Ann. Inst. Statist. Math.
Vol. 42, No. 3, 445–461 (1990)

ON THE STEREOLOGICAL ESTIMATION OF REDUCED MOMENT MEASURES

E. B. JENSEN[1], K. KIÊU[2] AND H. J. G. GUNDERSEN[3]

[1]*Institute of Mathematics, Aarhus University, Ny Munkegade, DK-8000 Aarhus C, Denmark*
[2]*Institut National de la Recherche Agronomique, Biométrie, Domain Saint-Paul, 84140 Montfavet, France*
[3]*Stereological Research Laboratory, Aarhus University, DK-8000 Aarhus C, Denmark*

(Received November 15, 1988; revised October 17, 1989)

Abstract. Recently, a new set of fundamental stereological formulae based on isotropically oriented probes through fixed points have been derived, the so-called "nucleator" estimation principle (cf. Jensen and Gundersen (1989, *J. Microsc.*, **153**, 249–267)). In the present paper, it is shown how a model-based version of these formulae leads to stereological estimators of reduced moment measures of stationary and isotropic random sets in $\mathbb{R}^n$.

Key words and phrases: Boolean model, Hausdorff measures, marked point processes, measure decomposition, moment measures, stereology, stochastic geometry.

1. Introduction

One of the recent trends in stereology has been towards the creation of more sophisticated sampling designs such as vertical sections (cf. Baddeley *et al.* (1986)) and the nucleator (cf. Gundersen (1988)). The latter device uses isotropic lines and planes through a restricted subset of space, which may reduce to a single fixed point. The theoretical foundation of the nucleator, which has been given in Jensen and Gundersen (1989), is based on a generalized version of the Blaschke-Petkantschin formula.

The East German School of stochastic geometry has developed stereological estimators of second-order properties of random spatial structures in the last decade (cf. e.g., Hanisch and Stoyan (1981), Stoyan (1981, 1984, 1985*a*, 1985*b*), Stoyan and Ohser (1982, 1985), Hanisch (1985), Schwandtke (1988)). Reading papers on stereological estimation of moment measures by this school, in particular the paper by Stoyan (1981) dealing with planar fibre processes, it became apparent to us that the theory presented in Jensen and Gundersen (1989) could also be used in the development of stereological estimators of reduced moment measures of random d-dimensional sets in $\mathbb{R}^n$, $d \leq n$. The critical observation was that the reduced moment measures can be expressed as moments with respect to a random set which is isotropic but not stationary. The geometric measure decomposition given in Jensen and Gundersen (1989) is just the one needed for treating such random sets.

In this paper, we try to give parts of a general theory of stereological estimation of reduced moment measures, generalizing earlier work by the East German School. The results were presented at the 18th European Meeting of Statisticians, East Berlin, August 1988. At that meeting, it became apparent that similar results had been obtained independently by Zähle (1990). Estimation of non-reduced moment measures has also been discussed in Nagel (1987).

In the present paper, we mainly discuss stereological estimation of reduced moment measures defined for general random sets under stationarity and isotropicity assumptions. In Section 2, we give a formal definition of a random d-dimensional set and its reduced moment measures. The stereological estimator of the measures is presented in Section 3. In Section 4, a modification of the reduced second moment measure is presented. The modified measure is the Lebesgue measure for a random set described by a Boolean model, and represents a generalization of the K-function known from point processes ($d = 0$) to general d-dimensional random sets. Stereological estimation of this modified reduced second moment measure is also discussed in Section 4.

2. Random (d, n)-sets and their moment measures

Let λ_n^d be the d-dimensional Hausdorff measure in $\mathbb{R}^n$, $d = 0, 1, \ldots, n$. In particular, $\lambda_n = \lambda_n^n$ is the Lebesgue measure in $\mathbb{R}^n$ and λ_n^0 is the counting measure in $\mathbb{R}^n$. A formal definition of the Hausdorff measure can be found in Zähle (1982), which also contains many of the geometric measure theoretic definitions referred to below. General references on geometric measure theory are Federer (1969) and Simon (1983).

Let $\mathscr{B}^n$ be the Borel σ-algebra in $\mathbb{R}^n$ and let N_d be the set of subsets $\phi \subseteq \mathbb{R}^n$ which are closed and λ_n^d-rectifiable. A set $\phi \subseteq \mathbb{R}^n$ is called λ_n^d-rectifiable if $\lambda_n^d(\phi \cap B) < \infty$ for any bounded Borel set B and if there exists a sequence $\{\phi_j\}_{j=1}^{\infty}$ of d-rectifiable subsets of $\mathbb{R}^n$ such that $\lambda_n^d\left(\phi \setminus \bigcup_{j=1}^{\infty} \phi_j\right) = 0$. (A set $\psi \subseteq \mathbb{R}^n$ is d-rectifiable if there exists a Lipschitzian function which maps some bounded subset of $\mathbb{R}^d$ onto ψ.) The closed λ_n^d-rectifiable sets are a subclass of the class $\mathscr{F}$ of all closed subsets of $\mathbb{R}^n$, which can be equipped with Matheron's σ-algebra $\mathscr{T}$. In Zähle ((1982), Theorem 2.2.1), it is shown that $N_d \in \mathscr{T}$. Let $\mathscr{N}_d$ be the restriction of $\mathscr{T}$ to N_d, i.e. $\mathscr{N}_d = N_d \cap \mathscr{T}$. The functions from N_d to $\mathbb{R}$ defined by $\phi \to \lambda_n^d(\phi \cap C)$, $C \in \mathscr{B}^n$, are $(\mathscr{N}_d$,

$\mathscr{B}(\mathbb{R} \cup \{+\infty\})$)-measurable (cf. Zähle (1982), Corollary 2.1.4).

DEFINITION 2.1. A random (d,n)-set is a random variable Φ with range $(N_d, \mathscr{N}_d)$, i.e. a measurable mapping from a basic probability space $(\Omega, \mathscr{A}, \mathbb{P})$ into $(N_d, \mathscr{N}_d)$.

A random (d,n)-set Φ is called stationary respectively isotropic, if its distribution is invariant under translations respectively rotations in $\mathbb{R}^n$.

A random (d,n)-set gives rise to a random measure on $(\mathbb{R}^n, \mathscr{B}^n)$

$$\mu_\Phi(B) = \lambda_n^d(\Phi \cap B), \qquad B \in \mathscr{B}^n . \tag{2.1}$$

The associated intensity measure of Φ is

$$\Lambda_\Phi(B) = E\mu_\Phi(B) = E\lambda_n^d(\Phi \cap B), \qquad B \in \mathscr{B}^n . \tag{2.2}$$

Observe that if Φ is stationary, then

$$\Lambda_\Phi(B) = \nu_n^d \lambda_n(B) , \tag{2.3}$$

where ν_n^d is called the intensity of Φ. In what follows, we assume that $0 < \nu_n^d < \infty$.

The intensity measure represents the first-order properties of the set. Higher-order properties can be studied by means of moment measures of order 2 or more. Let $\mu_\Phi^{(k)}$ be the k-fold product of μ_Φ with itself. Then, the k-th moment measure is defined as

$$\Lambda_\Phi^{(k)}(B) = E\mu_\Phi^{(k)}(B), \qquad B \in \mathscr{B}^{k \cdot n}, \quad k = 1, 2, \dots . \tag{2.4}$$

In particular,

$$\Lambda_\Phi^{(k)}(B_1 \times \cdots \times B_k) = E \prod_{i=1}^k \lambda_n^d(\Phi \cap B_i), \qquad B_i \in \mathscr{B}^n, \quad i = 1, \dots, k . \tag{2.5}$$

The moment measure of first order is simply the intensity measure.

If Φ is stationary, then Λ_Φ is completely described by ν_n^d. More generally, it is possible, still under the stationarity assumption, to describe the k-th moment measure by the so-called reduced k-th moment measure. To make this precise, we need to define the Palm distribution P_0 of the random (d,n)-set.

Heuristically, $P_0(A)$, $A \in \mathscr{N}_d$, can be interpreted as the probability that $\Phi \in A$ when the origin of $\mathbb{R}^n$ is chosen as a typical point of Φ. There are a number of equivalent ways of defining the Palm distribution. Probably the most direct one is the following. We let $B \in \mathscr{B}^n$ be a Borel set with $0 < \lambda_n(B) < \infty$. Then, (cf. Mecke (1967), (2.8))

$$(2.6) \qquad P_0(A) = \frac{1}{v_n^d \lambda_n(B)} \int_{N_d} \int_{B \cap \phi} I_A(\phi - x) \lambda_n^d(dx) P(d\phi)$$

$$= \frac{1}{v_n^d \lambda_n(B)} E \int_{B \cap \Phi} I_A(\Phi - x) \lambda_n^d(dx), \qquad A \in \mathscr{N}_d .$$

Because of the stationarity assumption, this definition of P_0 does not depend on $B \in \mathscr{B}^n$. Using the "standard" measure theoretic proof, it follows directly from the definition of the Palm distribution that

$$(2.7) \qquad E \int_{\Phi} h(x, \Phi) \lambda_n^d(dx) = v_n^d \int_{\mathbb{R}^n} \int_{N_d} h(x, \phi + x) P_0(d\phi) \lambda_n(dx)$$

for any $\mathscr{B}^n \times \mathscr{N}_d$-measurable non-negative function. This result is usually called the refined Campbell theorem. (The standard Campbell theorem concerns functions of the simpler type $h(x, \Phi) = f(x)$.) The distribution P_0 is not stationary. However, it is easy to prove that if the original process is isotropic then P_0 is isotropic too. Below, we let Φ_0 be a random (d, n)-set with distribution P_0 and E_0 is the mean operator with respect to P_0.

The reduced k-th moment measure is defined as the following measure on $(\mathbb{R}^{n(k-1)}, \mathscr{B}^{n(k-1)})$:

$$(2.8) \qquad \mathscr{K}_{\Phi}^{(k)}(B) = \Lambda_{\Phi_0}^{(k-1)}(B) / (v_n^d)^{k-1}, \qquad B \in \mathscr{B}^{n(k-1)} .$$

Using the refined Campbell theorem we can express $\Lambda_{\Phi}^{(k)}$ in terms of $\mathscr{K}_{\Phi}^{(k)}$: For $B_i \in \mathscr{B}^n$, $i = 1, \ldots, k$,

$$(2.9) \qquad \Lambda_{\Phi}^{(k)}(B_1 \times \cdots \times B_k) = v_n^d \int_{B_1} \int_{N_d} \lambda_n^d(\phi \cap (B_2 - x)) \cdots \lambda_n^d(\phi \cap (B_k - x)) \cdot P_0(d\phi) \lambda_n(dx)$$

$$= v_n^d \int_{B_1} \Lambda_{\Phi_0}^{(k-1)}(B_2 - x \times \cdots \times B_k - x) \lambda_n(dx)$$

$$= v_n^d \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n(k-1)}} I_{B_1 \times \cdots \times B_k}(x, x + h_1, \ldots, x + h_{k-1}) \cdot \Lambda_{\Phi_0}^{(k-1)}(dh_1, \ldots, dh_{k-1}) \lambda_n(dx) .$$

More generally, for $B \in \mathscr{B}^{nk}$,

$$(2.10) \qquad \Lambda_{\Phi}^{(k)}(B) = v_n^d \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n(k-1)}} I_B(x, x + h_1, \ldots, x + h_{k-1}) \cdot \Lambda_{\Phi_0}^{(k-1)}(dh_1, \ldots, dh_{k-1}) \lambda_n(dx)$$

$$= (v_n^d)^k \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n(k-1)}} I_B(x, x+h_1, \dots, x+h_{k-1})$$
$$\cdot \mathscr{K}_\Phi^{(k)}(dh_1, \dots, dh_{k-1}) \lambda_n(dx) ,$$

which is the announced relation between moment measures and reduced moment measures.

For $d = n$, the reduced second moment measure can be expressed in terms of the covariance C of the process which is defined by

$$C(x) = \mathbb{P}(0 \in \Phi, x \in \Phi), \qquad x \in \mathbb{R}^n . \tag{2.11}$$

It is thus easy to prove, using the definition (2.6) of the Palm distribution that

$$v_n^n \mathscr{K}_\Phi^{(2)}(B) = E_0 \lambda_n(\Phi \cap B) = \frac{1}{v_n^n} \int_B C(x) \lambda_n(dx), \qquad B \in \mathscr{B}^n . \tag{2.12}$$

The reduced second moment measure is thereby for $d = n$ a cumulative version of the covariance.

3. Stereological estimation of reduced moment measures

Below, we present a stereological estimator of $(v_n^d)^q \mathscr{K}_\Phi^{(q+1)}(B) = \Lambda_{\Phi_0}^{(q)}(B)$ for sets $B \in \mathscr{B}^{nq}$ of the form

$$B = B_1 \times \cdots \times B_q \tag{3.1}$$

where

$$B_i = \{x \in \mathbb{R}^n : r^{(i)} \leq \|x\| \leq R^{(i)}\} , \tag{3.2}$$

$i = 1, \dots, q$. For $n = 2$, B_i is an annulus. The estimators are based on measurements on or in the neighbourhood of a q-dimensional subspace of $\mathbb{R}^n$. In case $q = 1$, knowledge of $\mathscr{K}_\Phi^{(q+1)}$ on sets of the form (3.1) completely determines $\mathscr{K}_\Phi^{(q+1)}$ under the isotropy assumption of Φ. For $q > 1$, this is no longer true. Reduced third moment measures are discussed in Hanisch (1983).

The estimator is based on a generalized version of the Blaschke-Petkantschin formula which has been described in Jensen and Gundersen (1989). This formula gives a geometric measure decomposition which is just the one needed for our estimation problem. In order to present this formula, we need some extra notation and notions.

Let $L_{q(0)}$ denote a q-subspace of $\mathbb{R}^n$, i.e. a q-dimensional linear subspace of $\mathbb{R}^n$ and let $\mathscr{L}_{q(0)}$ be the set of all q-subspaces. Let $dL_{q(0)}$ be the (differential) element of the measure on $\mathscr{L}_{q(0)}$ which is invariant under rotations in $\mathbb{R}^n$ about the origin. This measure is unique up to multiplication with a positive constant and is here scaled such that the total integral is

$$\int_{\mathscr{L}_{q(0)}} dL_{q(0)} = \sigma_n \cdots \sigma_{n-q+1} / \sigma_1 \cdots \sigma_q \tag{3.3}$$
$$= c(n,q) \, ,$$

say, where

$$\sigma_n = 2\pi^{n/2}/\Gamma(n/2) \, , \tag{3.4}$$
$$\sigma_1 = 2, \quad \sigma_2 = 2\pi, \quad \sigma_3 = 4\pi, \quad \sigma_4 = 2\pi^2 \, ,$$

is the surface area of the unit ball in $\mathbb{R}^n$.

We can normalize the rotation invariant measure on $\mathscr{L}_{q(0)}$ to a probability measure. We let $\overline{dL_{q(0)}}$ be the element of this probability measure. Let μ_n be the unique invariant probability measure on the group $SO(n)$ of rotations in $\mathbb{R}^n$. Then, it is easy to show that if $L_{q(00)}$ is an arbitrary fixed q-subspace and T is the mapping

$$A \in SO(n) \to AL_{q(00)} \in \mathscr{L}_{q(0)} \tag{3.5}$$

then $\mu_n \cdot T$ is *the* rotation invariant probability measure on $\mathscr{L}_{q(0)}$.

The generalization of the Blaschke-Petkantschin formula concerns a geometric measure decomposition of the q-fold product measure of λ_n^d with itself, $d = 1,\dots,n$. As we shall see, the decomposition is only meaningful, if $q = n - d,\dots,n$. Let $x_1^d,\dots,x_q^d$ be q points on a λ_n^d-rectifiable closed set $\phi \in N_d$ in $\mathbb{R}^n$. The superscript of x_i^d is used to indicate the dimension of the space within which the point is regarded to lie. For brevity, we write, when convenient, dx_i^d instead of $\lambda_n^d(dx_i^d)$ in what follows. We need to assume that $x_1^d,\dots,x_q^d$ are linearly independent such that these q points determine a unique q-subspace $L_{q(0)}$. Let $x_1^{d-n+q},\dots,x_q^{d-n+q}$ represent the position of the points within $\phi \cap L_{q(0)}$. Then,

$$J_n^{d;q}(x_1^d,\dots,x_q^d;\phi)dx_1^d \cdots dx_q^d = dx_1^{d-n+q} \cdots dx_q^{d-n+q} dL_{q(0)} \tag{3.6}$$

where $J_n^{d;q}$ is the Jacobian of the mapping

$$(x_1^d,\dots,x_q^d) \to L_{q(0)} \tag{3.7}$$

(cf. Federer (1969), (3.2.22)). The existence of this Jacobian is a very deep result in geometric measure theory. The decomposition (3.6) is a special case of the so-called co-area formula. In Zähle (1990), an explicit expression for the Jacobian has been derived. In Jensen and Kiêu (1990), an alternative proof is presented.

Lemma 3.1 below represents the first step towards the development of a stereological estimator of

$$(\nu_n^d)^q \mathscr{K}_\Phi^{(q+1)}(B_1 \times \cdots \times B_q)\,. \tag{3.8}$$

In this lemma we express (3.8) as the mean value in the Palm distribution of a measurement along a fixed q-subspace $L_{q(00)}$. We will assume that q points $x_i^d \in \Phi \cap B_i$, $i = 1,\ldots,q$, are essentially linearly independent in the following manner

$$P_0\left(\left\{\phi : \prod_{i=1}^{q} \lambda_n^d(\phi \cap B_i) = \int_{\phi \cap B_1} \cdots \int_{\phi \cap B_q} I(x_1^d,\ldots,x_q^d \text{ linearly independent}) dx_1^d \cdots dx_q^d\right\}\right) = 1\,.$$

LEMMA 3.1. *Let Φ be a stationary and isotropic random (d,n)-set. Let $L_{q(00)}$ be a fixed q-subspace and let B_i be of the form* (3.2), $i = 1,\ldots,q$. *Furthermore, let*

$$\begin{aligned} &f(\phi; L_{q(00)}, B_1 \times \cdots \times B_q) \\ &\quad = c(n,q) \int_{\phi \cap B_1 \cap L_{q(00)}} \cdots \int_{\phi \cap B_q \cap L_{q(00)}} \\ &\qquad \cdot J_n^{d;q}(x_1^{d-n+q},\ldots,x_q^{d-n+q}; \phi)^{-1} dx_1^{d-n+q} \cdots dx_q^{d-n+q}\,. \end{aligned} \tag{3.9}$$

Then,

$$(\nu_n^d)^q \mathscr{K}_\Phi^{(q+1)}(B_1 \times \cdots \times B_q) = E_0 f(\phi; L_{q(00)}, B_1 \times \cdots \times B_q)\,, \tag{3.10}$$

$q = \max(1, n-d),\ldots,n.$

PROOF. Because of the isotropy of the Palm distribution

$$E_0 f(\Phi; L_{q(00)}, B_1 \times \cdots \times B_q) = E_0 f(A\Phi; L_{q(00)}, B_1 \times \cdots \times B_q)\,, \tag{3.11}$$

for any rotation $A \in SO(n)$. Using the transformation $x_i^{d-n+q} \to A' x_i^{d-n+q}$, $i = 1,\ldots,q$, we therefore get

$$\begin{aligned} &E_0 f(\Phi; L_{q(00)}, B_1 \times \cdots \times B_q) \\ &\quad = E_0 f(\Phi; A' L_{q(00)}, B_1 \times \cdots \times B_q) \\ &\quad = \int_{SO(n)} E_0 f(\Phi; A' L_{q(00)}, B_1 \times \cdots \times B_q) \mu_n(dA) \end{aligned} \tag{3.12}$$

$$= \int_{\mathscr{L}_{q(0)}} E_0 f(\Phi; L_{q(0)}, B_1 \times \cdots \times B_q) \overline{dL_{q(0)}}$$

$$= E_0 \int_{\mathscr{L}_{q(0)}} \int_{\Phi \cap B_1 \cap L_{q(0)}} \cdots \int_{\Phi \cap B_q \cap L_{q(0)}} (J_n^{d;q})^{-1}$$

$$\cdot dx_1^{d-n+q} \cdots dx_q^{d-n+q} dL_{q(0)}$$

$$= E_0 \int_{\Phi \cap B_1} \cdots \int_{\Phi \cap B_q} dx_1^d \cdots dx_q^d$$

$$= (v_n^d)^q \mathscr{K}_\Phi^{(q+1)}(B_1 \times \cdots \times B_q) . \qquad \square$$

Note that in Lemma 3.1 we have assumed that we can use the reciprocal of the Jacobian. This is true if we use the convention that f is an integral over the set, where $J_n^{d;q}$ is positive, and furthermore assume that

$$P_0\left(\left\{\phi \colon \prod_{i=1}^{q} \lambda_n^d(\phi \cap B_i) = \int_{\phi \cap B_1} \cdots \int_{\phi \cap B_q} I_{\mathbb{R}_+}(J_n^{d;q}(x_1^d, \ldots, x_q^d; \phi)) dx_1^d \cdots dx_q^d\right\}\right) = 1 ,$$

where $I_{\mathbb{R}_+}$ is the indicator function of $\mathbb{R}_+$.

Example 3.1. (Random planar fibres) Let us assume that $n = 2$, $d = 1$ and $q = 1$. The aim is thus to estimate the reduced second moment measure of a planar random set consisting of fibres. We let

$$B_1 = B(0, r) = \{x \in \mathbb{R}^2 \colon \|x\| \le r\} \tag{3.13}$$

and use the short notation

$$v_2^1 \mathscr{K}_\Phi^{(2)}(B(0, r)) = L_A \mathscr{K}(r) . \tag{3.14}$$

We assume that the fibres are piecewise smooth of class C^1. The Jacobian of the mapping $x^1 \to (L_{1(0)}, x^0)$ is then (cf. Jensen and Gundersen (1989)),

$$J(x^1) = J_2^{1;1}(x^1; \phi) = \sin \alpha / |x^1| , \tag{3.15}$$

where α is the angle between $L_{1(0)}$ and the tangent to ϕ at x^1. Thus, if $\phi \cap L_{1(00)} \cap B(0, r) = \{x_1, \ldots, x_{\nu(\phi)}\}$ with corresponding angles $\alpha_i(\phi)$ and distances $d_i(\phi) = |x_i|$

$$f(\phi; L_{1(00)}, B_1) = \pi \sum_{i=1}^{\nu(\phi)} d_i(\phi) / \sin \alpha_i(\phi) \tag{3.16}$$

under the assumption that

$$P_0\left(\left\{\phi \colon \lambda_2^1(\phi \cap B(0, r)) = \int_{\phi \cap B(0, r)} I_{\mathbb{R}_+}(\alpha(x^1)) dx^1\right\}\right) = 1 ,$$

where $\alpha(x^1)$ is the angle between the tangent to ϕ at x^1 and the line joining 0 and x^1. The result of Lemma 3.1 is therefore that under the above assumption

$$(3.17) \qquad L_A \mathscr{K}(r) = \pi E_0 \sum_{i=1}^{\nu(\phi)} d_i(\Phi)/\sin \alpha_i(\Phi) .$$

This is Stoyan ((1981), (4.1)) under the assumptions stated. Note that in that paper the coefficient 2π instead of π occurs, because in that paper half of the line $L_{1(00)}$ is used for observation.

In conclusion, it follows from Lemma 3.1 that $L_A \mathscr{K}(r)$ can be estimated by choosing a typical point of the fibre as the origin and performing the measurement (3.16).

In order to use the result in Lemma 3.1 it is necessary to select a set of typical points on ϕ. Evidently, it would be most convenient to use points on $\phi \cap L_{q(00)}$. Some caution should be taken here because $\phi \cap L_{q(00)}$ cannot be regarded as a collection of typical points from ϕ. We can solve the problem by means of Lemma 3.2 below which only requires stationarity of Φ. We collect points on $\phi \cap L_{r(00)}$, where $L_{r(00)} \subseteq L_{q(00)}$ is an r-subspace contained in $L_{q(00)}$. Typically, r is chosen such that $\phi \cap L_{r(00)}$ is 0-dimensional, i.e., $d - n + r = 0$. We let $G_n^{d;r}(\cdot\,; \phi)$ be the Jacobian of the mapping

$$(3.18) \qquad x^d \in \phi \to x^{n-r} ,$$

where x^{n-r} is the projection of x^d onto the orthogonal complement $L_{r(00)}^{\perp}$. Note that

$$(3.19) \qquad G_n^{d;r}(x; \phi) = G_n^{d;r}(0; \phi - x) .$$

LEMMA 3.2. *Let Φ be a stationary random (d, n)-set and let $L_{r(00)}$ be a fixed r-subspace. Let $T_r \subseteq L_{r(00)}$ be a bounded Borel subset of $L_{r(00)}$ and let $h(\cdot\,, \cdot)$: $L_{r(00)} \times N_d \to \mathbb{R}_+ \cup \{0\}$ be a measurable non-negative function. Then,*

$$(3.20) \qquad E\int_{\Phi \cap T_r} h(x, \Phi)\lambda_r^{d-n+r}(dx)$$
$$= \nu_n^d \int_{T_r} \int_{N_d} h(x, \phi + x) G_n^{d;r}(0; \phi) P_0(d\phi) \lambda_r(dx) ,$$

$r = \max(1, n - d), \ldots, n$.

PROOF. For any $y \in L_{r(00)}^{\perp}$ we have, because of the stationarity of Φ,

(3.21)
$$\begin{aligned} E\int_{\Phi\cap T_r} h(x,\Phi)\lambda_r^{d-n+r}(dx) &= E\int_{\Phi-y\cap T_r} h(x,\Phi-y)\lambda_r^{d-n+r}(dx) \\ &= E\int_{\Phi\cap(T_r+y)} h(x-y,\Phi-y)\lambda_r^{d-n+r}(dx)\,. \end{aligned}$$

If we let p and $p_\perp$ be the projections onto $L_{r(00)}$ and $L_{r(00)}^\perp$, respectively, we can define a function $\tilde{h}$ on $\mathbb{R}^n \times N_d$ by

$$\tilde{h}(x,\phi) = h(px, \phi - p_\perp x)\,. \tag{3.22}$$

The last line of (3.21) can thereby be expressed as

$$E\int_{\Phi\cap(T_r+y)} \tilde{h}(x,\Phi)\lambda_r^{d-n+r}(dx)\,. \tag{3.23}$$

Thus, if we let I_{n-r} be the unit cube in $L_{r(00)}^\perp$, we get, using the refined Campbell theorem,

(3.24)
$$\begin{aligned} E\int_{\Phi\cap T_r} h(x,\Phi)\lambda_r^{d-n+r}(dx) &= \int_{I_{n-r}} E\int_{\Phi\cap(T_r+y)} \tilde{h}(x,\Phi)\lambda_r^{d-n+r}(dx)\lambda_n^{n-r}(dy) \\ &= E\int_{I_{n-r}}\int_{\Phi\cap(T_r+y)} \tilde{h}(x,\Phi)\lambda_r^{d-n+r}(dx)\lambda_n^{n-r}(dy) \\ &= E\int_{\Phi\cap(T_r\times I_{n-r})} \tilde{h}(x,\Phi)G_n^{d;r}(x;\Phi)\lambda_n^d(dx) \\ &= v_n^d\int_{T_r\times I_{n-r}}\int_{N_d} \tilde{h}(x,\phi+x)G_n^{d;r}(x;\phi+x)P_0(d\phi)dx \\ &= v_n^d\int_{T_r\times I_{n-r}}\int_{N_d} h(px,\phi+px)G_n^{d;r}(0;\phi)P_0(d\phi)dx \\ &= v_n^d\int_{T_r}\int_{N_d} h(x,\phi+x)G_n^{d;r}(0;\phi)P_0(d\phi)\lambda_r(dx)\,. \end{aligned}$$ □

For $r = n$, Lemma 3.2 is simply the refined Campbell theorem. We are now ready to present the main result of this paper.

THEOREM 3.1. *Let Φ be a stationary and isotropic random (d, n) set. Let $L_{q(00)}$ and $L_{r(00)}$ be fixed q- and r-subspaces, such that $L_{r(00)} \subseteq L_{q(00)}$ and let $T_r \subseteq L_{r(00)}$ be a bounded Borel subset of $L_{r(00)}$. Let B_i be of the form* (3.2), *$i = 1,\ldots, q$, and let f be defined as in Lemma* 3.1. *Then,*

$$(3.25)\quad (\nu_n^d)^q \mathscr{K}_\Phi^{(q+1)}(B_1 \times \cdots \times B_q) \\ = \frac{1}{\nu_n^d \lambda_r(T_r)} E \int_{\Phi \cap T_r} f(\Phi - x;\ L_{q(00)}, B_1 \times \cdots \times B_q) / G_n^{d;r}(0;\ \Phi - x) \\ \cdot \lambda_r^{d-n+r}(dx)\ ,$$

$q, r = \max(1, n-d), \ldots, n$, $r \leq q$.

PROOF. Using Lemma 3.1 and Lemma 3.2, we get

$$(3.26)\quad E \int_{\Phi \cap T_r} f(\Phi - x;\ L_{q(00)}, B_1 \times \cdots \times B_q) / G_n^{d;r}(0;\ \Phi - x) \lambda_r^{d-n+r}(dx) \\ = \nu_n^d \lambda_r(T_r) E_0 f(\Phi;\ L_{q(00)}, B_1 \times \cdots \times B_q) \\ = \nu_n^d \lambda_r(T_r)(\nu_n^d)^q \mathscr{K}_\Phi^{(q+1)}(B_1 \times \cdots \times B_q)\ . \qquad \square$$

Note that we in Theorem 3.1 also have assumed that we can use the reciprocal of the G-Jacobian. This is true if we use the convention that the integral in (3.25) is over the set where $G_n^{d;r}$ is positive and furthermore assume that

$$P_0(\{\phi:\ G_n^{d;r}(0;\ \phi) > 0\}) = 1\ .$$

The right-hand side of (3.25) depends on the unknown intensity ν_n^d. Estimation of ν_n^d can be done by determining the lower-dimensional analogue, viz. the volume of $\Phi \cap T_r$. The result is formulated in the corollary below.

COROLLARY 3.1. *Let the situation be as in Theorem* 3.1 *and let*

$$(3.27)\qquad b(n, d, r) = \frac{\Gamma\left(\frac{r+1}{2}\right)\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d+r-n+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}\ .$$

Then,

$$(3.28)\quad (\nu_n^d)^q \mathscr{K}_\Phi^{(q+1)}(B_1 \times \cdots \times B_q) \\ = b(n, d, r)[E\lambda_r^{d-n+r}(\Phi \cap T_r)]^{-1} \\ \cdot E \int_{\Phi \cap T_r} f(\Phi - x;\ L_{q(00)}, B_1 \times \cdots \times B_q) / G_n^{d;r}(0;\ \Phi - x) \\ \cdot \lambda_r^{d-n+r}(dx)\ ,$$

$q, r = \max(1, n-d), \ldots, n, r \leq q$.

PROOF. The result follows immediately from the fact that

$$E\lambda_r^{d-n+r}(\Phi \cap T_r) = \nu_r^{d-n+r}\lambda_r(T_r) \tag{3.29}$$

and (cf. e.g., Zähle (1982), Theorem 3.2.2.3),

$$\nu_r^{d-n+r}/\nu_n^d = b(n, d, r) . \tag{3.30}$$

□

Example 3.1 (Random planar fibres, continued). Let $r = 1$. The Jacobian G of the mapping (3.18) is

$$G(x^1) = G_2^{1;1}(x^1; \phi) = \sin\alpha , \tag{3.31}$$

where α is the angle between $L_{1(00)}$ and the tangent to ϕ at x^1. Let $\Phi \cap L_{1(00)} = \{z_i\}_{i=1}^{\infty}$, and let α_i be the angle between $L_{1(00)}$ and the tangent to Φ at z_i, $i = 1, 2, \ldots$. If we let N be the number of the z_i's in T_1, the estimator of $L_A\mathscr{K}(r)$ obtained by replacing expected quantities by observed quantities in (3.28) is

$$\frac{2}{\pi}\frac{1}{N} \sum_{\substack{i \\ z_i \in T_1}} \sum_{\substack{j \\ |z_j - z_i| \leq r}} \pi |z_j - z_i| / \sin\alpha_i \sin\alpha_j . \tag{3.32}$$

This formula can also be found in Stoyan (1981); see also Mecke and Stoyan (1980) and Ambartzumian (1981).

Example 3.2. (Random spatial surfaces) Let us assume that $n = 3$, $d = 2$ and $q = r = 1$. For convenience, let

$$\nu_3^2 \mathscr{K}_\Phi^{(2)}(B(0, r)) = S_V\mathscr{K}(r) . \tag{3.33}$$

We assume that the surfaces are piecewise smooth of class C^1. Then, the J-and G-Jacobians are

$$J(x^2) = J_3^{2;1}(x^2; \phi) = \sin\alpha / |x^2|^2 , \tag{3.34}$$

$$G(x^2) = G_3^{2;1}(x^2; \phi) = \sin\alpha , \tag{3.35}$$

where α is the angle between the $L_{1(00)}$ and the tangent plane to ϕ at x^2. With the same notation as in Example 3.1 we get the following estimator of $S_V\mathscr{K}(r)$

$$\frac{1}{2}\frac{1}{N}\sum_{\substack{i \\ z_i \in T_1}}\sum_{\substack{j \\ |z_j - z_i| \le r}} 2\pi|z_j - z_i|^2/\sin\alpha_i \sin\alpha_j . \tag{3.36}$$

The measure decomposition which is used here provides stereological estimators only for the cases where $n - d \le q < n$. Another case ($n = 3$, $d = 1$, $q = 1$) is treated in Stoyan (1984, 1985*a*) where a stereological estimator of the reduced second moment measure of a spatial fibre process is presented. Again, spatial angles are needed. Furthermore, estimators of the reduced second moment measure of $(0,2)$- or $(0,3)$-sets (planar or spatial point processes) can be derived using disector-sampling. The results for spatial point processes are presented in Jensen (1990).

4. Modified reduced second moment measure

Reduced second moment measures are often used in the statistical analysis of $(0,n)$-sets, i.e. point processes. In particular, they are useful for comparison of a point process with a stationary Poisson point process. When Φ is a point process, the reduced second moment measure is

$$\mathscr{K}_\Phi^{(2)}(B) = E_0\lambda_n^0(\Phi \cap B)/v_n^0, \qquad B \in \mathscr{B}^n . \tag{4.1}$$

Usually, $B = B(0,r)$ is a ball of radius r centered at the origin, which under P_0 with probability one contains a point from Φ. It is usual that this point is not counted, leading to the following modification:

$$K_0(r) = \mathscr{K}_\Phi^{(2)}(B(0,r)\backslash\{0\}) = E_0\lambda_n^0(\Phi\backslash\{0\} \cap B(0,r))/v_n^0 . \tag{4.2}$$

For a stationary Poisson point process, we have

$$K_0(r) = \lambda_n(B(0,r)) = \omega_n r^n , \tag{4.3}$$

where $\omega_n = \pi^{n/2}/\Gamma(n/2+1)$ is the volume of the unit ball. A plot of $K_0(r)$ versus r or r^n is usually used in the second-order analysis of point processes. Deviations of an empirical plot from the Poisson curve can be interpreted as clustering or inhibition of the points (cf. e.g., Diggle (1983)).

The use of the moment measures in the analysis of random (d,n)-sets is for $d > 0$ less standard, because usually only a restricted class of random sets is considered. Each class implies its own definition of total randomness. Here we shall focus on the class of random sets which can be described by marked point processes. We shall call them (d,n)-marked point processes. The totally random (d,n)-marked point process is the Boolean model (cf. Stoyan *et al.* (1987), p. 65).

Deviations of a (d,n)-marked point process from the Boolean model can be analysed with the standard measures associated with marked point processes. But (d,n)-marked point processes also give rise to the random measure defined in (2.1) (note that we cannot in general identify the (d,n)-marked point process from the random measure (2.1)). Similarly, we can define moment measures, a Palm distribution, and reduced moment measures. Following the point process approach, we shall now propose a modification of the second reduced moment measure analogous to $K_0(r)$ which behaves in a similar manner for the Boolean model.

First we shall give a formal definition of a (d,n)-marked point process:

DEFINITION 4.1. A (d,n)-marked point process $\Psi = \{[x_i;\ \Xi_i]\}_{i=1}^{\infty}$ is a random point process on $\mathbb{R}^n \times N_d$ such that $\{x_i\}_{i=1}^{\infty}$ is a random point process on $\mathbb{R}^n$ and such that for any bounded Borel set $B \in \mathscr{B}^n$ the number of $\Phi_i = x_i + \Xi_i$ hitting B is finite.

Associated with Ψ, we have the random closed set $\Phi = \bigcup_{i=1}^{\infty} \Phi_i = \bigcup_{i=1}^{\infty} (x_i + \Xi_i)$ which is a random (d,n)-set.

DEFINITION 4.2. A (d,n)-marked point process $\Psi = \{[x_i;\ \Xi_i]\}_{i=1}^{\infty}$ is called a Boolean model, if $\{x_i\}_{i=1}^{\infty}$ is a stationary Poisson point process and $\{\Xi_i\}_{i=1}^{\infty}$ are i.i.d. and independent of $\{x_i\}_{i=1}^{\infty}$.

The reduced second moment measure of Φ is in general defined by

$$\mathscr{K}_{\Phi}^{(2)}(B) = E_0 \lambda_n^d(\Phi \cap B) / v_n^d, \qquad B \in \mathscr{B}^n. \tag{4.4}$$

In the present paper, we shall define a modification of this measure which requires that the Φ_i's are essentially non-overlapping, i.e.,

$$\sum_{i=1}^{\infty} \lambda_n^d(\Phi_i \cap B) = \lambda_n^d(\Phi \cap B) \quad \text{a.s.,} \qquad \text{for all} \quad B \in \mathscr{B}^n. \tag{4.5}$$

This is in the spirit of the present paper since we primarily consider random sets and not processes.

We assume that Ψ is stationary, i.e., Ψ_x has the same distribution as Ψ for all $x \in \mathbb{R}^n$, where $\Psi_x = \{[x_i + x;\ \Xi_i]\}_{i=1}^{\infty}$. Because of the stationarity we have for $B \in \mathscr{B}^n$ and $K \in \mathscr{N}_d$

$$E\,\#\{i\colon x_i \in B, \Xi_i \in K\} = \mu \lambda_n(B) P_m(K) \tag{4.6}$$

where μ is the intensity of the point process and P_m is the mark distribution. Under stationarity, the assumption (4.5) of non-overlapping is equivalent to the following simple relation between v_n^d, the first-order characteristic of

Φ, and μ, the first-order characteristic of the point process $\{x_i\}_{i=1}^{\infty}$, using the standard Campbell theorem for marked point processes

$$v_n^d = \mu E_m(\lambda_n^d(\Xi)) , \tag{4.7}$$

where E_m is the mean in the mark distribution (cf. Stoyan *et al.* (1987), p. 101). The assumption of non-overlapping is always satisfied for a Boolean model with $d < n$.

We shall define a Palm distribution of the marked point process Ψ, with another form of weighting than the one usually used for marked point processes,

$$P_0(A) = \frac{1}{v_n^d \lambda_n(W)} E\int_{\bigcup_{i=1}^{\infty}(x_i+\Xi_i)\cap W} I_A(\Psi_{-x})\lambda_n^d(dx) , \tag{4.8}$$

where A belongs to the σ-algebra associated with a (d,n)-marked point process and $W \in \mathscr{B}^n$ satisfies $\lambda_n(W) > 0$. The Palm distribution may be interpreted as the probability that $\Psi \in A$ when the origin of $\mathbb{R}^n$ is chosen as a typical point of $\Phi = \bigcup_{i=1}^{\infty} \Phi_i$. Note that the mapping

$$\Psi = \{[x_i; \Xi_i]\}_{i=1}^{\infty} \rightarrow \Phi = \bigcup_{i=1}^{\infty} (x_i + \Xi_i) \tag{4.9}$$

transforms the Palm distribution of this section into the Palm distribution of Section 2. It can be shown that for any measurable, nonnegative function h

$$E\int_{\Phi} h(x, \Psi)\lambda_n^d(dx) = v_n^d \int_{\mathbb{R}^n} E_0 h(x, \Psi_x)\lambda_n(dx) . \tag{4.10}$$

Because of the assumption of no overlapping exactly one Φ_i hits 0 under P_0. This is most easily seen by proving that

$$P_0(0 \in \cup \Phi_i) = E_0\left(\sum_i I_{\Phi_i}(0)\right) = 1 , \tag{4.11}$$

which can be done using the definition of P_0 and the non-overlap assumption.

We are now ready to define the modified reduced second moment measure.

DEFINITION 4.3. Let $\Psi = \{[x_i; \Xi_i]\}_{i=1}^{\infty}$ be a (d,n)-marked point process, distributed according to the Palm distribution P_0. Let Φ^0 be the unique $\Phi_i = x_i + \Xi_i$ hitting 0. Then, the modified reduced second moment measure is defined by

(4.12) $$K_d(r) = E_0\lambda_n^d(\Phi\backslash\Phi^0 \cap B(0,r))/v_n^d .$$

PROPOSITION 4.1. *If Ψ is a Boolean model then*

(4.13) $$K_d(r) = \omega_n r^n .$$

PROOF. Under the non-overlapping assumption, it can be shown that the distribution of $\Phi\backslash\Phi^0$ under P_0 is the same as the distribution of Φ under the original Boolean model. Therefore,

$$K_d(r) = E\lambda_n^d(\Phi \cap B(0,r))/v_n^d = \omega_n r^n . \qquad \square$$

In Stoyan (1985*a*), a result of this type was obtained for a planar fibre process.

The modified reduced second moment measure $K_d(r)$, defined for (d,n)-marked point processes, can also be expressed as a mean value of the type presented in Lemma 3.1. Thus, let Ψ be a stationary and isotropic (d,n)-marked point process. Isotropy means that $A\Psi$ has the same distribution as Ψ for any rotation A in $\mathbb{R}^n$ about the origin. Here, $A\Psi = \{[Ax_i; A\Xi_i]\}_{i=1}^{\infty}$. Under these assumptions $\Phi\backslash\Phi^0$ has the same distribution as $A(\Phi\backslash\Phi^0)$ under P_0. Therefore,

(4.14) $$v_n^d K_d(r) = E_0 f(\Phi\backslash\Phi^0;\, L_{1(00)}, B(0,r)) .$$

It is also easy to derive a modification of Theorem 3.1 and Corollary 3.1.

REFERENCES

Ambartzumian, R. V. (1981). Stereology of random planar segment processes, *Rend. Sem. Mat. Univ. Politec. Torino*, **39**, 147–159.

Baddeley, A., Gundersen, H. J. G. and Cruz-Orive, L. M. (1986). Estimation of surface area from vertical sections, *J. Microsc.*, **142**, 259–276.

Diggle, P. J. (1983). *Statistical Analysis of Point Processes*, Academic Press, London.

Federer, H. (1969). *Geometric Measure Theory*, Springer, Berlin-New York.

Gundersen, H. J. G. (1988). The nucleator, *J. Microsc.*, **151**, 3–21.

Hanisch, K.-H. (1983). Reduction of n-th moment measures and the special case of the third moment measure of stationary and isotropic planar point processes, *Math. Operationsforsch. u. Statist. Ser. Statist.*, **14**, 421–435.

Hanisch, K.-H. (1985). On the second order analysis of stationary and isotropic planar fibre processes by a line intercept method, *Geobild '85*, 141–146.

Hanisch, K.-H. and Stoyan, D. (1981). Stereological estimation of the radial distribution of centres of spheres, *J. Microsc.*, **122**, 131–141.

Jensen, E. B. (1990). Recent developments in the stereological analysis of particles, Research Report No. 181, Department of Theoretical Statistics, Institute of Mathematics, University of Aarhus, (submitted to *Ann. Inst. Statist. Math.*).

Jensen, E. B. and Gundersen, H. J. G. (1989). Fundamental stereological formulae based on isotropically oriented probes through fixed points—with applications to particle analysis, Proceedings of the 4th International Conference in Stereology and Stochastic Geometry, *J. Microsc.*, **153**, 249-267.

Jensen, E. B. and Kiêu, K. (1990). A new integral geometric formula of Blaschke-Petkantschin type, Research Report No. 180, Department of Theoretical Statistics, Institute of Mathematics, University of Aarhus.

Mecke, J. (1967). Stationäre zufällige Masse auf Lokalkompakten Abelschen Gruppen, *Z. Wahrsch. Verw. Gebiete*, **9**, 36-58.

Mecke, J. and Stoyan, D. (1980). Formulas for stationary planar fibre processes I—general theory, *Math. Operationsforsch. Statist., Ser. Statist.*, **11**, 267-279.

Nagel, W. (1987). An application of Crofton's formula to moment measures of random curvature measures, Forschungsergebnisse N/87/12, Friedrich-Schiller-Universität, Sektion Mathematik, Jena.

Schwandtke, A. (1988). Second-order quantities for stationary weighted fibre processes, *Math. Nachr.*, **139**, 321-334.

Simon, L. (1983). Lectures on Geometric Measure Theory, *Proc. Centre Math. Anal. Austral. Nat. Univ.*, Vol. 3, 1-272.

Stoyan, D. (1981). On the second-order analysis of stationary planar fibre processes, *Math. Nachr.*, **102**, 189-199.

Stoyan, D. (1984). Further stereological formulae for spatial fibre processes, *Math. Operationsforsch. u. Statist., Ser. Statist.*, **15**, 421-428.

Stoyan, D. (1985*a*). Stereological determination of orientations, second-order quantities and correlations for random fibre systems, *Biometrical J.*, **27**, 411-425.

Stoyan, D. (1985*b*). Practicable methods for the determination of the pair-correlation function of fibre process, *Geobild '85*, 131-140.

Stoyan, D. and Ohser, J. (1982). Correlations between planar random structures, with an ecological application, *Biometrical J.*, **24**, 631-647.

Stoyan, D. and Ohser, J. (1985). Cross-correlation measures of weighted random measures and their estimation, *Theory Probab. Appl.*, **29**, 345-355.

Stoyan, D., Kendall, W. S. and Mecke, J. (1987). *Stochastic Geometry and Its Applications*, Akademie-Verlag, Berlin.

Zähle, M. (1982). Random processes of Hausdorff rectifiable closed sets, *Math. Nachr.*, **108**, 49-72.

Zähle, M. (1990). A kinematic formula and moment measures of random sets, *Math. Nachr.* (to appear).