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INVARIANCE RELATIONS IN SINGLE SERVER QUEUES WITH LCFS SERVICE DISCIPLINE*

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Abstract. This paper is concerned with single server queues having LCFS service discipline. We give a condition to hold an invariance relation between time and customer average queue length distributions in the queues. The relation is a generalization of that in an ordinary GI/M/1 queue. We compare the queue length distributions for different single server queues with finite waiting space under the same arrival process and service requirement distribution of customer and derive invariance relations among them.

Key words and phrases: Queue, last-come-first-served, invariance relation, loss system.

1. Introduction

In recent years, the study of relations between characteristics in a queueing model and ordering relations among characteristics in different queueing models has been developed.

For the latter, see, for example, Stoyan (1983). As typical examples of the former, many relations between time and customer averages of quantities in queues have been derived based on the theory of point processes (see e.g., Miyazawa (1979, 1983) and Franken *et al.* (1981)). The relations were called "invariance relations" by Miyazawa (1983). In this paper we call both the relations between time and customer averages of quantities in a queue and the relations (not ordering) among characteristics such as queue length and sojourn time distributions in different queues "invariance relations."

This paper is concerned with single server queues having last-come-first-served (LCFS) service discipline in the steady state. Yamazaki (1984)

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derived a relationship between the queue length distributions for a GI/GI/1 with preemptive-resume (PR)-LCFS at any time and at an arrival instant. The relationship is the same as that in an ordinary GI/M/1 queue. Subsequently, Shanthikumar and Sumita (1986) showed that it remains true for G/GI/1 with PR-LCFS. Fakinos (1987) derived a similar relation for GI/GI/1 with PR-LCFS and service depending on queue size.

The first purpose of this paper is to unify and to extend the above results. This is done based on the equilibrium equations in the point process theory, called the basic equations (Miyazawa (1986)).

Our next concern is to obtain relations among the queue length distributions for different queues at arrival instants. For two kinds of single server queues with PR-LCFS and finite waiting space, these relations are derived.

This paper is composed of six sections. In Sections 2 and 3, we briefly introduce a model and the basic equations, respectively. In Section 4, we discuss relations between time- and customer-average queue length distributions in the model. In Sections 5 and 6, we deal with the single server queues with finite waiting space.

2. Model

Let us consider the following queue. Customers arrive at the queue in accordance with a stationary point process of rate λ. The service requirements of customers are stationary dependent random variables (r.v.'s). There is a single server and waiting space for N customers. The waiting space is partitioned into cells numbered from 1 to N. A cell with number j is called "position j." A customer present in the queue occupies one of the positions. When there are n customers in the queue, it operates in the following manner:

(i) Positions $1 - n$ are used.

(ii) The service effort by the server is supplied at the rate ϕ_n and it is directed to the customer in position 1.

(iii) When a customer arrives at the queue, he and the customers previously in positions $1 - n$ move to positions $1 - (n + 1)$ at the arrival instant according to a policy, called "reordering policy," where we use "position $(N + 1)$" as the "outside world," i.e., "a customer moves to position $(N + 1)$" implies that he leaves the queue (this customer is called a "lost customer"). When the customer in position 1 leaves the queue, his service completed, customers in positions $2 - n$ move to positions $1 - (n - 1)$ according to the reordering policy.

If the service discipline in the queue is FCFS, the reordering policy is as follows. Suppose that there are n customers in the queue. When a customer arrives at the queue, he moves into position $(n + 1)$; customers previously in positions $1 - n$ remain the same positions. When the customer in position 1 leaves the queue customers in positions $2-n$ move to positions $1-(n-1)$, respectively.

If the service discipline is LCFS, when there are n customers in the queue the reordering policy is as follows. When a customer arrives at the queue he moves into position 1; customers previously in positions $1-n$ move to positions $2-(n+1)$, respectively. When the customer in position 1 leaves the queue customers in positions $2-n$ move to positions $1-(n-1)$, respectively.

Under a service discipline such as LCFS, the service of a customer may be interrupted several times, owing to subsequent arriving customers. The restarting policy, that is, the way the server serves an interrupted customer, may depend on the history of the customer in the queue. For example, it may be preemptive-resume, that is, each customer continues his service just from the point he left it upon his last interruption and hence no loss of service is involved. Or again it may be preemptive-repeat.

3. Basic equations

For a queue in Section 2, the following characteristics are defined at time t:

$l(t)$ = the number of customers in the queue ,

$R_j(t)$ = the remaining service requirement of the customer in position j $(j = 1, 2, \ldots, l(t))$,

$U(t)$ = the remaining time until the next arrival of a customer .

Let $X(t) = \{l(t), R_1(t), \ldots, R_{l(t)}(t), U(t)\}$. Our essential assumption is that $X(t)$ forms a stationary process with respect to a suitably chosen probability measure P. Now, introduce two point processes consisting of the arrival and departure instants of customers, denoted by N_0 and N_1, respectively. Let P_i be conditional distribution P under the condition that there exists a point of N_i at time 0 for $i = 0, 1$. Let E, E_0 and E_1 be the expectations concerning P, P_0 and P_1, respectively.

Define

$$X_n(t) = \begin{cases} I\{l(t) = n\} \exp\{-\xi U(t)\} & \text{for} \quad n = 0 , \\ I\{l(t) = n\} \exp\left\{-\xi U(t) - \sum_{j=1}^{n} \theta_j R_j(t)\right\} & \text{for} \quad n \geq 1 , \end{cases}$$

where I_A is an indicator function of a set A, and ξ and θ_j are non-negative numbers.

When lost customers are counted as departures, the following equations, which are a version of Corollary 3.1 of Miyazawa (1983), hold.

$$(3.1)\qquad E(X_n'(0)) = \lambda \sum_{i=0}^{1} E_i[X_n(0-) - X_n(0+)] \quad \text{for} \quad 0 \le n \le N,$$

where $X_n'(0)$ is a right-hand derivative of $X_n(0)$.

For convenience, we use the following notation.

$$\begin{aligned}
&p_n = P(l(0) = n), \qquad q_n = P_0(l(0-) = n), \\
&r_n = P_1(l(0+) = n), \\
&\tilde{H}_n(\theta_1, \theta_2, \dots, \theta_n; \xi) = E\{X_n(0) \mid l(0) = n\}, \\
&\tilde{F}_n(\theta_1, \theta_2, \dots, \theta_n; \xi) = E_0\{X_n(0-) \mid l(0-) = n\}, \\
&\tilde{F}_n^*(\theta_1, \theta_2, \dots, \theta_n; \xi) = E_0\{X_n(0+) \mid l(0+) = n\}, \\
&\tilde{G}_n(\theta_1, \theta_2, \dots, \theta_n; \xi) = E_1\{X_n(0+) \mid l(0+) = n\}, \\
&\tilde{G}_n^*(\theta_1, \theta_2, \dots, \theta_n; \xi) = E_1\{X_n(0-) \mid l(0-) = n\}.
\end{aligned}$$

Then (3.1) can be rewritten as

$$(3.2)\qquad \begin{aligned}
&\xi p_0 \tilde{H}_0(\xi) = \lambda[q_0 - r_0 \tilde{G}_0(\xi)], \\
&(\xi + \phi_n \theta_1) p_n \tilde{H}_n(\theta_1, \theta_2, \dots, \theta_n; \xi) \\
&\quad = \lambda[q_n \tilde{F}_n(\theta_1, \theta_2, \dots, \theta_n; 0) - q_{n-1} \tilde{F}_n^*(\theta_1, \theta_2, \dots, \theta_n; \xi) \\
&\qquad + r_{n-1} \tilde{G}_n^*(\theta_1, \theta_2, \dots, \theta_n; \xi) - r_n \tilde{G}_n(\theta_1, \theta_2, \dots, \theta_n; \xi)]
\end{aligned}$$

for $n = 1, 2, \dots, N$. Equations (3.2) are called "the basic equations", following Miyazawa (1986). For details of the equations, please refer to Miyazawa and Yamazaki (1988).

Although p_n, q_n and r_n are defined for the queue, we use the same notations for other queues, i.e., $\{p_n\}$, $\{q_n\}$ and $\{r_n\}$ are the distributions of the number of customers in any queue at an arbitrary instant, just before an arrival instant and just after a departure instant, respectively.

For more general queues, it is well-known that

$$(3.3)\qquad q_n = r_n \quad \text{for all} \quad n.$$

For the queue in this section, (3.3) can be directly obtained by putting $\xi = \theta_1 = \theta_2 = \cdots = \theta_N = 0$ in (3.2).

4. Relationship between time- and customer-average queue length distributions

For G/M/1/N or G/GI/1/1 queue, the following relations hold (see, e.g., Franken *et al.* (1981)):

$$p_n = \lambda E(S) r_{n-1} \quad \text{for} \quad n \geq 1 , \tag{4.1}$$

where S is a genetic r.v. of the service requirement for their queues.

The main purpose of this section is to give a condition to hold a generalization of (4.1) for the queue described in Section 2.

By putting $\xi = \theta_2 = \theta_3 = \cdots = \theta_N = 0$ in (3.2) and using (3.3),

$$\begin{aligned} &\phi_n \theta_n p_n E\{\exp(-R_1(0)\theta_1 | l(0) = n)\} \\ &\quad = \lambda[r_n E_0\{\exp(-\theta_1 R_1(0-)| l(0-) = n)\} \\ &\qquad - r_{n-1} E_0\{\exp(-\theta_1 R_1(0+)| l(0+) = n)\} \\ &\qquad + r_{n-1} - r_n E_1\{\exp(-\theta_1 R_1(0+)| l(0+) = n)\}] \end{aligned} \tag{4.2}$$

for $n = 1, 2, \ldots, N$.

By dividing (4.2) by $\phi_n \theta_1$ and letting $\theta_1 \downarrow 0$, we can find that

$$p_n = \lambda \left[r_n \frac{E_1(R_1(0+)| l(0+) = n) - E_0(R_1(0-)| l(0-) = n)}{\phi_n} + r_{n-1} \frac{E_0(R_1(0+)| l(0+) = n)}{\phi_n} \right]. \tag{4.3}$$

In the derivation, the L'Hôspital theorem is used. Equations (4.3) give a relation between $\{p_n\}$ and $\{r_n\}$.

Let $LD(N-1)$ be a set of service disciplines under which a customer finding n customers in the queue upon his arrival instant moves into position 1 when $n \leq N-1$.

Let B_n and $E(S_n)$ be the distribution function and expectation of service requirements for customers who find n customers in the queue upon their arrival instants, respectively. Then we have the following.

PROPOSITION 4.1. *For the queue with a service discipline belonging to $LD(N-1)$, if,*

$$E_0(R_1(0-)| l(0-) = n) = E_1(R_1(0+)| l(0+) = n) , \tag{4.4}$$

then,

$$p_n = \frac{\lambda E(S_{n-1})}{\phi_n} r_{r-1} \quad \textit{for} \quad n = 1, 2, \ldots, N . \tag{4.5}$$

Furthermore, if,

$$P_0\{R_1(0-) > x \mid l(0-) = n\} = P_1\{R_1(0+) > x \mid l(0+) = n\} , \tag{4.6}$$

then,

$$P\{R_1(0) > x \mid l(0) = n\} = B^*_{n-1}(x) , \tag{4.7}$$

where

$$B^*_{n-1}(x) = 1 - \frac{1}{E(S_{n-1})} \int_0^x \{1 - B_{n-1}(y)\} dy .$$

PROOF. Because of the discipline, we have

$$E_0\{R_1(0+) \mid l(0+) = n\} = E(S_{n-1}) \quad \text{for} \quad n = 1, 2, \ldots, N .$$

By substituting this and (4.4) into (4.3), we have (4.5). Equations (4.7) can be obtained by substituting (4.5) into (4.2), dividing it by $\phi_n \theta_1$ and using (4.6).

Remark 4.1. Consider a queue operating under (i) and (ii) in Section 2 and the following (iii′) and (iv).

(iii′) If n, which is the number of customers in the queue, is less than N, the reordering policy is on LCFS basis (cf. Section 2). If $n = N$, an arriving customer moves to position $(N+1)$ upon his arrival instant; customers in positions $1 - N$ remain in the same positions, and when the customer in position 1 leaves the queue customers in positions $2 - N$ move to positions $1 - (N-1)$, respectively.

(iv) The restarting policy is preemptive-resume.

We refer to this queue as an N-loss queue with PR-LCFS. When $\phi_n = 1$ $(n = 1, 2, \ldots, N)$, the N-loss queue with PR-LCFS is denoted by G/G/1 (N; PR-LCFS). If the inter-arrival times or service requirements of customers are i.i.d. r.v.'s, we correspondingly use "GI" instead of G. Furthermore, we allow N to be infinite and use the notation "∞." Because of (iii′) and (iv), in the N-loss queue each customer leaves the queue in the same state that he finds it upon his arrival instant. Hence, this queue is a typical one for which (4.5) and (4.7) hold. A G/GI/1/1 is equivalent to G/GI/1 (1; PR-LCFS).

Remark 4.2. Consider a queue in Section 2 in which a customer finding n customers in the queue upon his arrival instant joins the queue with probability α_n and immediately leaves the queue with probability $1-\alpha_n$. When customers who immediately leave owing to this are not counted as both arrivals and departures, (3.2) remains true if λ in (3.2) is replaced by the effective arrival rate "λ^*." Therefore, Proposition 4.1 for the queue with a service discipline belonging to $LD(N-1)$ remains true if λ in the proposition is replaced by λ^*. Shanthikumar and Sumita (1986) proved (4.5) for G/GI/1 (∞; PR-LCFS) with the above lost customers in which the service requirements of customers are independent of the arrival process.

Remark 4.3. Fakinos (1987) proved (4.5) for GI/G/1 (∞; PR-LCFS) in which the service requirements of customers finding n customers in the queue are indentically distributed r.v.'s with d.f. B_n and the successive service requirements are stochastically independent of each other and of the arrival process.

Remark 4.4. Consider a queue in Section 2 in which the service requirements of customers are i.i.d. r.v.'s with an exponential d.f. and they are independent of the arrival process. Equations (4.4) and (4.6) hold for the queue under any reordering policy. Therefore, (4.5) and (4.7) hold for the queue, not depending on the reordering policy.

5. Loss queue with PR-LCFS

Consider GI/GI/1 (j; PR-LCFS) queue (cf. Remark 4.1), $j=1,2,\ldots,N$, in which the service requirements of customers are independent of the arrival process. Let $A(\cdot)$ and $B(\cdot)$ be d.f.'s of the inter-arrival times and service requirements, respectively. It is assumed that the expectations of $A(\cdot)$ and $B(\cdot)$ are finite, and that $A(0)=B(0)=0$. Throughout the paper the Laplace-Stieltjes transform (LST) and tail distribution of a given d.f. are distinguished by adding these marks ($\sim$) and ($-$) over the same letter: $\tilde{A}(\cdot)$ is the LST of $A(\cdot)$ and $\bar{A}(\cdot)=1-A(\cdot)$. This is the reason for using ($\sim$) in equations (3.2). We will use j for notations to specify "j" in this j-loss queue whenever it is needed: $p_n^j=P(l(0)=n$ in j-loss queue).

Let $\mathscr{A}$ ($\mathscr{D}$) be the set of arrival (departure) instants in the GI/GI/1 (j; PR-LCFS) queue. At instant $e\in\mathscr{E}=\mathscr{A}\cup\mathscr{D}$, let $K_e=+1$ or -1 according as $e\in\mathscr{A}$ or $e\in\mathscr{D}$. Suppose that the process $X(t;j)$ defined in Section 3 is observed exclusively at instants $e\in\mathscr{E}$: the process is observed just before or after instants e according to whether $K_e=+1$ or -1. We count lost customers as departures. The process $X_e^j=(K_e,l,R_1,R_2,\ldots,R_l,U)$ observed at successive instants $e\in\mathscr{E}$ is a Markov process.

We assume that, for each j, the process X_e^j has a stationary distribution, i.e., there exists the distribution of X_e^j which satisfies the following equations.

$$(5.1)\qquad \Pr(K_e = +1, l = 0) = \int_0^\infty \Pr(K_e = -1, l = 0, U \in dy),$$

$$(5.2)\qquad \Pr(K_e = +1, l = n, R_1 > x_1, R_2 > x_2, \dots, R_n > x_n)$$
$$= \int_0^\infty \Pr(K_e = +1, l = n-1, R_1 > x_2,$$
$$R_2 > x_3, \dots, R_{n-1} > x_n)\bar{B}(x_1 + y)dA(y)$$
$$+ \int_0^\infty \Pr(K_e = -1, l = n, R_1 > x_1 + y,$$
$$R_2 > x_2, \dots, R_n > x_n, U \in dy)$$

for $n = 1, 2, \dots, j$,

$$(5.3)\qquad \Pr(K_e = -1, l = n, R_1 > x_1, R_2 > x_2, \dots, R_n > x_n, U > y)$$
$$= \int_0^\infty \Pr(K_e = +1, l = n, R_1 > x_1,$$
$$R_2 > x_2, \dots, R_n > x_n)\bar{A}(y + z)dB(z)$$
$$+ \int_0^\infty \Pr(K_e = -1, l = n+1, R_1 \in dz, R_2 > x_1,$$
$$R_3 > x_2, \dots, R_{n-1} > x_n, U > y + z)$$

for $n = 0, 1, \dots, j-1$,

$$(5.4)\qquad \Pr(K_e = -1, l = j, R_1 > x_1, R_2 > x_2, \dots, R_j > x_j, U > y)$$
$$= \Pr(K_e = +1, l = j, R_1 > x_1, R_2 > x_2, \dots, R_j > x_j)\bar{A}(y).$$

To obtain (5.1) note that the event $\{K_e = +1, l = 0\}$ will occur at an instant if and only if at the preceding instant the event $\{K_e = -1, l = 0\}$ occurred. Equation (5.2) is obtained from the consideration that the event $\{K_e = +1, l = n\}$ can occur at an instant only if at the preceding instant either of the events $\{K_e = +1, l = n-1\}$ or $\{K_e = -1, l = n\}$ occurred. Equation (5.3) reflects the fact that the event $\{K_e = -1, l = n\}$ must be preceded by one of the events $\{K_e = +1, l = n\}$ or $\{K_e = -1, l = n+1\}$. Equation (5.4) follows the fact that for $l = j$ an arriving customer leaves the queue immediately. The stationary distribution is denoted by

$$
\begin{aligned}
&\Pr(K_e = +1, l = n, R_1 > x_1, R_2 > x_2, \dots, R_n > x_n) \\
&\qquad = \Pr(K_e = +1, l = n)\bar{F}_n^j(x_1, x_2, \dots, x_n), \\
&\Pr(K_e = -1, l = n, R_1 > x_1, R_2 > x_2, \dots, R_n > x_n, U > y) \\
&\qquad = \Pr(K_e = -1, l = n)\bar{G}_n^j(x_1, x_2, \dots, x_n, y)
\end{aligned} \tag{5.5}
$$

for $n = 0, 1, \dots, j$, where

$$
\begin{aligned}
&\bar{F}_n^j(x_1, x_2, \dots, x_n) = \Pr(R_1 > x_1, R_2 > x_2, \dots, R_n > x_n | K_e = +1, l = n), \\
&\bar{G}_n^j(x_1, x_2, \dots, x_n, y) \\
&\qquad = \Pr(R_1 > x_1, R_2 > x_2, \dots, R_n > x_n, U > y | K_e = -1, l = n).
\end{aligned}
$$

Since the lost customers for the j-loss queue are counted as departures, it is clear that $\Pr(K_e = +1) = \Pr(K_e = -1) = 1/2$. Therefore, using (3.3) we have

$$
\begin{aligned}
\Pr(K_e = \pm 1, l = n) &= \Pr(l = n | K_e = \pm 1)\Pr(K_e = \pm 1) \\
&= \frac{1}{2} r_n^j .
\end{aligned} \tag{5.6}
$$

THEOREM 5.1. *For a $GI/GI/1$ (N; PR-LCFS) queue,*

$$
\bar{F}_n^N(x_1, x_2, \dots, x_n) = \prod_{j=N-n+1}^{N} \bar{F}_1^j(x_{n+j-N}), \tag{5.7}
$$

$$
\bar{G}_n^N(x_1, x_2, \dots, x_n, y) = \bar{G}_0^{N-n}(y) \prod_{j=N-n+1}^{N} \bar{F}^j(x_{n+j-N}), \tag{5.8}
$$

$$
r_n^N = r_0^N \prod_{j=N-n+1}^{N} \beta_1^j , \tag{5.9}
$$

where $\bar{G}_0^0(y) = \bar{A}(y)$ and $\beta_1^j = r_1^j / r_0^j$.

PROOF. We start with deriving a relation among $\bar{F}_1^j(\cdot)$, $\bar{G}_0^j(\cdot)$ and $\bar{G}_1^j(\cdot)$. R_1^j and U^j under the condition that a customer, C_1^j, leaves 1 customer in the j-loss queue are independent of each other because U^j depends only on the arrival instants and service requirements of customers who arrive between the arrival and departure instants of C_1^j. Hence we have

$$
\begin{aligned}
\bar{G}_1^j(x, y) = \Pr(R_1^j > x | K_e^j = -1, l^j = 1) \\
\cdot \Pr(U^j > y | K_e^j = -1, l^j = 1).
\end{aligned} \tag{5.10}
$$

Because of the particular service discipline, each customer leaves the queue in the same state that he finds it upon his arrival instant. From this fact we can find that

$$\Pr(R_1^j > x \mid K_e^j = -1, l^j = 1) = \bar{F}_1^j(x). \tag{5.11}$$

For a $(j-1)$-loss queue, consider U^{j-1} under the condition that a customer, C_0^{j-1}, leaves the queue empty. This U^{j-1} depends only on the arrival instants and service requirements of customers who arrive between the arrival and departure instants of C_0^{j-1}. The stochastic behavior in the time interval between the arrival and departure instants of C_0^{j-1} is the same as that between the arrival and departure instants of C_1^j because of the assumption that the inter-arrival times and service requirements of customers are i.i.d. r.v.'s with d.f.'s $A(\cdot)$ and $B(\cdot)$, respectively, for both queues. This implies that

$$\bar{G}_0^{j-1}(y) = \Pr(U^j > y \mid K_e^j = -1, l^j = 1). \tag{5.12}$$

Combining (5.10), (5.11) and (5.12) yields

$$\bar{G}_1^j(x,y) = \bar{F}_1^j(x)\bar{G}_0^{j-1}(y) \qquad \text{for} \quad j = 1,2,\dots,N. \tag{5.13}$$

Using (5.6), (5.13) and $\beta_1^j = r_1^j/r_0^j$, (5.2) for $n=1$ and (5.3) for $n=0$ can be rewritten as

$$\begin{aligned} \beta_1^j \bar{F}_1^j(x) &= \int_0^\infty \bar{B}(x+y)\,dA(y) + \beta_1^j \int_0^\infty \bar{F}(x+y)\,dG_0^{j-1}(y), \\ \bar{G}_0^j(y) &= \int_0^\infty \bar{A}(y+z)\,dB(z) + \beta_1^j \int_0^\infty \bar{G}_0^{j-1}(y+z)\,dF_1^j(z). \end{aligned} \tag{5.14}$$

Suppose that $\bar{F}_n^N(\cdot)$ and $\bar{G}_n^N(\cdot)$ in (5.5) for $j = N$ are (5.7) and (5.8), respectively, i.e., the stationary distribution for X_e^N is

$$\begin{aligned} &\Pr(K_e = +1, l = n, R_1 > x_1, R_2 > x_2, \dots, R_n > x_n) \\ &\quad = \frac{1}{2} r_n^N \prod_{j=N-n+1}^{N} \bar{F}_1^j(x_{n+j-N}), \\ &\Pr(K_e = -1, l = n, R_1 > x_1, R_2 > x_2, \dots, R_n > x_n, U > y) \\ &\quad = \frac{1}{2} r_n^N \bar{G}_0^{N-n}(y) \prod_{j=N-n+1}^{N} \bar{F}_1^j(x_{n+j-N}). \end{aligned} \tag{5.15}$$

To prove Theorem 5.1, then, it is sufficient to show that (5.15) satisfies (5.2) for $n = 2,3,\dots,N$, (5.3) for $n = 1,2,\dots,N-1$ and (5.4). This can be checked by using (5.14).

Remark 5.1. The process X_e^N observed at successive instants $e \in \mathscr{A}$ ($e \in \mathscr{D}$), X_a^N (X_d^N), is an aperiodic Markov process. A stationary distribution for X_a^N (X_d^N) is identical with the distribution concerning P_0 (P_1) introduced in Section 3. The stationary distributions for X_a^N and X_d^N can be obtained from Theorem 5.1 (cf. Kelly (1976)) and hence we have that, for the N-loss queue,

$$\begin{aligned}
&P_0(l(0-)=n,\ R_1(0-)>x_1,\ R_2(0-)>x_2,\dots,R_n(0-)>x_n)\\
&\quad = r_n^N \prod_{j=N-n+1}^{N} \bar{F}_1^j(x_{n+j-N}),\\
(5.16)\qquad &P_1(l(0+)=n,\ R_1(0+)>x_1,\\
&\quad R_2(0+)>x_2,\dots,R_n(0+)>x_n,\ U(0+)>y)\\
&\quad = r_n^N \bar{G}_0^{N-n}(y) \prod_{j=N-n+1}^{N} \bar{F}_1^j(x_{n+j-N}).
\end{aligned}$$

Remark 5.2. For an ∞-loss queue with PR-LCFS, i.e., a single server queue with infinite waiting space and PR-LCFS service discipline, a similar result stronger than Theorem 5.1 has been obtained. For example, Fakinos (1981) and Yamazaki (1982) showed that (r_n) $n = 0, 1,\dots$ in a GI/GI/1 (∞; PR-LCFS) queue is a geometric distribution, and that the remaining service requirements of customers just before an arrival instant are i.i.d. r.v.'s. Shanthikumar and Sumita (1986) showed that (r_n) $n = 0, 1,\dots$ in a G/GI/1 (∞; PR-LCFS) queue is a geometric distribution.

A relation among p_n^j ($j = 1, 2,\dots, N$) can be obtained by combining (5.9) with Remark 4.1. An expression of LST of $P(R_1(0) > x_1,\ R_2(0) > x_2,\dots, R_n(0) > x_n,\ U(0) > y \mid l(0) = n)$, $\tilde{H}_n^N(\theta_1, \theta_2,\dots, \theta_n; \xi)$, for the GI/GI/1 ($N$; PR-LCFS) queue can be determined by using (3.2) and (5.16) and it becomes

$$\begin{aligned}
(5.17)\qquad &\tilde{H}_n^N(\theta_1,\theta_2,\dots,\theta_n;\xi)\\
&= \frac{\prod_{j=N-n+2}^{N} \tilde{F}_1^j(\theta_{n+j-N})}{E(S)(\xi+\theta_1)}\\
&\quad \cdot [\beta_1^{N-n+1}\tilde{F}_1^{N-n+1}(\theta_1)(1-\tilde{G}_0^{N-n}(\xi)) + \tilde{G}_0^{N-n+1}(\xi) - \tilde{B}(\theta_1)\tilde{A}(\xi)]
\end{aligned}$$

for $n = 1, 2,\dots, N$, where S is a generic service requirement.

6. Push-out queue

A queue having N positions and operating under (i) and (ii) in Section 2, (iv) in Section 4 and LCFS service discipline (cf. Section 2) is called an

"N-push-out queue". Note that the N-push-out queue and N-loss queue are the same except for who leaves the queue when an arriving customer finds N customers in the queue; i.e., the customer in position N leaves at the arrival instant in the N-push-out queue, whereas the arriving customer leaves immediately upon his arrival instant in the N-loss queue.

In this section we only consider an N-push-out queue with $\phi_n = 1$ under the same assumptions as those in Section 5. We refer to this queue as GI/GI/1 (N-push-out). We also use the same notations as those in Section 5 for the GI/GI/1 (N-push-out) queue.

Consider the process X_e^j for GI/GI/1 (j-push-out) queue, $j = 1, 2, \ldots, N$. Then, for each j the stationary distribution for X_e^j satisfies (5.1), (5.2), (5.3) and the following equation.

$$\begin{aligned}(6.1)\qquad &\Pr(K_e = -1, l = j, R_1 > x_1, R_2 > x_2, \ldots, R_j > x_j, U > y) \\ &= \Pr(K_e = +1, l = j, R_1 > x_2, \\ &\qquad R_2 > x_3, \ldots, R_{j-1} > x_j, R_j > 0)\bar{B}(x_1)\bar{A}(y) .\end{aligned}$$

Equation (6.1) follows the fact that when an arriving customer finds j customers in the queue, he enters into position 1 and the customer previously in position j leaves the queue.

THEOREM 6.1. *For a GI/GI/1 (N-push-out) queue,*

$$(6.2)\qquad \bar{F}_n^N(x_1, x_2, \ldots, x_{n-1}, 0) = \bar{F}_{n-1}^{N-1}(x_1, x_2, \ldots, x_{n-1}) ,$$

$$(6.3)\qquad \bar{G}_n^N(x_1, x_2, \ldots, x_{n-1}, 0, y) = \bar{G}_{n-1}^{N-1}(x_1, x_2, \ldots, x_{n-1}, y)$$

for $n = 2, 3, \ldots, N$,

$$(6.4)\qquad r_n^N = r_0^N \prod_{j=N-n+1}^{N} \beta_1^j .$$

PROOF. For a GI/GI/1 (j-push-out) queue, let

D_n^j = the time interval that begins when an arrival finds n customers in the queue and ends when, for the first time after that, a departure leaves n customers .

Because of the particular service discipline and the assumptions of the inter-arrival times and service requirements of customers, the stochastic behavior in D_1^j is the same as that in D_0^{j-1}. From this, (6.2) and (6.3) follow.

Equation (5.2) can be rewritten as

(6.5) $$\beta_n^j \bar{F}_n^j(x_1, x_2, \ldots, x_n)$$
$$= \bar{F}_{n-1}^j(x_2, x_3, \ldots, x_n)\int_0^\infty \bar{B}(x_1 + y)dA(y)$$
$$+ \beta_n^j \int_0^\infty \Pr(R_1^j > x_1 + y, R_2^j > x_2, \ldots, R_n^j > x_n,$$
$$U^j \in dy | K_e^j = -1, l^j = n)$$

for $n = 1, 2, \ldots, j$, where $\beta_n^j = r_n^j / r_{n-1}^j$. When $n \geq 2$, by putting $x_n = 0$ in (6.5) and using (6.2) and (6.3) we can obtain

(6.6) $$\beta_n^j \bar{F}_{n-1}^{j-1}(x_1, x_2, \ldots, x_{n-1})$$
$$= \bar{F}_{n-2}^{j-1}(x_2, x_3, \ldots, x_{n-2})\int_0^\infty \bar{B}(x_1 + y)dA(y)$$
$$+ \beta_n^j \int_0^\infty \Pr(R_1^{j-1} > x_1 + y, R_2^{j-1} > x_2, \ldots, R_{n-1}^{j-1} > x_{n-1},$$
$$U^{j-1} \in dy | K_e^{j-1} = -1, l^{j-1} = n - 1).$$

Similarly, using (6.2) and (6.3) in (5.3) and (6.1) we find that

(6.7) $$\bar{G}_{n-2}^{j-1}(x_1, x_2, \ldots, x_{n-2}, y)$$
$$= \bar{F}_{n-2}^{j-1}(x_1, x_2, \ldots, x_{n-2})\int_0^\infty \bar{A}(y + z)dB(z)$$
$$+ \beta_n^j \int_0^\infty \Pr(R_1^{j-1} \in dz, R_2^{j-1} > x_1, \ldots, R_{n-1}^{j-1} > x_{n-2},$$
$$U^{j-1} > y | K_e^{j-1} = -1, l^{j-1} = n - 1)$$

for $n = 2, 3, \ldots, j$,

(6.8) $$\bar{G}_{j-1}^{j-1}(x_1, x_2, \ldots, x_{j-1}, y) = \bar{F}_{j-1}^{j-1}(x_2, x_3, \ldots, x_{j-1})\bar{B}(x_1)\bar{A}(y).$$

Noting that (6.6), (6.7) and (6.8) correspond to (5.2), (5.3) and (6.1), respectively, for X_e^{j-1}, we find that

(6.9) $$\beta_n^j = \beta_{n-1}^{j-1} \quad \text{for} \quad n = 2, 3, \ldots, j.$$

From (6.9), (6.4) follows.

From Theorem 6.1 we can obtain expressions of the distributions of $X^N(t)$ concerning P_0 and P_1 (cf. Remark 5.1).

Much work has been performed with service disciplines in relation to insensitivity (with respect to queue length distributions) in queues, under the assumption that the arrival process to the queues is a Poisson one. One of the common properties resulting from the service disciplines is that the remaining service requirements of customers in the queue at any time or an arrival instant are i.i.d. r.v.'s. The loss queue in Section 5 has a similar property weaker than this (cf. (5.16) and (5.17)). On the other hand, for the push-out queue, the independence among remaining service requirements of customers such as (5.16) and the equation (4.5) no longer hold. This can be checked by using an $M/E_2/1$ (3-push-out) queue. It should be noted, however, that an expression of r_n^N in the N-push-out queue is the same as that in an N-loss queue (cf. (5.9) and (6.4)).

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