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PARAMETRIC STOCHASTIC CONVEXITY AND CONCAVITY OF STOCHASTIC PROCESSES*

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Abstract. A collection of random variables $\{X(\theta), \theta \in \Theta\}$ is said to be parametrically stochastically increasing and convex (concave) in $\theta \in \Theta$ if $X(\theta)$ is stochastically increasing in θ, and if for any increasing convex (concave) function ϕ, $E\phi(X(\theta))$ is increasing and convex (concave) in $\theta \in \Theta$ whenever these expectations exist. In this paper a notion of directional convexity (concavity) is introduced and its stochastic analog is studied. Using the notion of stochastic directional convexity (concavity), a sufficient condition, on the transition matrix of a discrete time Markov process $\{X_n(\theta), n = 0, 1, 2, \ldots\}$, which implies the stochastic monotonicity and convexity of $\{X_n(\theta), \theta \in \Theta\}$, for any n, is found. Through uniformization these kinds of results extend to the continuous time case. Some illustrative applications in queueing theory, reliability theory and branching processes are given.

Key words and phrases: Sample path convexity and concavity, Markov processes, directional convexity and concavity, single stage queues, supermodular and submodular functions, L-superadditive functions, reliability theory, branching processes, shock models, total positivity.

1. Introduction and summary

Shaked and Shanthikumar (1988*a*) introduced a notion of stochastic convexity and concavity and illustrated (in Shaked and Shanthikumar (1988*a*, 1988*b* and 1990)) its prevalence in numerous applications. In using the approach developed there, it is often required to establish a sample path convexity, with respect to some parameter, for various collections of random variables. There is no single formal way of doing this. Therefore in Shaked and Shanthikumar (1990) we attempted to develop a general

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approach to identify sample path convexity and concavity. In the present paper we develop another general approach which can be used to obtain parametric sample path convexity and concavity for some Markov processes.

Some preliminaries regarding directional convexity and sample path convexity are given in Sections 2 and 3. A new notion of sample path stochastic directional convexity and concavity is introduced in Section 4. This new notion can be used to obtain parametric sample path stochastic convexity and concavity of some Markov processes. The main results describing this idea are given in Section 5. Some applications of the main results, in the areas of queueing theory, reliability theory and branching processes, are described in Section 6.

In this paper we tacitly assume that whenever we talk about an expectation, this expectation is well defined. Also, "increasing" means "nondecreasing" and "decreasing" means "nonincreasing".

2. Directional convexity

Suppose g is a real-valued function on $S = S_1 \times S_2 \times \cdots \times S_m$ $(m \geq 2)$ where each S_i is a convex subset of the real line or of $\{\dots, -1, 0, 1, \dots\}$. In most of the applications which follow each S_i is either $\boldsymbol{R} \equiv (-\infty, \infty)$ or $\boldsymbol{R}_+ \equiv [0, \infty)$ or $\boldsymbol{N} \equiv \{0, 1, 2, \dots\}$ or $\boldsymbol{N}_+ = \{1, 2, 3, \dots\}$.

DEFINITION 2.1. The function g is said to be *supermodular* (*submodular*) if

$$g(\boldsymbol{x} \wedge \boldsymbol{y}) + g(\boldsymbol{x} \vee \boldsymbol{y}) \geq (\leq) g(\boldsymbol{x}) + g(\boldsymbol{y}) \tag{2.1}$$

for all $\boldsymbol{x} = (x_1, \dots, x_m)$ and $\boldsymbol{y} = (y_1, \dots, y_m)$ in S.

Here, $\boldsymbol{x} \wedge \boldsymbol{y}$ stands for $(\min(x_1, y_1), \dots, \min(x_m, y_m))$ and $\boldsymbol{x} \vee \boldsymbol{y}$ stands for $(\max(x_1, y_1), \dots, \max(x_m, y_m))$.

Functions which satisfy (2.1) are described by a variety of names. Here, we have adapted the terminology of Topkis (1978). For further information regarding these functions see, e.g., Block *et al.* (1987) and references therein.

A function g is supermodular (submodular) if and only if e^g is multivariate TP_2 (RR_2) as defined in Karlin and Rinott (1980*a*, 1980*b*) for example. It follows from Kemperman (1977) that g is supermodular (submodular) if and only if e^g is TP_2 (RR_2) in pairs, that is,

$$g(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_m)$$
$$+ g(x_1, \dots, x_{i-1}, x_i', x_{i+1}, \dots, x_{j-1}, x_j', x_{j+1}, \dots, x_m)$$

$$\geq (\leq)\; g(x_1,\ldots,x_{i-1},x_i',x_{i+1},\ldots,x_{j-1},x_j,x_{j+1},\ldots,x_m)$$
$$+\, g(x_1,\ldots,x_{i-1},x_i,x_{i+1},\ldots,x_{j-1},x_j',x_{j+1},\ldots,x_m)$$

whenever $x_i \leq x_i'$, $x_j \leq x_j'$ for $i \neq j$.

DEFINITION 2.2. A real-valued function g on S is said to be *directionally convex* (*concave*) if for every choice of $\boldsymbol{x}_i \in S$, $i = 1,2,3,4$, such that $\boldsymbol{x}_1 \leq \boldsymbol{x}_2 \leq \boldsymbol{x}_4$, $\boldsymbol{x}_1 \leq \boldsymbol{x}_3 \leq \boldsymbol{x}_4$ and $\boldsymbol{x}_1 + \boldsymbol{x}_4 = \boldsymbol{x}_2 + \boldsymbol{x}_3$, one has

$$g(\boldsymbol{x}_1) + g(\boldsymbol{x}_4) \geq (\leq)\; g(\boldsymbol{x}_2) + g(\boldsymbol{x}_3)\,.$$

Such functions were mentioned in Ruschendorf (1983).

The following characterizations of directional convexity and concavity will be useful.

PROPOSITION 2.1. *The following statements are equivalent*:

(i) *The function g is directionally convex* (*concave*).

(ii) *For any $\boldsymbol{x}_i \in S$, $i = 1,2$, and $\boldsymbol{y} \geq \boldsymbol{0}$ such that $\boldsymbol{x}_1 \leq \boldsymbol{x}_2$ and $\boldsymbol{x}_i + \boldsymbol{y} \in S$, $i = 1,2$, one has*

$$g(\boldsymbol{x}_1 + \boldsymbol{y}) - g(\boldsymbol{x}_1) \leq (\geq)\; g(\boldsymbol{x}_2 + \boldsymbol{y}) - g(\boldsymbol{x}_2)\,.$$

(iii) *The function g is supermodular* (*submodular*) *and is convex* (*concave*) *in each coordinate when the other $m-1$ coordinates are held fixed.*

PROOF. The equivalence of (i) and (ii) is immediate. Suppose (iii) holds for the convexity case. Let $\boldsymbol{x}_1 = (x_{11},\ldots,x_{1m})$, $\boldsymbol{x}_2 = (x_{21},\ldots,x_{2m})$ and $\boldsymbol{y} = (y_1,\ldots,y_m) \geq \boldsymbol{0}$ where $\boldsymbol{x}_1 \leq \boldsymbol{x}_2$. Then

$$\begin{aligned}
g(x_{11},\ldots,x_{1,j-1},x_{1j}+y_j,x_{1,j+1},\ldots,x_{1m}) - g(\boldsymbol{x}_1)&\\
\leq g(x_{21},\ldots,x_{2,j-1},x_{1j}+y_j,x_{2,j+1},\ldots,x_{2m})&\\
- g(x_{21},\ldots,x_{2,j-1},x_{1j},x_{2,j+1},\ldots,x_{2m})&\\
\leq g(x_{21},\ldots,x_{2,j-1},x_{2j}+y_j,x_{2,j+1},\ldots,x_{2m}) - g(\boldsymbol{x}_2)&\,,
\end{aligned}$$

where the first inequality follows from (2.1) and the second inequality follows from the componentwise convexity of g. That is, denoting $\boldsymbol{e}_j = (0,\ldots,0,1,0,\ldots,0)$ where 1 appears in the j-th coordinate, we have

$$g(\boldsymbol{x}_2 + y_j\boldsymbol{e}_j) - g(\boldsymbol{x}_1 + y_j\boldsymbol{e}_j) \geq g(\boldsymbol{x}_2) - g(\boldsymbol{x}_1)\,.$$

Continuing these inequalities for all $j = 1, 2, \dots, m$, one gets $g(\boldsymbol{x}_2 + \boldsymbol{y}) - g(\boldsymbol{x}_1 + \boldsymbol{y}) \geq g(\boldsymbol{x}_2) - g(\boldsymbol{x}_1)$. That is, (iii) implies (ii) in the convexity case. The proof of the concave analog is similar.

Now suppose (ii) holds for the convex case. Let

$$\boldsymbol{x}_1 = (x_1, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_m) ,$$
$$\boldsymbol{x}_2 = (x_1, \dots, x_{j-1}, x_j', x_{j+1}, \dots, x_m)$$

and $\boldsymbol{y} = y_j \boldsymbol{e}_j$ where $x_j \leq x_j'$ and $y_j \geq 0$. Then, from (ii) it is seen that g is convex in x_j when the other $m-1$ coordinates are held fixed. Finally, let

$$\boldsymbol{x}_1 = (x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_m) ,$$
$$\boldsymbol{x}_2 = (x_1, \dots, x_{i-1}, x_i', x_{i+1}, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_m)$$

and $\boldsymbol{y} = (x_j' - x_j)\boldsymbol{e}_j$ were $x_i \leq x_i'$, $x_j \leq x_j'$ for $i \neq j$. Then (2.2) follows from (ii). Therefore g is supermodular. The proof of the concave case is similar. □

Remarks. (1) If g satisfies (ii), then in any positive direction in S, the increase in $g(\boldsymbol{x})$ per unit increases (decreases) in $\boldsymbol{x}$. This is why we called g directionally convex (concave).

(2) If g is twice differentiable, then it is directionally convex (concave) if and only if

$$\frac{\partial^2}{\partial x_j^2} g(\boldsymbol{x}) \geq (\leq)\, 0 , \qquad j = 1, \dots, m ,$$

and

$$\frac{\partial^2}{\partial x_i \partial x_j} g(\boldsymbol{x}) \geq (\leq)\, 0 , \quad i \neq j .$$

(3) Usual convexity (concavity) neither implies nor is implied by directional convexity (concavity).

3. Preliminary: Stochastic convexity and concavity

Let $\{P_\theta, \theta \in \Theta\}$ be a family of univariate distributions. Throughout this paper Θ is a convex set (that is, an interval) of the real line or of the set $\{0, 1, 2, \dots\}$. Let $X(\theta)$ denote a random variable with distribution P_θ. We find it convenient and intuitive to replace the notation $\{P_\theta, \theta \in \Theta\}$ by $\{X(\theta), \theta \in \Theta\}$ and this notation will be used throughout this paper. Note that when we write $\{X(\theta), \theta \in \Theta\}$ we do not assume (and often we are not concerned with) any dependence (or independence) properties among the $X(\theta)$'s. We are only interested in the "marginal distributions" $\{P_\theta, \theta \in \Theta\}$ even when in some circumstances $\{X(\theta), \theta \in \Theta\}$ is a well defined stochastic process. Note also that $X(\theta)$ *does not mean* that X is a function of θ; it only indicates that the distribution of $X(\theta)$ is P_θ. Thus, for example, for ϕ: $\boldsymbol{R} \to \boldsymbol{R}$, the notation $E\phi(X(\theta))$ stands for $\int \phi dP_\theta$—this is usually denoted in the literature by $E_\theta\phi(X)$. When $\{X(\theta), \theta \in \Theta\}$ is a well defined stochastic process then the notation $E\phi(X(\theta))$ is often justifiably used.

In the following definition (Shaked and Shanthikumar (1988*a*)), the abbreviations *SI*, *SICX*, *SD*, *SDCV*, etc., stand, respectively, for stochastically increasing, stochastically increasing and convex, stochastically decreasing, stochastically decreasing and concave, etc.

DEFINITION 3.1. Let $\{X(\theta), \theta \in \Theta\}$ be a set of random variables. Denote:

(a) $\{X(\theta), \theta \in \Theta\} \in SI[SD]$ if

$$\phi \in C \Rightarrow E\phi(X(\cdot)) \in C_\Theta \tag{3.1}$$

for C—the class of all increasing real functions on $\boldsymbol{R}$ and C_Θ—the class of all increasing (decreasing) real functions on Θ.

(b) $\{X(\theta), \theta \in \Theta\} \in SICX\ [SICV]$ if $\{X(\theta), \theta \in \Theta\} \in SI$, and if (3.1) holds for C—the class of all increasing and convex (concave) real functions on $\boldsymbol{R}$ and C_Θ—the class of all increasing and convex (concave) real functions on Θ.

(c) $\{X(\theta), \theta \in \Theta\} \in SDCX\ [SDCV]$ if $\{X(\theta), \theta \in \Theta\} \in SD$, and if (3.1) holds for C—the class of all *increasing* and convex (concave) real functions on $\boldsymbol{R}$ and C_Θ—the class of all *decreasing* and convex (concave) real functions on Θ.

In Definition 3.1(a) and (b) we require C and C_Θ to be "similar" classes (e.g., for *SICX*, both are classes of increasing convex functions). In general, these need not be "similar". For example, of particular interest is the class of processes $\{X(\theta), \theta \in \Theta\}$ such that $E\phi(X(\theta))$ is increasing and convex in θ for all increasing functions ϕ. Such classes are considered in Shaked and Shanthikumar (1990).

Shaked and Shanthikumar (1988*a*) found sufficient conditions which imply that a process $\{X(\theta), \theta \in \Theta\}$ satisfies some of these notions. Their approach was to "put" some (more explicitly four) of the random variables $\{X(\theta), \theta \in \Theta\}$ on a common probability space and then obtain "almost sure" results which carry back to the whole process $\{X(\theta), \theta \in \Theta\}$. In general, the technique of constructing new random variables having a certain relationship with probability one, but the same marginal distributions as the original random variables, is often very powerful (see, e.g., Cambanis and Simons (1982) and references therein).

For any four real (or vectors) x_1, x_2, x_3 and x_4 we abbreviate the conditions $x_1 \le \min(x_2, x_3) \le \max(x_2, x_3) \le x_4$ by $x_1 \le [x_2, x_3] \le x_4$. Also, $x_1 \le [x_2, x_3]$ denotes $x_1 \le \min(x_2, x_3)$ and $[x_1, x_2] \le x_3$ denotes $\max(x_1, x_2) \le x_3$. Similarly, $[x_1, x_2, x_3] \le x_4$ denotes $\max(x_1, x_2, x_3) \le x_4$.

Consider a family $\{X(\theta), \theta \in \Theta\}$. Let $\theta_i \in \Theta$, $i = 1, 2, 3, 4$ be four values such that $\theta_1 \le \theta_2 \le \theta_3 \le \theta_4$ and $\theta_1 + \theta_4 = \theta_2 + \theta_3$. Sometimes (see examples in Shaked and Shanthikumar (1988*a*, 1988*b*)), there exist four random variables $\hat{X}_i$, $i = 1, 2, 3, 4$ defined on the same probability space such that

$$(st) \qquad \hat{X}_i \stackrel{st}{=} X(\theta_i), \qquad i = 1, 2, 3, 4 ,$$

and which satisfy some of the following inequalities. (The notation (*st*) above stands for the stochastic equality which it states. Below, (*cx*), (*cv*), (*i-cx*), (*d-cx*), etc., stand for the conditions of convexity, concavity, increasingness and convexity, decreasingness and convexity, etc., which these state.)

$$(cx) \qquad \hat{X}_2 + \hat{X}_3 \le \hat{X}_1 + \hat{X}_4 \quad \text{a.s.},$$

$$(cv) \qquad \hat{X}_1 + \hat{X}_4 \le \hat{X}_2 + \hat{X}_3 \quad \text{a.s.},$$

$$(i\text{-}cx) \qquad [\hat{X}_2, \hat{X}_3] \le \hat{X}_4 \quad \text{a.s.},$$

$$(d\text{-}cx) \qquad [\hat{X}_2, \hat{X}_3] \le \hat{X}_1 \quad \text{a.s.},$$

$$(i\text{-}cv) \qquad \hat{X}_1 \le [\hat{X}_2, \hat{X}_3] \quad \text{a.s.}$$

and

$$(d\text{-}cv) \qquad \hat{X}_4 \le [\hat{X}_2, \hat{X}_3] \quad \text{a.s.}$$

In the following definition $\{X(\theta), \theta \in \Theta\}$ is classified according to the almost sure inequalities which $\hat{X}_i$, $i = 1, 2, 3, 4$, satisfy.

DEFINITION 3.2. Let $\{X(\theta), \theta \in \Theta\}$ be a family as described above. If for any $\theta_i \in \Theta$, $i = 1, 2, 3, 4$, such that $\theta_1 \le \theta_2 \le \theta_3 \le \theta_4$ and $\theta_1 + \theta_4 = \theta_2 + \theta_3$, there exist four random variables $\hat{X}_i$, $i = 1, 2, 3, 4$, which satisfy:

(a) conditions (*st*), (*cx*) and (*i-cx*), then $\{X(\theta), \theta \in \Theta\}$ is said to be *stochastically increasing and convex in the sample path sense* (denoted by $SICX(sp)$);

(b) conditions (*st*), (*cv*) and (*i-cv*), then $\{X(\theta), \theta \in \Theta\}$ is said to be *stochastically increasing and concave in the sample path sense* (denoted by $SICV(sp)$);

(c) conditions (*st*), (*cx*) and (*d-cx*), then $\{X(\theta), \theta \in \Theta\}$ is said to be *stochastically decreasing and convex in the sample path sense* (denoted by $SDCX(sp)$);

(d) conditions (*st*), (*cv*) and (*d-cv*), then $\{X(\theta), \theta \in \Theta\}$ is said to be *stochastically decreasing and concave in the sample path sense* (denoted by $SDCV(sp)$).

Remark 3.1. The definition of sample path convexity and concavity given in Definition 3.2 here differs slightly from a similar definition in Shaked and Shanthikumar (1988*a*, 1988*b* and 1990). In those papers, for example, (*i-cx*) states $[\hat{X}_1, \hat{X}_2, \hat{X}_3] \le \hat{X}_4$ a.s. and not just $[\hat{X}_2, \hat{X}_3] \le \hat{X}_4$ a.s. as in the present paper. However, all the results which appear in those papers are valid for the modified definitions of $SICX(sp)$, $SICV(sp)$, $SDCX(sp)$ and $SDCV(sp)$ given here.

For example, Theorem 3.1 in Shaked and Shanthikumar (1988*b*) is stated for the $SICX(sp)$ notion which requires (*i-cx*) to be $[\hat{X}_1, \hat{X}_2, \hat{X}_3] \le \hat{X}_4$ a.s. and not just $[\hat{X}_2, \hat{X}_3] \le \hat{X}_4$ a.s. as in the present paper. However, that theorem is valid also for the present definition of $SICX(sp)$, but in order to prove it one needs to modify the proof in Shaked and Shanthikumar (1988*b*) along the lines of Theorem 5.1 below. In particular, instead of using Result 5.A.9 of Marshall and Olkin (1979) (which increases $(\hat{X}_2, \hat{X}_3)$ to (X_2^*, X_3^*)), as is done in Shaked and Shanthikumar (1988*b*), one should decrease $\hat{X}_1$ to X_1^* as in the proof of Theorem 5.1 in the present paper.

Shaked and Shanthikumar (1988*a*) showed that if $\{X(\theta), \theta \in \Theta\} \in SICX(sp)$ $(SICV(sp), SDCX(sp), SDCV(sp))$, then $\{X(\theta), \theta \in \Theta\} \in SICX$ $(SICV, SDCX, SDCV)$. They used these results to obtain useful stochastic convexity and concavity properties of output random variables in various stochastic systems.

Counterexample 3.1. It should be mentioned that whereas, for example, $SICX(sp) \Rightarrow SICX$ the reverse implication is not true in general. To see this, let $\Theta = \{1, 2, 3\}$ and let the distributions of $X(1)$, $X(2)$ and $X(3)$ be as follows:

$$P\{X(1) = 0\} = \frac{1}{4}, \qquad P\{X(1) = 4\} = \frac{3}{4},$$

$$P\{X(2) = 4\} = 1, \qquad P\{X(3) = 6\} = 1.$$

Then, clearly, $\{X(\theta), \theta \in \{1, 2, 3\}\} \in SI$ and, for each x, the value of $E[X(\theta) - x]^+ \left(= \int_x^\infty P\{X(\theta) > u\} du \right)$ is a convex increasing function in $\theta \in \{1, 2, 3\}$. Approximating a convex increasing function $\phi(\cdot)$ by a constant plus a sum of functions of the form $a_i[\cdot + x_i]^+$ where $a_i \geq 0$ for each i (see, e.g., Stoyan (1983), p.9), it is seen that the convexity of $E[X(\theta) - x]^+$ in $\theta \in \{1,2,3\}$, for each x, implies the convexity of $E\phi(X(\theta))$ in $\theta \in \{1,2,3\}$ for each convex increasing function ϕ. Therefore $\{X(\theta), \theta \in \{1,2,3\}\} \in SICX$.

To show that $\{X(\theta), \theta \in \{1,2,3\}\} \notin SICX(sp)$, let $\theta_1 = 1$, $\theta_2 = \theta_3 = 2$ and $\theta_4 = 3$. Then, indeed $\theta_1 \leq \theta_2 \leq \theta_3 \leq \theta_4$ and $\theta_1 + \theta_4 = \theta_2 + \theta_3$. Suppose we construct on some probability space four random variables $\hat{X}_i$, $i = 1,2,3,4$, such that $\hat{X}_i \overset{st}{=} X(\theta_i)$, $i = 1,2,3,4$. Then $\hat{X}_2 = \hat{X}_3 = 4$ a.s. and $\hat{X}_4 = 6$ a.s. whereas $\hat{X}_1$ equals either 0 (with probability 1/4) or 4 (with probability 3/4). Clearly, $[\hat{X}_2, \hat{X}_3] \leq \hat{X}_4$ a.s. However, with probability 1/4, $\hat{X}_1 + \hat{X}_4 = 6 < 4 + 4 = \hat{X}_2 + \hat{X}_3$. Therefore $\{X(\theta), \theta \in \{1,2,3\}\} \notin SICX(sp)$.

In a similar way it can be shown that the implications $SDCX(sp) \Rightarrow SDCX$, $SICV(sp) \Rightarrow SICV$ and $SDCV(sp) \Rightarrow SDCV$ are strict.

4. Stochastic directional convexity and concavity

In this section we first define a regular stochastic directional convexity (concavity) analogue of Definition 3.1. Then a definition of sample path stochastic directional convexity (concavity) which is analogous to Definition 3.2 is given. Using these we establish, in Section 5, parametric stochastic convexity (concavity) for some Markov processes.

DEFINITION 4.1. A collection of random variables $\{Z(\mathbf{x}), \mathbf{x} \in S\}$, $S = S_1 \times \cdots \times S_m$ (where each S_i is as described in Section 2) is said to be:

(a) *Stochastically increasing and directionally convex (concave) in* $\mathbf{x}$ if $\{Z(\mathbf{x}), \mathbf{x} \in S\} \in SI$ and if $E\phi(Z(\mathbf{x}))$ is directionally convex (concave) in $\mathbf{x}$ for every increasing convex (concave) function ϕ. We then write $\{Z(\mathbf{x}), \mathbf{x} \in S\} \in SI\text{-}DCX$ $(SI\text{-}DCV)$.

(b) *Stochastically decreasing and directionally convex (concave) in* $\mathbf{x}$ if $\{Z(\mathbf{x}), \mathbf{x} \in S\} \in SD$ and if $E\phi(Z(\mathbf{x}))$ is directionally convex (concave) in $\mathbf{x}$ for any increasing convex (concave) function ϕ. We then write $\{Z(\mathbf{x}), \mathbf{x} \in S\} \in SD\text{-}DCX$ $(SD\text{-}DCV)$.

DEFINITION 4.2. A collection of random variables $\{Z(\mathbf{x}), \mathbf{x} \in S\}$, $S = S_1 \times S_2 \times \cdots \times S_m$ (as in Definition 4.1) is said to be:

(a) *Stochastically increasing and directionally convex (concave) in the sample path sense* if, for every choice of $\mathbf{x}_i \in S$, $i = 1,2,3,4$, such that $\mathbf{x}_1 \leq [\mathbf{x}_2, \mathbf{x}_3] \leq \mathbf{x}_4$ and $\mathbf{x}_1 + \mathbf{x}_4 = \mathbf{x}_2 + \mathbf{x}_3$, there exist four random variables $\hat{Z}_i$, $i = 1,2,3,4$, defined on a common probability space such that

$$\hat{Z}_i \overset{st}{=} Z(\mathbf{x}_i), \qquad i = 1,2,3,4\,, \tag{4.1}$$

$$[\hat{Z}_2, \hat{Z}_3] \leq \hat{Z}_4 \quad \text{a.s.} \tag{4.2}$$

(4.3) $$(\hat{Z}_1 \leq [\hat{Z}_2, \hat{Z}_3] \quad \text{a.s.})$$

and

(4.4) $$\hat{Z}_1 + \hat{Z}_4 \geq (\leq)\, \hat{Z}_2 + \hat{Z}_3 \,.$$

We then write $\{Z(\boldsymbol{x}), \boldsymbol{x} \in S\} \in SI\text{-}DCX(sp)$ $(SI\text{-}DCV(sp))$.

(b) *Stochastically decreasing and directionally convex (concave) in the sample path sense* if, for any choice of $\boldsymbol{x}_i \in S$, $i = 1, 2, 3, 4$, such that $\boldsymbol{x}_1 \geq [\boldsymbol{x}_2, \boldsymbol{x}_3] \geq \boldsymbol{x}_4$ and $\boldsymbol{x}_1 + \boldsymbol{x}_4 = \boldsymbol{x}_2 + \boldsymbol{x}_3$, there exist four random variables as in (a) which satisfy (4.1), (4.2), (4.3) and (4.4). We then write $\{Z(\boldsymbol{x}), \boldsymbol{x} \in S\}$ $\in SD\text{-}DCX(sp)$ $(SD\text{-}DCV(sp))$.

Remark 4.1. It is not hard to show that

$$SI\text{-}DCX(sp) \Rightarrow SI\text{-}DCX \,,$$

$$SI\text{-}DCV(sp) \Rightarrow SI\text{-}DCV \,,$$

$$SD\text{-}DCX(sp) \Rightarrow SD\text{-}DCX$$

and that

$$SD\text{-}DCV(sp) \Rightarrow SD\text{-}DCV \,.$$

The reverse implications need not be true. For example, it is shown in Counterexample 3.1 that $SI\text{-}DCX \not\Rightarrow SI\text{-}DCX(sp)$.

5. Parametric stochastic convexity and concavity for Markov processes

In this section we consider temporally homogeneous discrete time Markov processes. Denote such a process by $X(\theta) = \{X_n(\theta), n = 0, 1, 2, \ldots\}$. Let its state space T be a convex subset of $\boldsymbol{R}$ or $\boldsymbol{N}$. The notation indicates that the distribution of the initial state $X_0(\theta)$ and the transition matrix of the process are allowed to depend on a parameter $\theta \in \Theta$ where Θ is also a convex subset of $\boldsymbol{R}$ or $\boldsymbol{N}$.

For any random variable U and an event A, denote by $[U \mid A]$ any random variable whose distribution is the conditional distribution of U given A.

For the Markov process $X(\theta)$, let $Z(x, \theta)$ denote any random variable which satisfies

$$Z(x, \theta) \overset{st}{=} [X_{n+1}(\theta) \mid X_n(\theta) = x], \qquad x \in T, \quad \theta \in \Theta \,.$$

In Theorems 5.1 and 5.2 and in the related results below, it is assumed that $\{Z(x,\theta), (x,\theta) \in T\times\Theta\}$ is *SI-DCX(sp)* or *SI-DCX*. This means, explicitly, that the *SI-DCX* property of $Z(x,\theta)$ is assumed to hold in terms of the *pair* (x,θ).

THEOREM 5.1. *Suppose* $\{Z(x,\theta), (x,\theta) \in T\times\Theta\} \in SI\text{-}DCX(sp)$ $(SI\text{-}DCV(sp))$. *If* $\{X_0(\theta), \theta\in\Theta\} \in SICX(sp)$ $(SICV(sp))$, *then* $\{X_n(\theta), \theta\in\Theta\} \in SICX(sp)$ $(SICV(sp))$ *for each* $n = 0, 1, 2, \ldots$.

PROOF. Only the convex case will be proved. The concave case is proven similarly.

As an induction hypothesis assume that for some n

$$\{X_n(\theta), \theta\in\Theta\} \in SICX(sp)\,. \tag{5.1}$$

We just have to show that

$$\{X_{n+1}(\theta), \theta\in\Theta\} \in SICX(sp) \tag{5.2}$$

and the proof will be complete.

From (5.1) it follows that for any $\theta_i \in \Theta$, $i = 1,2,3,4$, such that $\theta_1 \le \theta_2 \le \theta_3 \le \theta_4$ and $\theta_1 + \theta_4 = \theta_2 + \theta_3$, there exist four random variables $\hat{X}_i$, $i = 1,2,3,4$, defined on a common probability space such that

$$\hat{X}_i \overset{st}{=} X(\theta_i)\,, \qquad i = 1,2,3,4\,,$$

$$[\hat{X}_2, \hat{X}_3] \le \hat{X}_4 \qquad \text{a.s.}$$

and

$$\hat{X}_2 + \hat{X}_3 \le \hat{X}_1 + \hat{X}_4 \qquad \text{a.s.}$$

Define $X_1^* = \min(\hat{X}_1, \hat{X}_2, \hat{X}_3)$ and $X_4^* = \hat{X}_2 + \hat{X}_3 - X_1^*$ and notice that

$$X_1^* \le \hat{X}_1 \quad \text{and} \quad X_4^* \le \hat{X}_4 \qquad \text{a.s.} \tag{5.3}$$

Then it is not hard to see that $X_1^* \le [\hat{X}_2, \hat{X}_3] \le X_4^*$ a.s. and that $X_1^* + X_4^* = \hat{X}_2 + \hat{X}_3$ a.s. Therefore $(X_1^*, \theta_1) \le [(\hat{X}_2, \theta_2), (\hat{X}_3, \theta_3)] \le (X_4^*, \theta_4)$ a.s. and $(X_1^*, \theta_1) + (X_4^*, \theta_4) = (\hat{X}_2, \theta_2) + (\hat{X}_3, \theta_3)$ a.s. So if $Y_i \overset{st}{=} Z(\hat{X}_i, \theta_i)$, $i = 2,3$, and $Y_i^* = Z(X_i^*, \theta_i)$, $i = 1,4$, then from the fact that $\{Z(X,\theta), (X,\theta) \in T\times\Theta\} \in SI\text{-}DCX(sp)$ it follows that there exist four random variables $\hat{Y}_i$, $i = 2,3$, and $\hat{Y}_i^*$, $i = 1,4$, defined on a common probability space, such that

$$\hat{Y}_i \overset{st}{=} Y_i, \quad i = 2,3, \qquad \hat{Y}_i^* \overset{st}{=} Y_i^*, \quad i = 1,4\,,$$

$$[\hat{Y}_2, \hat{Y}_3] \le \hat{Y}_4^* \quad \text{a.s.}$$

and

$$\hat{Y}_1^* + \hat{Y}_1^* \ge \hat{Y}_2 + \hat{Y}_3 \quad \text{a.s.}$$

From the stochastic monotonicity of $Z(x,\theta)$ in x, using (5.3), it follows that random variables $\hat{Y}_i$, $i = 1,4$, can be constructed such that $\hat{Y}_i \overset{st}{=} Y_i$ and $\hat{Y}_i \ge \hat{Y}_i^*$ a.s., $i = 1,4$.

Summarizing, we have shown the existence of $\hat{Y}_i$, $i = 1,2,3,4$, such that

$$\hat{Y}_i \overset{st}{=} Z(X_n(\theta_i), \theta_i)\,, \qquad i = 1,2,3,4,$$

$$[\hat{Y}_2, \hat{Y}_3] \le \hat{Y}_4 \quad \text{a.s.}$$

and

$$\hat{Y}_2 + \hat{Y}_3 \le \hat{Y}_1 + \hat{Y}_4 \quad \text{a.s.}$$

But $Z(X_n(\theta_i), \theta_i) \overset{st}{=} X_{n+1}(\theta_i)$, $i = 1,2,3,4$. This proves (5.2). □

Remark 5.1. In Section 6 we will need a modification of Theorem 5.1. The modification corresponds to the restriction that in the choice of θ_1, θ_2, θ_3, θ_4 it is required that $\theta_2 = \theta_3$. Using a proof similar to the proof of Theorem 5.1, one can show the following result: If for any choice of (x_i, θ_i), $i = 1,2,3,4$, such that $\theta_1 \le \theta_2 = \theta_3 \le \theta_4$, $x_1 \le x_2 \le x_3 \le x_4$ and $(x_1, \theta_1) + (x_4, \theta_4) = (x_2, \theta_2] + (x_3, \theta_3)$, there exist random variables $\hat{Z}_i$, $\hat{X}_i$, $i = 1,2,3,4$, such that $\hat{Z}_i \overset{st}{=} Z(x_i, \theta_i)$, $\hat{X}_i \overset{st}{=} X_0(\theta_i)$, $i = 1,2,3,4$, and a.s., $[\hat{Z}_2, \hat{Z}_3] \le \hat{Z}_4$, $\hat{Z}_1 + \hat{Z}_4 \ge \hat{Z}_2 + \hat{Z}_3$, $[\hat{X}_2, \hat{X}_3] \le \hat{X}_4$ and $\hat{X}_1 + \hat{X}_4 \ge \hat{X}_2 + \hat{X}_3$, then for every $n \in N$ there exist four random variables $\hat{Y}_i$, $i = 1,2,3,4$, such that $\hat{Y}_i \overset{st}{=} X_n(\theta_i)$, $i = 1,2,3,4$, $[\hat{Y}_2, \hat{Y}_3] \le \hat{Y}_4$ a.s. and $\hat{Y}_1 + \hat{Y}_4 \ge \hat{Y}_2 + \hat{Y}_3$ a.s. A similar statement corresponding to the increasing concave case can also be stated and proved.

In order to apply Theorem 5.1, one needs to verify that $\{Z(x,\theta), (x,\theta) \in T \times \Theta\}$ is either *SI-DCX*(*sp*) or *SI-DCV*(*sp*). This can be achieved only by coming up with a sample path construction. So far there is no standard way of doing it. Hence it seems advantageous to provide a sufficient condition, for the parametric stochastic convexity (concavity) of discrete time Markov processes, that would be easy to verify. This is done in the next result. This result assumes less than Theorem 5.1 but its conclusion is weaker.

THEOREM 5.2. *Suppose* $\{Z(x,\theta),(x,\theta)\in T\times\Theta\}\in$ *SI-DCX* (*SI-DCV*). *If* $\{X_0(\theta),\theta\in\Theta\}\in$ *SICX* (*SICV*), *then* $\{X_n(\theta),\theta\in\Theta\}\in$ *SICX* (*SICV*) *for each* $n=0,1,2,\ldots$.

PROOF. We prove the convex case only. The concave case is similarly proven. As an induction hypothesis assume that $\{X_n(\theta),\theta\in\Theta\}\in SICX$ for some $n\in N$. For any $\theta_i\in\Theta$, $i=1,2,3,4$, such that $\theta_1\le\theta_2=\theta_3\le\theta_4$ and $\theta_1+\theta_4=\theta_2+\theta_3$ consider $X_n(\theta_i)$, $i=1,2,3,4$. The stochastic monotonicity of $X_n(\theta)$ implies that there exist two random variables $\hat{X}_1$ and $\hat{X}_4$ defined on a common probability space such that $\hat{X}_i\stackrel{st}{=}X_n(\theta_i)$, $i=1,4$ and $\hat{X}_1\le\hat{X}_4$ a.s. Let I be a random variable independent of $\hat{X}_1$ and $\hat{X}_4$ such that $P\{I=0\}=P\{I=1\}=1/2$. Define $\hat{X}_2=(1-I)\hat{X}_1+I\hat{X}_4$ and $\hat{X}_3=I\hat{X}_1+(1-I)\hat{X}_4$. Clearly, $\hat{X}_2\stackrel{st}{=}\hat{X}_3$, $(\hat{X}_1,\theta_1)\le(\min(\hat{X}_2,\hat{X}_3),\theta_2)\le(\max(\hat{X}_2,\hat{X}_3),\theta_3)\le(\hat{X}_4,\theta_4)$ a.s. and $(\hat{X}_1,\theta_1)+(\hat{X}_4,\theta_4)=(\min(\hat{X}_2,\hat{X}_3),\theta_2)+(\max(\hat{X}_2,\hat{X}_3),\theta_3)$ a.s. So using the fact that $\{Z(x,\theta),(x,\theta)\in T\times\Theta\}\in SI\text{-}DCX$, it follows that for any increasing and convex function ϕ, one has

$$
\begin{aligned}
(5.4)\qquad & E\phi(Z(\hat{X}_1,\theta_1))+E\phi(Z(\hat{X}_4,\theta_4))\\
&\ge E\phi(Z(\min(\hat{X}_2,\hat{X}_3),\theta_2))+E\phi(Z(\max(\hat{X}_2,\hat{X}_3),\theta_3))\\
&=E\phi(Z(\hat{X}_2,\theta_2))+E\phi(Z(\hat{X}_3,\theta_3))\ ,
\end{aligned}
$$

where the last equality follows from $\theta_2=\theta_3$.

By the definition of $\hat{X}_2$,

$$P\{\hat{X}_2>x\}=\frac{1}{2}P\{X_n(\theta_1)>x\}+\frac{1}{2}P\{X_n(\theta_4)>x\}\ .$$

From the fact that $\{X_n(\theta),\theta\in\Theta\}\in SICX$ it follows that $\int_x^\infty P\{X_n(\theta)>u\}du$ is a convex function of θ for each x. Therefore using the fact that $\theta_2=\theta_1/2+\theta_4/2$, we have

$$
\begin{aligned}
\int_x^\infty P\{\hat{X}_2>u\}du &= \frac{1}{2}\int_x^\infty P\{X_n(\theta_1)>u\}du\\
&\quad+\frac{1}{2}\int_x^\infty P\{X_n(\theta_4)>u\}du\ge\int_x^\infty P\{X_n(\theta_2)>u\}du\ .
\end{aligned}
$$

That is, $\hat{X}_2\ge_c X_n(\theta_2)$ where the ordering $\ge_c$ is defined, e.g., in Stoyan ((1983), Subsection 1.3). Therefore, using the fact that $\{Z(x,\theta),(x,\theta)\in T\times\Theta\}\in SI\text{-}DCX$, it is seen that

$$(5.5)\qquad E\phi(Z(\hat{X}_2,\theta_2))\ge E\phi(Z(X_n(\theta_2),\theta_2))$$

since $\psi(x) \equiv E\phi(Z(x, \theta_2))$ is increasing and convex in x by Proposition 2.1(iii). Combining (5.4) and (5.5) and observing that $Z(\hat{X}_i, \theta_i) \overset{st}{=} X_{n+1}(\theta_i)$, $i = 1, 4$ and $Z(X_n(\theta_2), \theta_2) = X_{n+1}(\theta_2) \overset{st}{=} X_{n+1}(\theta_3)$, one obtains $\{X_{n+1}(\theta), \theta \in \Theta\} \in SICX$. □

Remark 5.2. Analogous to the observation in Remark 5.1, it is sufficient in Theorem 5.2 to require the *SI-DCX* (*SI-DCV*) property of $\{Z(x, \theta), (x, \theta) \in T \times \Theta\}$ with the restriction $\theta_2 = \theta_3$. That is, using a proof similar to the proof of Theorem 5.2 the following result can be obtained: If $\{Z(x, \theta), (x, \theta) \in T \times \Theta\} \in SI$, and if for any choice of (x_i, θ_i), $i = 1, 2, 3, 4$, such that $\theta_1 \le \theta_2 = \theta_3 \le \theta_4$, $x_1 \le x_2 \le x_3 \le x_4$ and $(x_1, \theta_1) + (x_4, \theta_4) = (x_2, \theta_2) + (x_3, \theta_3)$, we have $E\phi(Z(x_1, \theta_1)) + E\phi(Z(x_4, \theta_4)) \ge (\le)\ E\phi(Z(x_2, \theta_2)) + E\phi(Z(x_3, \theta_3))$ for every increasing convex function ϕ, and if $\{X_0(\theta), \theta \in \Theta\} \in SICX$ (*SICV*), then $\{X_n(\theta), \theta \in \Theta\} \in SICX$ (*SICV*) for each $n = 0, 1, 2, \ldots$.

Remark 5.3. Notice that in order to verify that $\{Z(x, \theta), (x, \theta) \in T \times \Theta\} \in SI\text{-}DCX$ (*SI-DCV*), or its weaker version described in Remark 5.2, one only needs to show that, for each y, $\int_y^\infty P\{Z(x, \theta) > u\}du$ is increasing and directionally convex (concave) in (x, θ) or its weaker version.

Using a proof similar to that of Theorem 5.2, the following closure property can be established.

THEOREM 5.3. *Suppose $\{X(\theta), \theta \in \Theta\}$ and $\{Y(\theta), \theta \in \Theta\}$ are two collections of random variables such that, for each θ, $X(\theta)$ and $Y(\theta)$ are independent. If $\{X(\theta), \theta \in \Theta\} \in SICX$ (SICV) and $\{Y(\theta), \theta \in \Theta\} \in SICX$ (SICV), then $\{X(\theta) + Y(\theta), \theta \in \Theta\} \in SICX$ (SICV).*

PROOF. We prove the convex case only. The concave case can be similarly proven. Let $\theta_i \in \Theta$, $i = 1, 2, 3, 4$, be such that $\theta_1 \le \theta_2 = \theta_3 \le \theta_4$ and $\theta_1 + \theta_4 = \theta_2 + \theta_3$. The stochastic monotonicity of $X(\theta)$ and $Y(\theta)$ can be used to construct four random variables $\hat{X}_1$, $\hat{X}_4$, $\hat{Y}_1$, $\hat{Y}_4$ such that $\hat{X}_i \overset{st}{=} X(\theta_i)$, $\hat{Y}_i \overset{st}{=} Y(\theta_i)$, $i = 1, 4$, $\hat{X}_1 \le \hat{X}_4$ a.s. and $\hat{Y}_1 \le \hat{Y}_4$ a.s. Furthermore, $(\hat{X}_1, \hat{X}_4)$ and $(\hat{Y}_1, \hat{Y}_4)$ can be constructed so that they are independent. Let I_1 and I_2 be independent random variables, independent of $\hat{X}_1$, $\hat{X}_4$, $\hat{Y}_1$, $\hat{Y}_4$, such that $P\{I_1 = 0\} = P\{I_1 = 1\} = P\{I_2 = 0\} = P\{I_2 = 1\} = 1/2$. Define $\hat{X}_2 = (1 - I_1)\hat{X}_1 + I_1\hat{X}_4$, $\hat{X}_3 = I_1\hat{X}_1 + (1 - I_1)\hat{X}_4$, $\hat{Y}_2 = (1 - I_1)\hat{Y}_1 + I_2\hat{Y}_4$ and $\hat{Y}_3 = I_2\hat{Y}_1 + (1 - I_2)\hat{Y}_4$. It is then not hard to see that $\hat{X}_2 \overset{st}{=} \hat{X}_3$, $\hat{Y}_2 \overset{st}{=} \hat{Y}_3$,

$$[\hat{X}_1, \hat{Y}_1) \le [(\hat{X}_2, \hat{Y}_2), (\hat{X}_3, \hat{Y}_3)] \le (\hat{X}_4, \hat{Y}_4) \qquad \text{a.s.}$$

and

$$(\hat{X}_1 + \hat{Y}_1) + (\hat{X}_4 + \hat{Y}_4) = (\hat{X}_2 + \hat{Y}_2) + (\hat{X}_3 + \hat{Y}_3) \qquad \text{a.s.}$$

Then, for any increasing convex function ϕ, one has

$$E\phi(\hat{X}_1 + \hat{Y}_1) + E\phi(\hat{X}_4 + \hat{Y}_4)$$
$$\geq E\phi(\hat{X}_2 + \hat{Y}_2) + E\phi(\hat{X}_3 + \hat{Y}_3) .$$

Observe, as in the proof of Theorem 5.2, that $\hat{X}_2 \geq_c X(\theta_2)$ and $\hat{Y}_2 \geq_c Y(\theta_2)$. So by the preservation of the ordering $\geq_c$ under convolution (see, e.g., Ross (1983)), it follows that $\hat{X}_2 + \hat{Y}_2 \geq_c X(\theta_2) + Y(\theta_2)$. That is, for any increasing convex function ϕ, one has

$$E\phi(\hat{X}_2 + \hat{Y}_2) \geq E\phi(X(\theta_2) + Y(\theta_2)) .$$

Therefore

$$E\phi(X(\theta_1) + Y(\theta_1)) + E\phi(X(\theta_4) + Y(\theta_4))$$
$$\geq E\phi(X(\theta_2) + Y(\theta_2)) + E\phi(X(\theta_3) + Y(\theta_3)) .$$

Combining this with the preservation of stochastic monotonicity under convolution, one has $\{X(\theta) + Y(\theta), \theta \in \Theta\} \in SICX$. □

6. Applications

6.1 $M^B/M(n)/1$ *queues*

Consider a single stage queueing system at which customers arrive according to a Poisson process with rate $\lambda(\theta) > 0$ which depends on a parameter $\theta \in \Theta$ where Θ is a convex subset of $\boldsymbol{R}$ or $\boldsymbol{N}$. Customer n brings a random number B_n of tasks, $n = 1, 2, \ldots$. The B_n's are independent and identically distributed with a common distribution function Q. The service rate of a task, when there are x tasks in the system, also depends on θ. Denote it by $\mu(x, \theta)$, $x \in \boldsymbol{N}$, $\theta \in \Theta$. Suppose $\mu(0, \theta) = 0$ and $\mu(x, \theta) > 0$. Let $Y_t(\theta)$ denote the number of tasks in the system at time t.

Denote $\mu_s \equiv \sup_{x \in N, \theta \in \Theta} \{\mu(x, \theta)\}$ and $\lambda_s \equiv \sup_{\theta \in \Theta} \{\lambda(\theta)\}$, and suppose that $\mu_s < \infty$ and $\lambda_s < \infty$.

THEOREM 6.1. *Suppose that $\lambda(\theta)$ is increasing and convex in $\theta \in \Theta$ and that $\mu(x, \theta)$ is decreasing in θ for each x and is directionally concave in (x, θ). If $\{Y_0(\theta), \theta \in \Theta\} \in SICX(sp)$, then $\{Y_t(\theta), \theta \in \Theta\} \in SICX(sp)$ for each $t \geq 0$.*

PROOF. Define $\Lambda \equiv 2(\mu_s + \lambda_s)$. Consider the discrete time Markov chain $\{X_n(\theta), n = 0, 1, 2, \dots\}$ with state space $\boldsymbol{N}$ and

$$Z(x,\theta) = \begin{cases} x + B & \text{with probability} \quad \lambda(\theta)/\Lambda\,, \\ x & \text{with probability} \quad 1 - \dfrac{1}{\Lambda}(\lambda(\theta) + \mu(x,\theta))\,, \\ x - 1 & \text{with probability} \quad \mu(x,\theta)/\Lambda\,, \end{cases}$$

where B has the distribution function Q. Suppose $X_0(\theta) \overset{st}{=} Y_0(\theta)$. Let $\{N(t), t \geq 0\}$ be a Poisson process with rate Λ defined on the same probability space as that of $\{X_n(\theta), n = 0, 1, 2, \dots\}$ and independent of it. Then the uniformized process $\{X_{N(t)}(\theta), t \geq 0\}$ satisfies

$$\{Y_t(\theta), t \geq 0\} \overset{st}{=} \{X_{N(t)}(\theta), t \geq 0\} \qquad \text{for each} \quad \theta \in \Theta$$

(see, e.g., Keilson (1979)).

Now, for any $x_i \in \boldsymbol{N}$, $i = 1, 2, 3, 4$, such that $x_1 \leq [x_2, x_3] \leq x_4$ and $x_1 + x_4 = x_2 + x_3$ and for any $\theta_i \in \Theta$, $i = 1, 2, 3, 4$, such that $\theta_1 \leq [\theta_2, \theta_3] \leq \theta_4$ and $\theta_1 + \theta_4 = \theta_2 + \theta_3$ construct $\hat{X}_i$, $i = 1, 2, 3, 4$ on a common probability space using two independent uniform $(0, 1)$ random variables U_1 and U_2 as follows:

(i) If $U_1 \in (0, \lambda(\theta_i)/\Lambda)$, then set $\hat{X}_i = x_i + \bar{Q}^{-1}(U_2)$, $i = 1, 2, 4$ (here $\bar{Q} \equiv 1 - Q$) and if

$$U_1 \in (0, \lambda(\theta_i)/\Lambda) \cup (\lambda(\theta_2)/\Lambda\,,\, (\lambda(\theta_2) + \lambda(\theta_3) - \lambda(\theta_1))/\Lambda)\,,$$

then set $\hat{X}_3 = x_3 + \bar{Q}^{-1}(U_2)$. Note that by the convexity of λ, one has $\lambda(\theta_2) + \lambda(\theta_3) - \lambda(\theta_1) \leq \lambda(\theta_4)$.

(ii) If $U_1 \in (\lambda(\theta_i)/\Lambda, 1 - \mu_s/\Lambda)$, then set $\hat{X}_i = x_i$, $i = 1, 2, 4$ and if $U_1 \in (\lambda(\theta_1)/\Lambda, \lambda(\theta_2)/\Lambda) \cup ((\lambda(\theta_2) + \lambda(\theta_3) - \lambda(\theta_1))/\Lambda, 1 - \mu_s/\Lambda)$, then set $\hat{X}_3 = x_3$.

(iii) Denote $r_i \equiv \mu(x_i, \theta_i)$, $i = 1, 2, 3, 4$, $r_{12} \equiv \min(r_1, r_2)$, $r_{34} \equiv \min(r_3, r_4)$, $R_{13} \equiv \max(r_1, r_3)$ and $R_{24} \equiv \max(r_2, r_4)$. If $U_1 \in (1 - \mu_s/\Lambda, 1)$, then set $\hat{X}_i = x_i - \Delta_i$, $i = 1, 2, 3, 4$, where

$$\Delta_1 = I_{(0, r_{12}/\mu_s)}(U_2) + I_{((R_{13} - r_1 + r_{12})/\mu_s,\, R_{13}/\mu_s)}(U_2)\,,$$

$$\Delta_2 = I_{(0, r_{12}/\mu_s)}(U_2) + I_{((R_{24} - r_2 + r_{12})/\mu_s,\, R_{24}/\mu_s)}(U_2)\,,$$

$$\Delta_3 = I_{(0, r_{34}/\mu_s)}(U_2) + I_{((R_{13} - r_3 + r_{34})/\mu_s,\, R_{13}/\mu_s)}(U_2)$$

and

$$\Delta_4 = I_{(0, r_{34}/\mu_s)}(U_2) + I_{((R_{24} - r_4 + r_{34})/\mu_s,\, R_{24}/\mu_s)}(U_2)\,.$$

It is now easily verified that $\hat{X}_i \stackrel{st}{=} Z(x_i, \theta_i)$, $i = 1, 2, 3, 4$. It is not hard to verify that $[\hat{X}_2, \hat{X}_3] \le \hat{X}_4$ and $\hat{X}_1 + \hat{X}_4 \ge \hat{X}_2 + \hat{X}_3$ a.s. in case (i) and also in case (ii) (that is, when $U_1 \in (0, 1 - \mu_s/\Lambda)$). For case (iii) notice that by the directional concavity of $\mu(x, \theta)$ in (x, θ), one has $r_1 + r_4 \le r_2 + r_3$. Therefore we have only the following three cases to consider (see, e.g., Appendix of Shanthikumar and Yao (1987*a*)): (a) $r_1 \le r_2$ and $r_3 < r_4$, (b) $r_1 > r_2$ and $r_3 \ge r_4$ and (c) $r_1 \le r_2$ and $r_3 \ge r_4$.

In case (a):

$$(\Delta_2 + \Delta_3) - (\Delta_1 + \Delta_4) = I_{((R_{24} - r_2 + r_1)/\mu_s, R_{24}/\mu_s)}(U_2) - I_{((R_{24} - r_4 + r_3)/\mu_s, R_{24}/\mu_s)}(U_2) \ge 0 \quad \text{a.s.}$$

since $r_1 - r_2 \le r_3 - r_4$.

In case (b):

$$(\Delta_2 + \Delta_3) - (\Delta_1 + \Delta_4) = I_{((R_{13} - r_3 + r_4)/\mu_s, R_{13}/\mu_s)}(U_2) - I_{((R_{13} - r_1 + r_2)/\mu_s, R_{13}/\mu_s)}(U_2) \ge 0 \quad \text{a.s.}$$

since $r_4 - r_3 \le r_2 - r_1$.

In case (c):

$$(\Delta_2 + \Delta_3) - (\Delta_1 + \Delta_4) = I_{((R_{24} - r_2 + r_1)/\mu_s, R_{24}/\mu_s)}(U_2) - I_{((R_{13} - r_3 + r_4)/\mu_s, R_{13}/\mu_s)}(U_2) \ge 0 \quad \text{a.s.}$$

So in case (iii), $(\hat{X}_1 + \hat{X}_4) - (\hat{X}_2 + \hat{X}_3) = (x_1 + x_4) - (x_2 + x_3) + (\Delta_2 + \Delta_3) - (\Delta_1 + \Delta_4) \ge 0$ a.s.

We will now show that in case (iii), $[\hat{X}_2, \hat{X}_3] \le \hat{X}_4$ a.s. Since each Δ_i is either 0 or 1 we only need to consider the cases: (α) $x_4 = x_2$ (and show that then $\Delta_4 \le \Delta_2$) and (β) $x_4 = x_3$ (and show that then $\Delta_4 \le \Delta_3$).

Consider case (α). Then, since $\mu(x, \theta) \downarrow \theta$ it follows that $r_2 \ge r_4$. Therefore $R_{24} = r_2$.

In case (a) (that is, when $r_1 \le r_2$ and $r_3 < r_4$):

$$\Delta_2 = I_{(0, r_2/\mu_s)}(U_2)$$

and

$$\Delta_4 = I_{(0, r_3/\mu_s)}(U_2) + I_{((r_2 - r_4 + r_3)/\mu_s, r_2/\mu_s)}(U_2) .$$

So $\Delta_4 \le \Delta_2$.

In case (b) (that is, when $r_1 > r_2$, and $r_3 \ge r_4$):

$$\Delta_2 = I_{(0, r_2/\mu_s)}(U_2)$$

and

$$\Delta_4 = I_{(0, r_4/\mu_s)}(U_2)\ .$$

Since $r_2 \geq r_4$ it follows that $\Delta_4 \leq \Delta_2$.

In case (c) (that is, when $r_1 \leq r_2$, $r_3 \geq r_4$):

$$\Delta_2 = I_{(0, r_2/\mu_s)}(U_2)$$

and

$$\Delta_4 = I_{(0, r_4/\mu_s)}(U_2)\ .$$

Since $r_2 \geq r_4$ it follows that $\Delta_4 \leq \Delta_2$.

Consider now case (β). Then since $\mu(x, \theta) \downarrow \theta$ and $\theta_3 \leq \theta_4$ it follows that $r_3 \geq r_4$. So, only case (b) and (c) are to be considered. But in these cases it is easy to see that $\Delta_4 \leq \Delta_3$.

In summary, we have shown that $\hat{X}_i \overset{st}{=} Z(x_i, \theta_i)$, $i = 1, 2, 3, 4$, $[\hat{X}_2, \hat{X}_3] \leq \hat{X}_4$ a.s. and $\hat{X}_1 + \hat{X}_4 \geq \hat{X}_2 + \hat{X}_3$ a.s. Therefore $\{Z(x, \theta), (x, \theta) \in \boldsymbol{N} \times \Theta\} \in$ *SI-DCX*(*sp*). Then, by Theorem 5.1, one has $\{X_n(\theta), \theta \in \Theta\} \in$ *SICX*(*sp*). The required convexity of $Y_t(\theta)$ now follows from the preservation of the *SICX*(*sp*) property under mixtures (see Theorem 3.9 of Shaked and Shanthikumar (1988*a*)). □

Consider now an $M/M/c$ queue with arrival rate λ and mean service time μ^{-1} where $\lambda < c\mu$. Let $N(\lambda, \mu)$ be the stationary number of customers in the system. Then one has

COROLLARY 6.1. *If* (λ_i, μ_i), $i = 1, 4$, *are such that* $\lambda_1 \leq \lambda_4$, $\mu_1 \geq \mu_4$ *and* $\lambda_4 \leq c\mu_4$, *then*

$$E\phi(N(\lambda, \mu)) \leq [E\phi(N(\lambda_1, \mu_1)) + E\phi(N(\lambda_4, \mu_4))]/2$$

for any increasing convex function ϕ, *where* $(\lambda, \mu) = [(\lambda_1, \mu_1) + (\lambda_4, \mu_4)]/2$.

PROOF. Let $\lambda(\theta) \equiv \lambda_1 + (\lambda_2 - \lambda_1)\theta$, $\theta = 0, 1, 2$ and $\mu(x, \theta) = \min(x, c)\mu(\theta)$ where $\mu(\theta) = \mu_1 - (\mu_1 - \mu_2)\theta$, $\theta = 0, 1, 2$. Observe that $\lambda(\theta)$ is increasing and convex in θ and $\mu(x, \theta)$ is increasing and concave in x and is decreasing and concave in θ. Furthermore, a lengthy verification shows that for any choice of x_i, $i = 1, 2, 3, 4$, such that $x_1 \leq x_2 \leq x_3 \leq x_4$ and $x_1 + x_4 = x_2 + x_3$, one has

$$\mu(x_1, 0) + \mu(x_4, 2) = \min(x_1, c)\mu_1 + \min(x_4, c)\mu_4$$

$$\leq \min(x_2, c)\mu_2 + \min(x_3, c)\mu_3 = \mu(x_2, 1) + \mu(x_3, 1) .$$

That is, $\mu(x,\theta)$ is directionally concave on $\boldsymbol{N}\times\{0,1,2\}$ (for $\theta_2=\theta_3$). Now the desired result follows from the proof of Theorem 6.1 combined with Remark 5.1. □

Remark 6.1. The above result does not give the joint convexity of $N(\lambda,\mu)$ in (λ,μ). It only gives the joint convexity of $N(\lambda,\mu)$ in the diagonal decreasing direction (i.e., along the line $\lambda = a - b\mu$ for $b>0$). Anyway, the joint convexity of $N(\lambda,\mu)$ in the general context is not possible. For example, where $c\to\infty$, i.e., for the $M/M/\infty$ queue, one has $EN(\lambda,\mu)=\lambda/\mu$. This is clearly not jointly convex in (λ,μ). Harel and Zipkin (1987), however, show that $E[W(\lambda,\mu)]=E[N(\lambda,\mu)]/\lambda$ is jointly convex in (λ,μ) where $W(\lambda,\mu)$ is the corresponding stationary waiting time.

6.2 *GI/M/c queue*

Consider a c-server queueing system with a renewal arrival process and exponentially distributed service time. Let $A(\theta)$ be a generic random variable, with distribution depending on a parameter $\theta\in\Theta$, representing the interarrival times, and let μ^{-1} be the mean service time. Let $N_k(\theta)$ denote the number of customers in the system at the k-th arrival epoch and let $W_k(\theta)$ denote the waiting time of the k-th customer. Then one has

THEOREM 6.2. *Suppose* $\{A(\theta),\theta\in\Theta\}\in SDCV(sp)$. *If* $\{N_0(\theta),\theta\in\Theta\}\in SICX(sp)$, *then*

(a) $\{N_k(\theta),\theta\in\Theta\}\in SICX(sp)$ *and*

(b) $\{W_k(\theta),\theta\in\Theta\}\in SICX(sp)$.

In order to prove the above theorem, we need the following result of Shanthikumar and Yao (1987*b*). Let $D(x,t)$ be the number of survivors at time t in a pure death process starting with x survivors at time 0, that is, $x-D(x,t)$ is the number of deaths during $(0,t]$. Let $\gamma(n)$ be the death rate when the number of survivors is n. Then one has (Shanthikumar and Yao (1987*b*)).

THEOREM 6.3. *Suppose $\gamma(n)$ is increasing and concave in n. Then for any choice of* (x_i,t_i), $i=1,2,3,4$, *such that* $x_1\leq x_2\leq x_3\leq x_4$, $x_1+x_4=x_2+x_3$, $t_1\leq[t_2,t_3]\leq t_4$ *and* $t_1+t_4=t_2+t_3$, *there exist four random variables* $\hat{X}_i$, $i=1,2,3,4$, *defined on a common probability space, such that*

$$\hat{X}_i \overset{st}{=} D(x_i,t_i), \qquad i=1,2,3,4,$$

$$[\hat{X}_1,\hat{X}_2,\hat{X}_3]\leq\hat{X}_4 \qquad a.s.$$

and

$$\hat{X}_1 + \hat{X}_4 \geq \hat{X}_2 + \hat{X}_3 \qquad a.s.$$

PROOF OF THEOREM 6.2. Observe that $\{N_k(\theta), k = 0, 1, 2, \ldots\}$ is a Markov chain with

$$Z(x, \theta) \stackrel{st}{=} [N_{k+1}(\theta) | N_k(\theta) = x] \stackrel{st}{=} D(x + 1, A(\theta)) ,$$

where $D(x, t)$ is the number of survivors at time t in a pure death process starting with x survivors and the death rate, when the number of survivors is n, is $\gamma(n) = \min(n, c)\mu$, $n = 0, 1, 2, \ldots$. Clearly, $\gamma(n)$ is increasing and concave in n. So combining Theorem 6.3 with the observation that $\{A(\theta), \theta \in \Theta\} \in SDCV(sp)$ and $D(x, t)$ is stochastically decreasing in t, one can conclude that

$$\{Z(x, \theta), (x, \theta) \in \boldsymbol{N} \times \Theta\} \in SI\text{-}DCX(sp) .$$

(In order to verify this statement, one needs to use an argument similar to the one used in the proof of Theorem 5.1.) Part (a) now follows from Theorem 5.1. For part (b) observe that

$$W_k(\theta) \stackrel{st}{=} \sum_{i=1}^{[N_k(\theta) - c]^+} X_i ,$$

where X_i, $i = 1, 2, \ldots$, are independent and identically distributed exponential random variables with mean $(\mu c)^{-1}$. The closure properties of the $SICX(sp)$ property, given in Shaked and Shanthikumar (1988*a*, 1988*b*) guarantee that $\{W_k(\theta), \theta \in \Theta\} \in SICX(sp)$ for each k. □

Remark 6.2. Consider a $GI/M(n)/1$ queue with service rate $\gamma(n)$ when there are n customers in the queue. Then from the proof of Theorem 6.2 it is clear that part (a) will still hold true for this system as long as $\gamma(n)$ is increasing and concave.

6.3 *Cumulative damage shock models in reliability theory*

Esary *et al.* (1973) considered the following model for wear processes to which we here add a parameter θ.

Model 6.1. Suppose an item is subjected to shocks occurring randomly in time according to a Poisson process $\{N(t), t \geq 0\}$ with rate λ. The i-th shock causes a nonnegative random damage $X_i(\theta)$ where $\theta \in \Theta$ and Θ is a convex subset of $\boldsymbol{R}$ or $\boldsymbol{N}$. The damages are independent and identically distributed and accumulate additively.

Let $S_0(\theta)$ be the damage at time 0 and define $S_n(\theta) = S_{n-1}(\theta) + X_n(\theta)$, $n = 1, 2, \ldots$. Thus at time $t > 0$ the accumulated damage is $S_{N(t)}(\theta)$.

THEOREM 6.4. *Suppose* $\{X_1(\theta), \theta \in \Theta\} \in SICX(sp)$ $(SICV(sp))$. *If* $\{S_0(\theta), \theta \in \Theta\} \in SICX(sp)$ $(SICV(sp))$, *then for each* t, $\{S_{N(t)}(\theta), \theta \in \Theta\} \in SICX(sp)$ $(SICV(sp))$.

PROOF. By Theorem 3.9 of Shaked and Shanthikumar (1988*a*) it suffices to show that

$$\{S_n(\theta), \theta \in \Theta\} \in SICX(sp)\ (SICV(sp)) \tag{6.1}$$

for each $n = 0, 1, 2, \ldots$.

Let

$$Z(x, \theta) \stackrel{st}{=} [S_n(\theta) \mid S_{n-1}(\theta) = x]\,. \tag{6.2}$$

Then $Z(x, \theta) = X(\theta) + x$ where $X(\theta) \stackrel{st}{=} X_1(\theta)$.

Fix (x_i, θ_i), $i = 1, 2, 3, 4$, such that $(x_1, \theta_1) \le [(x_2, \theta_2), (x_3, \theta_3)] \le (x_4, \theta_4)$ and $(x_1, \theta_1) + (x_4, \theta_4) = (x_2, \theta_2) + (x_3, \theta_3)$. Then there exist four random variables $\hat{X}_i$, $i = 1, 2, 3, 4$, defined on a common probability space such that

$$X_i \stackrel{st}{=} X(\theta_i), \qquad i = 1, 2, 3, 4\,,$$

$$[\hat{X}_2, \hat{X}_3] \le \hat{X}_4 \quad (\hat{X}_1 \le [\hat{X}_2, \hat{X}_3]) \qquad \text{a.s.}$$

and

$$\hat{X}_1 + \hat{X}_4 \ge (\le)\ \hat{X}_2 + \hat{X}_3 \qquad \text{a.s.}$$

Define $\hat{Z}_i = \hat{X}_i + x_i$, $i = 1, 2, 3, 4$. Then

$$\hat{Z}_i \stackrel{st}{=} Z(x_i, \theta_i), \qquad i = 1, 2, 3, 4\,,$$

$$[\hat{Z}_2, \hat{Z}_3] \le \hat{Z}_4 \quad (\hat{Z}_1 \le [\hat{Z}_2, \hat{Z}_3]) \qquad \text{a.s.}$$

and

$$\hat{Z}_2 + \hat{Z}_3 \le (\ge)\ \hat{Z}_1 + \hat{Z}_4 \qquad \text{a.s.}$$

That is,

$$\{Z(x, \theta), (x, \theta) \in [0, \infty) \times \Theta\} \in SI\text{-}DCX(sp)\ (SI\text{-}DCV(sp))$$

and (6.1) is obtained from Theorem 5.1. □

Remark 6.3. A simple modification of Theorem 6.4 shows that the conclusion $\{S_{N(t)}(\theta), \theta \in \Theta\} \in SICX(sp)$ $(SICV(sp))$ holds for models which are more general than Model 6.1. All that is needed is that $Z(x, \theta)$, defined in (6.2), satisfies the *SI-DCX(sp)* (*SI-DCV(sp)*) property. For example, if

$$Z(x, \theta) = \psi(X(\theta), x) + x$$

where ψ is a directionally convex (concave) function, then $\{S_{N(t)}(\theta), \theta \in \Theta\} \in SICX(sp)$ $(SICV(sp))$. Here the distribution of the new damage is allowed to depend on the present accumulated damage.

6.4 *Branching processes*

Consider a Galton-Watson discrete time branching process $\{X_i(\theta), i = 0, 1, 2, \ldots\}$ depending on a parameter $\theta \in (0, \infty)$ with the offspring discrete probability function $f(\cdot; \theta)$. If

$$Z(x, \theta) \overset{st}{=} [X_n(\theta) \mid X_{n-1}(\theta) = x]\,, \tag{6.3}$$

then

$$P\{Z(x, \theta) = y\} = f^{(x)}(y; \theta)$$

where $f^{(x)}(\cdot; \theta)$ denotes the x-th convolution of $f(\cdot; \theta)$. The following result is a generalization of Result 5.9 of Shaked and Shanthikumar (1988*a*) and its proof is simpler than in that paper.

THEOREM 6.5. *Suppose* $f(\cdot; \theta)$, $\theta > 0$, *has the semigroup property, that is,* $f(\cdot; \theta_1) * f(\cdot; \theta_2) = f(\cdot; \theta_1 + \theta_2)$ *where "*" denotes convolution. If* $\{X_0(\theta), \theta \in (0, \infty)\} \in SICX(sp)$ *then* $\{X_n(\theta), \theta \in (0, \infty)\} \in SICX(sp)$ *for each* $n = 0, 1, 2, \ldots$.

PROOF. Fix (x_i, θ_i), $i = 1, 2, 3, 4$, such that $(x_1, \theta_1) \le [(x_2, \theta_2), (x_3, \theta_3)] \le (x_4, \theta_4)$ and $(x_1, \theta_1) + (x_4, \theta_4) = (x_2, \theta_2) + (x_3, \theta_3)$. On some probability space define the following mutually independent random variables

Y_j^1 having probability function $f(\cdot; \theta_1)$, $j = 1, 2, \ldots$,

Y_j^2 having probability function $f(\cdot; \theta_2 - \theta_1)$, $j = 1, 2, \ldots$ and

Y_j^3 having probability function $f(\cdot; \theta_4 - \theta_2)$, $j = 1, 2, \ldots$.

Let

$$\hat{Z}_1 = \sum_{j=1}^{x_1} Y_j^1\,,$$

$$\hat{Z}_2 = \sum_{j=1}^{x_2} (Y_j^1 + Y_j^2)\,,$$

$$\hat{Z}_3 = \sum_{j=1}^{x_1} (Y_j^1 + Y_j^3) + \sum_{j=x_2+1}^{x_4} (Y_j^1 + Y_j^3)$$

and

$$\hat{Z}_4 = \sum_{j=1}^{x_4} (Y_j^1 + Y_j^2 + Y_j^3) .$$

Then it is easy to verify, using the semigroup property of $f(\cdot;\theta)$, that

$$\hat{Z}_i \stackrel{st}{=} Z(x_i, \theta_i), \qquad i = 1, 2, 3, 4 .$$

It is also easy to see that

$$[\hat{Z}_2, \hat{Z}_3] \leq \hat{Z}_4 \qquad \text{a.s.}$$

and

$$\hat{Z}_1 + \hat{Z}_4 \geq \hat{Z}_2 + \hat{Z}_3 \qquad \text{a.s.}$$

That is, $\{Z(x,\theta), (x,\theta) \in \boldsymbol{N} \times (0,\infty)\} \in SI\text{-}DCX(sp)$. The desired result now follows from Theorem 5.1. □

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