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THE GROWTH CURVE MODEL WITH AN AUTOREGRESSIVE COVARIANCE STRUCTURE

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Abstract. The growth curve model with an autoregressive covariance structure is considered. An iterative algorithm for finding the MLE's of the parameters in the model is presented, based on the modified likelihood equations. Asymptotic distributions of the MLE's are obtained when the sample size is large. A likelihood ratio statistic for testing the autoregressive covariance structure is presented.

Key words and phrases: Growth curve model, autoregressive covariance structure, MLE's, asymptotic distributions, likelihood ratio statistic.

1. Introduction

The growth curve model (Potthoff and Roy (1964)) is given by

$$\begin{aligned} E(Y) &= A\Xi B\,, \\ V(\mathrm{vec}\,(Y')) &= I_N \otimes \Sigma\,, \end{aligned} \tag{1.1}$$

where $Y=(y_1,\dots,y_N)'$: $N\times p$ is an observation matrix, $\mathrm{vec}\,(Y')=(y_1',\dots,y_N')'$, A: $N\times k$ is a known design matrix of rank k, Ξ: $k\times q$ is a matrix of unknown parameters, B: $q\times p$ is a known matrix of rank q, Σ: $p\times p$ is positive definite and the rows of Y are independently normally distributed. This model has been considered by many authors, including Potthoff and Roy (1964), Rao (1965, 1967), Khatri (1966) and Grizzle and Allen (1969). In general, p is the number of time points observed for each of the N subjects, $(q-1)$ is the polynomial degree, and k is the number of groups. In most applications of the model, p is small, i.e., the data consist of very short series for each subject. Most theoretical results are for the case when the correlation structure is arbitrary.

In this paper we consider the case when Σ has an autoregressive structure of the first order, i.e.,

$$\Sigma = \sigma^2(\rho^{|i-j|}), \quad i,j = 1,2,\dots,p\,, \\ = \sigma^2 G(\rho)\,. \tag{1.2}$$

This structure has been considered by Potthoff and Roy (1964). However, it may be noted that the structure has received little attention in the theory of the growth curve model. Sandland and McGilchrist (1979) discussed the problem of modeling growth and considered a growth curve model with autoregressive errors. Glasbey (1988) pointed out the need for caution in assuming particular correlated error models. It has been noted (Sandland and McGilchrist (1979) and Lee (1988)) that often the best initial choice of model for repeated measurements of very short series will be to assume the autoregressive structure of the first order for Σ. Lee (1988) has considered the prediction of future observations in the model (1.1) with the autoregressive covariance structure (1.2). He noted some advantages of the restrictive covariance structure when it is appropriate, and examined its appropriateness for three sets of real data.

This paper is concerned with inferential problems in the model (1.1) with the autoregressive covariance structure (1.2). In Section 2 we present an iterative algorithm for finding the MLE's of Ξ, ρ and σ^2, based on the modified likelihood equations. Asymptotic distributions of the MLE's are obtained when p and k are fixed and $N \to \infty$. The asymptotic situation, in which p and k are small and N is large in the comparison with p and k, is important for the growth curve data given in Potthoff and Roy (1964) and Grizzle and Allen (1969). The asymptotic results are presented in Section 3. Hudson (1983) has discussed asymptotic theory for the growth curve model with an autoregressive structure, but his asymptotic results are only for the case when N is fixed and $p \to \infty$. In Section 4 we derive asymptotic distribution of the likelihood ratio (LR) statistic for testing the covariance structure (1.2). An example is presented in Section 5.

2. The MLE's

We consider the MLE's of Ξ, ρ and σ^2 under the model (1.1) with the covariance structure (1.2). Under (1.1) it is well known that

$$|\Sigma| = (\sigma^2)^p(1-\rho^2)^{p-1}\,, \\ \Sigma^{-1} = \{\sigma^2(1-\rho^2)\}^{-1}(\rho^2 C_1 - 2\rho C_2 + I_p)\,, \tag{2.1}$$

where

$$C_1 = \begin{pmatrix} 0 & & & & \\ & 1 & & 0 & \\ & & \ddots & & \\ & 0 & & 1 & \\ & & & & 0 \end{pmatrix}, \quad C_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 & & & 0 \\ 1 & 0 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & 0 & \ddots & & 1 \\ & & & 1 & 0 \end{pmatrix}.$$

Therefore, we can write the log likelihood of Ξ, ρ and σ^2 based on Y as

$$(2.2) \quad l(\Xi, \sigma^2 G(\rho)) = -\frac{1}{2}[N \log |\Sigma| + Np \log 2\pi + \operatorname{tr} \Sigma^{-1}(Y - A\Xi B)'(Y - A\Xi B)]$$
$$= -\frac{1}{2}[Np \log \sigma^2 + N(p-1) \log (1-\rho^2) + Np \log 2\pi + \{\sigma^2(1-\rho^2)\}^{-1} \cdot \operatorname{tr} (\rho^2 C_1 - 2\rho C_2 + I_p)(Y - A\Xi B)'(Y - A\Xi B)] .$$

THEOREM 2.1. *The MLE's of* Ξ, ρ *and* σ^2 *in the model* (1.1) *with the covariance structure* (1.2) *are the solutions of the following equations* (1)–(3):

(1) $\hat{\Xi} = \Xi(\hat{\rho}) = (A'A)^{-1}A'Y\hat{G}^{-1}B'(B\hat{G}^{-1}B')^{-1}$,

(2) $\hat{\sigma}^2 = \sigma^2(\hat{\Xi}, \hat{\rho}) = (n/N)\{p(1-\hat{\rho}^2)\}^{-1}(a_1\hat{\rho}^2 - 2a_2\hat{\rho} + a_3)$,

(3) $(p-1)a_1\hat{\rho}^3 - (p-2)a_2\hat{\rho}^2 - (pa_1 + a_3)\hat{\rho} + pa_2 = 0$,

where $\hat{G} = G(\hat{\rho})$, $a_i = \operatorname{tr} C_i R$, $i = 1, 2, 3$, $C_3 = I_p$, $n = N - k$ *and* $R = n^{-1}(Y - A\hat{\Xi}B)'(Y - A\hat{\Xi}B)$.

PROOF. First we consider the maximum of $l(\Xi, \sigma^2 G(\rho))$ with respect to Ξ when ρ and σ^2 are fixed. It is seen (Khatri (1966)) that this maximum is achieved at $\Xi = \Xi(\hat{\rho})$. Next we consider the maximum of $l(\Xi, \sigma^2 G(\rho))$ with respect to ρ and σ^2 when Ξ are fixed. Maximizing with respect to σ^2 yields

$$\max_{\sigma^2} l(\Xi, \sigma^2 G(\rho)) = l(\Xi, \sigma^2(\Xi, \rho) G(\rho))$$
$$= -\frac{N}{2}[p \log (\tilde{a}_1\rho^2 - 2\tilde{a}_2\rho + \tilde{a}_3) - \log (1-\rho^2) + p(1 + \log 2\pi n(Np)^{-1})] ,$$

where $\tilde{a}_i = n^{-1} \operatorname{tr} C_i(Y - A\Xi B)'(Y - A\Xi B)$. It is easily checked that the last expression is maximized when ρ is the solution of equation (3) with a_i

replaced by $\tilde{a}_i$. Consequently we obtain the fact that the MLE's of Ξ, ρ and σ^2 are the solutions of (1)–(3).

It is easy to obtain the maximum $\hat{l}(\rho)$ of $l(\Xi, \sigma^2 G(\rho))$ with respect to Ξ and σ^2 when ρ is fixed. However, it is difficult to obtain the MLE of ρ in an explicit form by maximizing $\hat{l}(\rho)$. Lee (1988) has pointed that the maximum of $\hat{l}(\rho)$ with respect to ρ can be numerically obtained by a one-dimensional search. Using (2.2) it is shown that the likelihood equations are given by (1), (2) and

$$(3') \quad (p-1)Nn^{-1}\hat{\sigma}^2\hat{\rho}^3 - a_2\hat{\rho}^2 + \{a_1 + a_3 - (p-1)Nn^{-1}\hat{\sigma}^2\}\hat{\rho} - a_2 = 0 .$$

It is easily checked that (1), (2), (3) $\Leftrightarrow$ (1), (2), (3′). So, the equations (1), (2) and (3) can be regarded as a set of modified likelihood equations. Each of the two sets of the equations gives an iterative scheme. However, we note that the iterative scheme based on (1), (2) and (3) is much simpler than the one based on (1), (2) and (3′), since (3) does not involve $\hat{\sigma}^2$, but (3′) involves $\hat{\sigma}^2$. We suggest the iterative scheme based on (1), (2) and (3) wherein from an initial estimate of ρ, $\hat{\rho}$, one can get $\hat{\Xi}$ from (1) and $\hat{\sigma}^2$ from (2), and then solve the equation (3) in $\hat{\rho}$ to yield the next estimate of $\hat{\rho}$. We suggest the solution of

$$(2.3) \qquad \begin{aligned}(p-1)(\mathrm{tr}\, C_1S)\hat{\rho}^3 - (p-2)(\mathrm{tr}\, C_2S)\hat{\rho}^2 \\ - (p\,\mathrm{tr}\, C_1S + \mathrm{tr}\, S)\hat{\rho} + p(\mathrm{tr}\, C_2S) = 0\end{aligned}$$

as an initial estimate of ρ, where

$$(2.4) \qquad S = n^{-1}Y'(I_N - A(A'A)^{-1}A')Y .$$

It may be noted that the $\hat{\rho}$ in (2.3) is the MLE of ρ in a MANOVA model, i.e., in the model (1.1) with $E(Y) = A\Xi B$ replaced by $E(Y) = A\Theta$, where Θ: $k \times p$ is the matrix of unknown parameters.

3. Asymptotic distributions of the MLE's

We consider the asymptotic distributions of the MLE's $\hat{\Xi}$, $\hat{\rho}$ and $\hat{\sigma}^2$ when p and k are fixed and $n = N - k \to \infty$. Let

$$(3.1) \qquad \begin{aligned} U &= (A'A)^{-1/2}A'(Y - A\Xi B) , \\ V &= \sqrt{n}(\Sigma^{-1/2}S\Sigma^{-1/2} - I_p) . \end{aligned}$$

Then U and V are independent, $\mathrm{vec}\,(U') \sim N_{kp}(\mathbf{0}, I_k \otimes \Sigma)$, and the limiting distribution of $V=(v_{ij})$ is normal with mean zero and $\mathrm{Var}(v_{ii})=2$, $\mathrm{Var}(v_{ij})=1$, $i \neq j$. Further the $(1/2)p(p+1)$ elements v_{ij} $(i \leq j)$ are independent in the limiting distribution. The limiting distribution of $\mathrm{vec}(V)$ is expressed (see, e.g., Muirhead (1982), p. 90) as $N_{p^2}(\mathbf{0}, \Psi)$, where

$$\Psi = I_p \otimes I_p + \sum_{i,j=1}^{p} H_{ij} \otimes H_{ij}' \tag{3.2}$$

and H_{ij} is the matrix whose (i,j)-th element is one and whose other elements are zero.

LEMMA 3.1. *Let $\hat{\Xi}$, $\hat{\rho}$ and $\hat{\sigma}^2$ be the MLE's of Ξ, ρ and σ^2 based on Y. Then*

$$\text{(i)}\quad (A'A)^{1/2}(\hat{\Xi}-\Xi) = U\Sigma^{-1}B'(B\Sigma^{-1}B')^{-1} + O_p(n^{-1/2}),$$

$$\text{(ii)}\quad \hat{\rho} = \rho + n^{-1/2}\rho_1 + O_p(n^{-1}),$$

$$\text{(iii)}\quad \hat{\sigma}^2 = \sigma^2 + n^{-1/2}\delta_1 + O_p(n^{-1}),$$

where

$$\begin{aligned} &\rho_1 = -[(1-\rho^2)/\{2(p-1)\rho\}]\,\mathrm{tr}\,DV, \qquad \delta_1 = r^{-1}\,\mathrm{tr}\,QV, \\ &D = I_p - \{p/(r\sigma^2)\}Q, \qquad Q = \Sigma - \rho^2\Sigma^{1/2}C_1\Sigma^{1/2}, \\ &r = 2\rho^2 + p(1-\rho^2). \end{aligned} \tag{3.3}$$

PROOF. We can write

$$\hat{\Xi} = \Xi + (A'A)^{-1/2}U\hat{\Sigma}^{-1}B'(B\hat{\Sigma}^{-1}B')^{-1}, \tag{3.4}$$

and

$$R = S + n^{-1}W(\hat{\Sigma}), \tag{3.5}$$

where $W(\hat{\Sigma}) = \{I_p - \hat{\Sigma}^{-1}B'(B\hat{\Sigma}^{-1}B')^{-1}B\}'U'U\{I_p - \hat{\Sigma}^{-1}B'(B\hat{\Sigma}^{-1}B')^{-1}B\}$ and $\hat{\Sigma} = \hat{\sigma}^2 G(\hat{\rho})$. Result (i) follows from (3.4), (ii) and (iii). Substituting (3.1) and (3.5) into equation (3) and finding the solution of $\hat{\rho}$ in an expanded form, we obtain (ii). Result (iii) follows from equation (2) and result (ii).

THEOREM 3.1. *When p and k are fixed and $n \to \infty$, it holds that*

$$\text{(i)}\quad \mathrm{vec}(\{(A'A)^{1/2}(\hat{\Xi}-\Xi)\}') \xrightarrow{d} N_{kq}(\mathbf{0}, I_k \otimes (B\Sigma^{-1}B')^{-1}),$$

(ii) $\sqrt{n}\begin{pmatrix} \hat{\rho}-\rho \\ \hat{\sigma}^2-\sigma^2 \end{pmatrix} \xrightarrow{d} N_2\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \frac{1}{r}\begin{pmatrix} \alpha & \gamma \\ \gamma & \beta \end{pmatrix}\right]$,

(iii) $\hat{\Xi}$ *and* $(\hat{\rho}, \hat{\sigma}^2)$ *are independent*,

where $r = 2\rho^2 + p(1-\rho^2)$, $\alpha = p(1-\rho^2)^2/(p-1)$, $\beta = 2(1+\rho^2)\sigma^4$ *and* $\gamma = -2\rho(1-\rho^2)\sigma^2$.

PROOF. Lemma 3.1(i) implies result (i). From Lemma 3.1(ii) and (iii) it follows that the limiting distribution of $\sqrt{n}(\hat{\rho}-\rho, \hat{\sigma}^2-\sigma^2)$ is the same as that of (ρ_1, δ_1). Consider the characteristic function of (ρ_1, δ_1), which is expressed as

$$\begin{aligned} C(t_1, t_2) &= E\{\exp(it_1\rho_1 + it_2\delta_1)\} \\ &= E\{\exp(\mathrm{tr}\, MV)\} \\ &= \exp\{\mathrm{tr}\, M^2\} + O(n^{-1/2}), \end{aligned}$$

where $M = -(it_1)[(1-\rho^2)/\{2(p-1)\rho\}]D + it_2 r^{-1}Q$. Result (ii) is proven by showing that

$$\mathrm{tr}\, M^2 = \frac{1}{2}\{\alpha(it_1)^2 + 2\gamma(it_1)(it_2) + \beta(it_2)^2\}.$$

This identity follows by noting that

$$\mathrm{tr}\, Q = r\sigma^2, \qquad \mathrm{tr}\, Q^2 = r(1+\rho^2)\sigma^4.$$

Result (iii) follows from Lemma 3.1 and the independence of U and V.

4. The LR test for an autoregressive covariance structure

It is important to examine whether Σ has the autoregressive structure (1.2) or not. We consider the problem of testing

(4.1) $$H\colon \Sigma = \sigma^2 G(\rho) \quad \text{against} \quad K\colon \Sigma \text{ unrestricted}.$$

The maximum of the log likelihood when Ξ and Σ are unrestricted, which was obtained by Khatri (1966), is given by

(4.2) $$\begin{aligned} \max l(\Xi, \Sigma) &= l(\hat{\Xi}_\Omega, \hat{\Sigma}_\Omega) \\ &= -\frac{1}{2}N\left[\log|\hat{\Sigma}_\Omega| + p(1+\log 2\pi)\right], \end{aligned}$$

where $\hat{\Xi}_\Omega = (A'A)^{-1}A'YS^{-1}B'(BS^{-1}B')^{-1}$ and $N\hat{\Sigma}_\Omega = (Y - A\hat{\Xi}_\Omega B)'(Y - A\hat{\Xi}_\Omega B)$. From (2.2) it follows that

$$\text{(4.3)} \qquad \max_H l(\Xi, \Sigma) = l(\hat{\Xi}_\omega, \hat{\Sigma}_\omega) = -\frac{1}{2}N[p \log \hat{\sigma}^2 + (p-1)\log(1-\hat{\rho}^2) + p(1 + \log 2\pi)] ,$$

where $\hat{\Xi}_\omega = \hat{\Xi}$ and $\hat{\Sigma}_\omega = \hat{\sigma}^2 G(\hat{\rho})$. Therefore, the LR test is to reject the hypothesis H for large values of

$$\text{(4.4)} \qquad T = -2(n/N)\{l(\hat{\Xi}_\omega, \hat{\Sigma}_\omega) - l(\hat{\Xi}_\Omega, \hat{\Sigma}_\Omega)\} = -n \log\{|\tilde{\Sigma}_\Omega| / |\tilde{\Sigma}_\omega|\} ,$$

where $\tilde{\Sigma}_\Omega = (N/n)\hat{\Sigma}_\Omega$ and $\tilde{\Sigma}_\omega = (N/n)\hat{\Sigma}_\omega$.

LEMMA 4.1.

$$\text{(i)} \quad \tilde{\Sigma}_\Omega \tilde{\Sigma}_\omega^{-1} - I_p = O_p(n^{-1/2}) ,$$
$$\text{(ii)} \quad \operatorname{tr}(\tilde{\Sigma}_\Omega \tilde{\Sigma}_\omega^{-1} - I_p) = O_p(n^{-3/2}) .$$

PROOF. We can write

$$\tilde{\Sigma}_\Omega = S + n^{-1}W(S)$$

and

$$\tilde{\Sigma}_\omega^{-1} = ((N/n)\hat{\sigma}^2)^{-1}(1 - \hat{\rho}^2)^{-1}(\hat{\rho}^2 C_1 - 2\hat{\rho} C_2 + I_p) .$$

Result (i) follows from these expressions and Lemma 3.1. Using equation (2) of Theorem 2.1 we obtain

$$\text{(4.5)} \qquad \operatorname{tr} \tilde{\Sigma}_\Omega \tilde{\Sigma}_\omega^{-1} = p\{a_1\hat{\rho}^2 - 2a_2\hat{\rho} + a_3\}^{-1} \operatorname{tr} \tilde{\Sigma}_\Omega(\hat{\rho}^2 C_1 - 2\hat{\rho} C_2 + I_p) .$$

From (3.5) we have $\tilde{\Sigma}_\Omega = R + O_p(n^{-3/2})$. Substituting this result into the right-hand side of (4.5) we obtain result (ii).

THEOREM 4.1. *The asymptotic null distribution of the LR statistic T given by* (4.4) *when p and k are fixed and $n \to \infty$ is a central chi-square distribution with the degrees of freedom $f = (1/2)p(p+1) - 2$.*

PROOF. We can expand T as

$$T = -n \log |I_p + (\tilde{\Sigma}_\Omega \tilde{\Sigma}_\omega^{-1} - I_p)|$$

$$= -n \left[\operatorname{tr} (\tilde{\Sigma}_\Omega \tilde{\Sigma}_\omega^{-1} - I_p) - \frac{1}{2} \operatorname{tr} (\tilde{\Sigma}_\Omega \tilde{\Sigma}_\omega^{-1} - I_p)^2 + \frac{1}{3} \operatorname{tr} (\tilde{\Sigma}_\Omega \tilde{\Sigma}_\omega^{-1} - I_p)^3 + \cdots \right].$$

Using Lemma 4.1 we obtain the fact that the asymptotic distribution of T is the same as that of

$$\tilde{T} = \frac{n}{2} \operatorname{tr} (\tilde{\Sigma}_\Omega \tilde{\Sigma}_\omega^{-1} - I_p)^2 . \tag{4.6}$$

Using Lemma 3.1(ii) and (iii) it can be seen that $\tilde{T}$ is asymptotically equivalent to

$$T_0 = \frac{1}{2} \operatorname{tr} V^2 - \frac{1}{2p} (\operatorname{tr} V)^2 - d(\operatorname{tr} DV)^2$$

$$= \{\operatorname{vec} (V)\}' J \operatorname{vec} (V) ,$$

where $J = (1/2) I_p \otimes I_p - (1/2) p^{-1} \operatorname{vec} (I_p) \{\operatorname{vec} (I_p)\}' - d \operatorname{vec} (D) \{\operatorname{vec} (D)\}'$ and $d = r / \{4p(p-1)\rho^2\}$. Our conclusion is obtained by showing that $\Psi J \Psi J \Psi = \Psi J \Psi$ and $\operatorname{tr} \Psi J = (1/2) p(p+1) - 2$ (see, e.g., Rao (1973), p. 188). These are easily checked.

We note that as a competitor to the LR statistic T, we may use the statistic $\tilde{T}$ given by (4.6) whose asymptotic null distribution is the same as that of T.

As an alternative method for examining whether Σ has the autoregressive covariance structure (1.2) or not, we can use Akaike's information criterion (Akaike (1973)). This is equivalent to choosing the arbitrary covariance matrix or the autoregressive covariance structure (1.2) according to whether the value of

$$-N \log \{|\hat{\Sigma}_\Omega| / |\hat{\Sigma}_\omega|\} - p(p+1) + 4 \tag{4.7}$$

is positive or negative.

5. An example

We now examine the data (see, e.g., Grizzle and Allen (1969)) of the ramus heights, measured in *mm.* of 20 boys at 8, 8.5, 9 and 9.5 years of age. For the observation matrix Y: 20×4, we assume the model (1.1) with

$$E(Y) = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} (\xi_0, \xi_1) \begin{pmatrix} 1 & 1 & 1 & 1 \\ -3 & -1 & 1 & 3 \end{pmatrix}$$

and the autoregressive covariance structure (1.2) of the first order. Then, we obtain the MLE's of (ξ_0, ξ_1), ρ and σ^2 as follows:

$$(\hat{\xi}_0, \hat{\xi}_1) = (50.0057, 0.4650), \qquad \hat{\rho} = 0.9526, \qquad \sigma^2 = 6.5354 .$$

These values are obtained in a few iterations by using the solution $\rho = 0.9527$ in (2.3) as the initial value of ρ. The values are also obtained in a few iterations by using any value of $\rho = -0.1(0.1)1.0$ as the initial value of ρ. On the other hand, the MLE of (ξ_0, ξ_1) when Σ is unknown positive definite is (50.05, 0.4654). It may be noted that the MLE's of (ξ_0, ξ_1) in the two models are very similar. The value of the LR statistic T for testing the autoregressive covariance structure is 9.5. This value is fairly below the critical value of a chi-square distribution with the degree of freedom $f = 8$. Further, the value of (4.7) is -6.0. Hence it does not seem unreasonable to assume the autoregressive covariance structure of the first order in this example.

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