$C_1(u, v) = 1 + \delta D_1(u, v) + \delta^2[\log(1-u)\log(1-v)],$$
$$D_1(u, v) = 1 + \log(1-u) + \log(1-v).$$

Note that the third term of C_1 is always positive. But, as $u \to 1$ or $v \to 1$, the D_1 function tends to $-\infty$ so that δ *must be negative* to guarantee $C_1 \geq 0$ or $g(u, v) \geq 0$ for (3.1). Positive values of δ being precluded, model (3.1) can only be NQD.

Example 6. For the model (2.8), let us denote $r = -\log(1-u)$ and $s = -\log(1-v)$. Then, the corresponding $k^*(u, v)$ function becomes

$$k^*(u, v) = r + s - (r^{1/m} + s^{1/m})^m,$$

and we obtain $C(u, v)$ and $D(u, v)$ as follows:

$$C_2(u, v) = 1 + \delta D_2(u, v) + \delta^2 Z(r, s) Z(s, r)$$

and

$$D_2(u, v) = [(1/m) - 1] r^{(1/m)-1} s^{(1/m)-1} (r^{1/m} + s^{1/m})^{m-2} - Z(r, s) - Z(s, r),$$

where $Z(r, s) = 1 - r^{(1/m)-1}(r^{1/m} + s^{1/m})^{m-1}$. Since $r, s \geq 0$, we have $(r^{1/m} + s^{1/m})^{1-m} \geq r^{(1/m)-1}$, so $Z(r, s)$ and $Z(s, r)$ are both positive. Hence, we conclude that the third term of C_2 is positive. Taking $u = v \to 0^+$, i.e., $r = s \to 0^+$, we have

$$D_2(u, u) = 2^{m-2}[(1/m) - 1] r^{-1} + (2^m - 2) \to \infty,$$

which restricts the dependence parameter δ *to be positive* to insure $C_2(u, v) \geq 0$. Consequently, the model (2.8) can only be PQD.

To construct a bivariate model with both PQD and NQD properties, we look for a $k^*(u,v)$ function such that $C(u,v)$ is positive for negative and positive values of δ. In order to allow negative values of δ, we shall look for positive $[\partial k^*/\partial u][\partial k^*/\partial v]$, and this implies bounded $D(u,v)$. With this in mind, we now proceed to construct a $k^*(u,v)$ function such that (3.4) is a bivariate distribution with uniform marginals. The simplest construction of a $k^*(u,v)$ is to require that $\partial^2 k^*/(\partial u \partial v) = 1$, which implies $\partial k^*/\partial u = v$ and $\partial k^*/\partial v = u$. For this choice, we not only get positive $[\partial k^*/\partial u][\partial k^*/\partial v] = uv$ but also have bounded $D(u,v)$ of the form

$$D(u,v) = (1-u)(1-v) - v(1-u) - u(1-v) = 1 - uv \in [0,1] .$$

Since $1 + \delta D(u,v)$ is never negative if $-1 \le \delta$, the density $g(u,v)$ is positive on the unit square. The resulting bivariate uniform distribution has the survival function

$$\bar{G}(u,v) = (1-u)(1-v)\exp(\delta uv), \quad 0 \le u, v \le 1, \quad -1 \le \delta , \tag{3.7}$$

which satisfies all boundary conditions (2.2) (without conditioning on s), and is PQD if $\delta \ge 0$ and NQD if $\delta \le 0$. Finally, the bivariate Weibull distribution, whose uniform representation is (3.7), is given by

$$\begin{aligned}\bar{F}(x,y) = \exp\{ & -(x/\theta_1)^{\beta_1} - (y/\theta_2)^{\beta_2} \\ & + \delta\{1 - \exp[-(x/\theta_1)^{\beta_1}]\}\{1 - \exp[-(y/\theta_2)^{\beta_2}]\}\} , \\ & -1 \le \delta, \quad 0 < \theta_i, \beta_i, \quad i = 1,2 .\end{aligned} \tag{3.8}$$

The special case, $\beta_1 = \beta_2 = 1$, leads to a new bivariate exponential model

$$\begin{aligned}\bar{F}(x,y) = \exp\{ & -(x/\theta_1) - (y/\theta_2) \\ & + \delta\{1 - \exp[-(x/\theta_1)]\}\{1 - \exp[-(y/\theta_2)]\}\} .\end{aligned}$$

Remarks. (1) Instead of Weibull, if we specify arbitrary marginals, (F_X, F_Y), and require (3.7) to be the uniform representation of $\bar{F}(x,y)$, then we arrive at

$$\bar{F}(x,y) = \bar{F}_X(x)\bar{F}_Y(y)\exp[\delta F_X(x)F_Y(y)], \quad -1 \le \delta . \tag{3.9}$$

This provides a new family of bivariate distributions with specified marginals. It is different from the Farlie-Gumbel-Morgenstern (FGM) family

$$\bar{F}(x,y) = \bar{F}_X(x)\bar{F}_Y(y)[1 + \delta F_X(x)F_Y(y)], \quad |\delta| \le 1 . \tag{3.10}$$

(2) Let us denote the product-moment correlations (ρ) of the bivariate

distribution obtained from the families (3.9) and FGM (3.10) as ρ_1 and ρ_2, respectively. Using the expansion

$$\exp(z) = 1 + z + z^2/2! + z^3/3! + \cdots ,$$

we observe that $E(XY) = \int_0^\infty \int_0^\infty \bar{F}(x, y) dx dy$ obtained from the new family (3.9) is larger than the one obtained from the FGM family (3.10) for all δ. Since ρ is a monotonic function of $E(XY)$, we conclude that, for the same marginal distributions, ρ_1 is always greater than ρ_2 except in the independent case.

(3) In the case of exponential marginals, we can obtain closed form expressions of the product-moment correlation for the families (3.9) and (3.10). We have

$$\rho_1 = \sum_{j=2}^{\infty} \delta^{j-1}/[j(j!)], \quad \rho_2 = \delta/4, \qquad -1 \le \delta \le 1 .$$

Evaluating the summation in ρ_1 to five terms ($j = 6$), we thus obtain the approximated range of ρ_1 as $(-0.20342, 0.31787)$ (the next omitted term is of size $1/35280$). This compares with the range $-0.25 \le \rho_2 \le 0.25$ obtained from the FGM family (2.9).

4. Properties of the bivariate Weibull distribution BVW

As remarked earlier, the BVW has several basic features: its derivation from random hazards lends a physical basis; it is absolutely continuous and has Weibull marginals and minimum. Some additional properties of this model are discussed in this section.

The BVW (2.4) has a pdf of the form

$$\begin{aligned} f(x, y) = {} & \theta_1^{-1}\theta_2^{-1}\beta_1\beta_2(x/\theta_1)^{\beta_1/\delta - 1}(y/\theta_2)^{\beta_2/\delta - 1}[(x/\theta_1)^{\beta_1/\delta} + (y/\theta_2)^{\beta_2/\delta}]^{\delta - 2} \\ & \times \{[(x/\theta_1)^{\beta_1/\delta} + (y/\theta_2)^{\beta_2/\delta}]^{\delta} + 1/\delta - 1\} \\ & \times \exp\{-[(x/\theta_1)^{\beta_1/\delta} + (y/\theta_2)^{\beta_2/\delta}]^{\delta}\} . \end{aligned}$$

An interesting feature of the BVW is that the random variable (X, Y) can be represented (cf. Lee (1979)) in terms of the independent random variable (U, V), where $(U, V) = [Z_1(Z_1 + Z_2)^{-1}, (Z_1 + Z_2)^{\delta}]$ and $(Z_1, Z_2) = [(X/\theta_1)^{\beta_1/\delta}, (Y/\theta_2)^{\beta_2/\delta}]$. Specifically, we have

$$X = U^{\delta/\beta_1} V^{1/\beta_1}\theta_1, \qquad Y = (1 - U)^{\delta/\beta_2} V^{1/\beta_2}\theta_2 , \tag{4.1}$$

where U has a uniform distribution on the unit interval and V has a mixture of exponential and gamma distributions with density $h(v) = [(1-\delta) + \delta v] \exp(-v), v > 0$.

The representation (4.1) can be an aid in evaluating the moments of (X, Y), generating BVW samples for simulation study, and deriving distributional results. For examples, we apply (4.1) to obtain the general moment of (X, Y),

$$E(X^i Y^j) = \theta_1^j \theta_2^j E[U^{i\delta/\beta_1}(1-U)^{j\delta/\beta_2}] E[V^{i/\beta_1 + j/\beta_2}] .$$

The first expectation is readily evaluated from the Beta function,

$$E[U^{i\delta/\beta_1}(1-U)^{j\delta/\beta_2}] = \Gamma(i\delta/\beta_1 + 1)\Gamma(i\delta/\beta_2 + 1)/\Gamma[(i/\beta_1 + j/\beta_2)\delta + 2] .$$

The second expectation can be evaluated in terms of Gamma functions,

$$\begin{aligned} E[V^{i\delta/\beta_1 + j/\beta_2}] &= (1-\delta)\int_0^\infty v^c \exp(-v)dv + \delta\int_0^\infty v^{c+1} \exp(-v)dv \\ &= (1+\delta c)\Gamma(c+1) , \end{aligned}$$

where $c = i/\beta_1 + j/\beta_2$. We then have

$$E(X^i Y^j) = \frac{\Gamma(i\delta/\beta_1 + 1)\Gamma(j\delta/\beta_2 + 1)\Gamma[i/\beta_1 + \beta_2 + 1]\theta_1^i \theta_2^j}{\Gamma(i/\beta_1 + j/\beta_2)\delta + 1} , \tag{4.2}$$

where i and j are positive integers. The mean, variance, and the covariance are as follows:

$$E(X) = \theta_1 \Gamma(1/\beta_1 + 1), \qquad \mathrm{Var}(X) = \theta_1^2\{\Gamma(2/\beta_1 + 1) - [\Gamma(1/\beta_1 + 1)]^2\} ,$$

$$\begin{aligned} \mathrm{Cov}(X, Y) = \theta_1\theta_2 \Bigg\{ & \frac{\Gamma(\delta/\beta_1 + 1)\Gamma(\delta/\beta_2 + 1)\Gamma(1/\beta_1 + 1/\beta_2 + 1)}{\Gamma[\delta(1/\beta_1 + 1/\beta_2) + 1]} \\ & - \Gamma(1/\beta_1 + 1)\Gamma(1/\beta_2 + 1) \Bigg\} . \end{aligned}$$

For the equal shape parameters case, $\beta = \beta_1 = \beta_2$, the correlation of X and Y is

$$\begin{aligned} \mathrm{Corr}(X, Y) = & \{\Gamma(2/\beta + 1) - [\Gamma(1/\beta + 1)]^2\}^{-1} \\ & \times \left\{ \frac{[\Gamma(\delta/\beta + 1)]^2 \Gamma(2/\beta + 1)}{\Gamma(2\delta/\beta + 1)} - [\Gamma(1/\beta + 1)]^2 \right\} . \end{aligned}$$

Figure 1 gives various plots of the correlation coefficient ρ against the

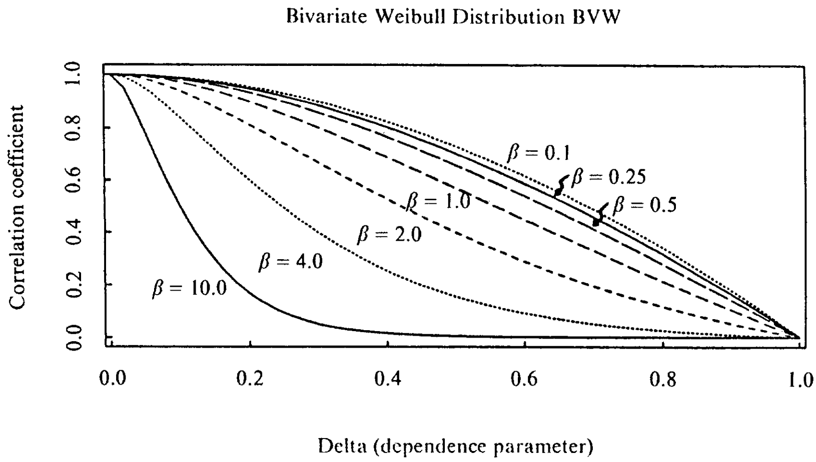

Fig. 1. Correlation versus dependence plot (equal shape parameters) for the BVW.

dependence parameter δ. As the shape parameters approach 1.0, the relationship between ρ and δ is closer to linear. In the bivariate exponential case ($\beta_1 = \beta_2 = 1$), we have

$$\operatorname{Corr}(X, Y) = 2[\Gamma(\delta + 1)]^2 / \Gamma(2\delta + 1) - 1 ,$$

which is free of the parameters in the marginal distribution of X and Y. This is convenient for simulation (cf. Raftery (1984)) and estimating the dependence parameter by using the sample correlation. In contrast, the correlation of the Marshall-Olkin bivariate exponential model is $\rho = \delta/(\lambda_1 + \lambda_2 + \delta)$, which involves the parameters of marginal distributions. This is the case of other bivariate exponential models such as those of Freund (1961), Downton (1970), Hawkes (1972), Paulson (1973) and Block and Basu (1974).

The conditional expectation of Y given $X = x$ is of the form

$$\delta\theta_2 (x/\theta_1)^{\beta_1(1/\delta - 1)} \exp\{(x/\theta_1)^{\beta_1}\} \times \left\{ \int_0^\infty w^c (w + t)^{2\delta - 2} \exp\{-(w + t)^\delta\} dw + (1/\delta - 1) \int_0^\infty w^c (w + t)^{\delta - 2} \exp\{-(w + t)^\delta\} dw \right\},$$

where $c = \delta/\beta_2$ and $t = (x/\theta_1)^{\beta_1/\delta}$. The integration $\int_0^\infty w^c (w + t)^d \exp\{-(w + t)^\delta\} dw$ (where $c > 0$ and $-2 < d \leq 0$) can be obtained by using numerical methods. We note that the conditional expectation Y given $X = x$ is not a linear function of x.

To the remainder of this section, we consider the case of equal shape parameters. Based on the representation (4.1), we have

$$\log (X/Y) = (\delta/\beta) \log [U/(1-U)] + \log (\theta_1/\theta_2) ,$$

and $\log [U/(1-U)]$ has the standard logistic distribution. We conclude that $\log (X/Y)$ has a logistic distribution with location and scale parameters $\log (\theta_1/\theta_2)$ and δ/β, respectively. This remarkable property leads to the construction of a simple estimator of the dependence parameter δ and a nearly exact test of equality of marginals in the equal shape parameter case. In the exponential case, the simple estimator turns out to be quite efficient (cf. Lu and Bhattacharyya (1988*b*)), especially for $\delta \le 0.6$.

Utilizing (4.1), we obtain the probability of $X < Y$ in a simple expression

$$P(X<Y) = \theta_1^{-\beta/\delta}(\theta_1^{-\beta/\delta} + \theta_2^{-\beta/\delta})^{-1} . \tag{4.3}$$

Let us denote $I[\cdot]$ as the indicator function. We show that the minimum (T) and the failure-indicator $(D = I[X<Y])$ are independent. The conditional density of T given $D=1$ is written as:

$$\lim_{\Delta \to 0} \frac{P(t \le T \le t+\Delta \mid D=1)}{\Delta} = \lim_{\Delta \to 0} \frac{P(t \le \min (X,Y) \le t+\Delta \mid X<Y)}{\Delta}$$
$$= \lim_{\Delta \to 0} \frac{P(t \le X \le t+\Delta < Y)}{\Delta} \frac{1}{P(X<Y)} .$$

The first term is evaluated as $-\partial \bar{F}(x,y)/\partial x$ at $x=y=t$ where $\bar{F}(x,y)$ is the survival function (2.4). The second term $P(X<Y)$ is given in (4.3). Hence, the conditional density becomes

$$\frac{\beta}{\theta_1}\left[\frac{t}{\theta_1}\right]^{\beta/\delta-1}\left[\left[\frac{t}{\theta_1}\right]^{\beta/\delta}+\left[\frac{t}{\theta_2}\right]^{\beta/\delta}\right]^{\delta-1} \tag{4.4}$$
$$\times \exp\left\{\left[\left[\frac{t}{\theta_1}\right]^{\beta/\delta}+\left[\frac{t}{\theta_2}\right]^{\beta/\delta}\right]^{\delta}\right\} \frac{\theta_1^{-\beta/\delta}+\theta_2^{-\beta/\delta}}{\theta_1^{-\beta/\delta}}$$
$$= \beta t^{\beta-1}[\theta_1^{-\beta/\delta}+\theta_2^{-\beta/\delta}]^{\delta}$$
$$\times \exp\{-t^{\beta}[\theta_1^{-\beta/\delta}+\theta_2^{-\beta/\delta}]^{\delta}\} .$$

On the other hand, differentiating the survival function of T,

$$P(T>t) = P(X>t, Y>t) = \exp\{-t^{\beta}[\theta_1^{-\beta/\delta}+\theta_2^{-\beta/\delta}]^{\delta}\} ,$$

with respect to t, we obtain the pdf of T as (4.4). Therefore, T and D are independent.

Instead of using the non-linear relationship of ρ and δ, one can use the distributional results of T and D to construct closed form simple (moment type) estimators of δ. The expectations of the products between X, Y, T and D are essential to establish the joint asymptotic distribution of the simple estimators of the parameters of BVW distribution. Here, we provide the evaluation of the expectation $E(TX)$ as an example. Let us define

$$w=\delta/\beta+1,\qquad c=\frac{\theta_1^{-\beta/\delta}}{\theta_1^{-\beta/\delta}+\theta_2^{-\beta/\delta}},\qquad B_c(a,b)=\int_0^c u^{a-1}(1-u)^{b-1}du\,.$$

Splitting the range of (X, Y) into $[X<Y]$, $[X>Y]$, and using (4.1), we have

$$\begin{aligned}E(TX)=\ &\theta_1^2\int_0^c u^{2\delta/\beta}du\int_0^\infty v^{2/\beta}[(1-\delta)+\delta v]\exp(-v)dv\\&+\theta_1\theta_2\int_c^1 u^{\delta/\beta}(1-u)^{\delta/\beta}du\int_0^\infty v^{2/\beta}[(1-\delta)+\delta v]\exp(-v)dv\,.\end{aligned}$$

The integration with respect to v and u can be evaluated in terms of Gamma functions and an incomplete Beta function, respectively. We have

$$\begin{aligned}E(TX)=\ &[(1-\delta)\Gamma(2/\beta+1)+\delta\Gamma(2/\beta+2)]\\&\times\{\theta_1^2c^{2\delta/\beta+1}(2\delta/\beta+1)^{-1}+\theta_1\theta_2[B_1(w,w)-B_c(w,w)]\}\,.\end{aligned}$$

In the exponential case, the result can be further simplified. We have

$$E(TX)=2\theta_1\{\theta_1c^{2\delta+1}+\theta_2(2\delta+1)[B_1(\delta+1,\delta+1)-B_c(\delta+1,\delta+1)]\}\,.$$

Similar calculation leads to the following results:

$$E(XI[X<Y])=\theta_1c^{\delta+1},\qquad E(YI[X<Y])=\theta_2[1-(1-c)^{\delta+1}]\,,$$

in the exponential case.

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