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ON THE CONVERGENCE OF BROYDEN-LIKE METHODS FOR NONLINEAR EQUATIONS WITH NONDIFFERENTIABLE TERMS

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Abstract. In this paper, the author studies a Broyden-like method for solving nonlinear equations with nondifferentiable terms, which uses as updating matrices, approximations for Jacobian matrices of differentiable terms. Local and semilocal convergence theorems are proved. The results generalize those of Broyden, Dennis and Moré.

Key words and phrases: Convergence theorems, Broyden-like methods, nonlinear equations, nondifferentiable terms.

1. Introduction

Broyden (1965) described a method

$$\begin{aligned} &x_{k+1} = x_k - \tilde{B}_k^{-1} f(x_k) , \\ (1.1)\qquad &\tilde{B}_{k+1} = \tilde{B}_k + (t_k - \tilde{B}_k s_k) s_k^t / s_k^t s_k , \\ &s_k = x_{k+1} - x_k, \qquad t_k = f(x_{k+1}) - f(x_k) , \end{aligned}$$

for solving the equation

$$(1.2)\qquad f(x) = 0, \quad x \in D \subset R^n ,$$

where f is a Fréchet differentiable operator defined in a domain D of R^n.

Furthermore, Broyden *et al.* (1973) and Dennis and Moré (1977) derived local and superlinear convergence theorems of (1.1). Dennis (1971) gave a semilocal convergence theorem under Kantorovich-type assumptions.

In this paper, we shall consider the equation

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$$F(x) = f(x) + g(x) = 0, \quad x \in D \subset R^n , \tag{1.3}$$

with nondifferentiable operator g: $D \subset R^n \to R^n$, and a Broyden-like method for solving (1.3):

$$\begin{aligned} x_{k+1} &= x_k - B_k^{-1}(f(x_k) + g(x_k)) , \\ B_{k+1} &= B_k + (t_k - B_k s_k)s_k^t / s_k^t s_k , \\ s_k &= x_{k+1} - x_k, \quad t_k = f(x_{k+1}) - f(x_k) . \end{aligned} \tag{1.4}$$

Note that the matrices B_k do not satisfy the quasi-Newton equations, $B_{k+1}(x_{k+1} - x_k) = F(x_{k+1}) - F(x_k)$, $k = 0, 1, 2, \ldots$.

We shall analyze the convergence of the iteration (1.4) by considering it as a Newton-like iteration

$$x_{k+1} = x_k - A(x_k)^{-1}(f(x_k) + g(x_k)), \quad k = 0, 1, \ldots \tag{1.5}$$

applied to (1.3), where $A(x)$ denotes a linear operator which approximates the Fréchet derivative $f'(x)$ of f at $x \in D$. Convergence analysis for the case where $A(x) = f'(x)$ has been given by Rheinboldt (1968), Yamamoto (1987) and Zabrejko and Nguen (1987). Furthermore, in Chen and Yamamoto (1989), Yamamoto and Chen (1990), Yamamoto and the author have studied the local and semilocal convergence of (1.5) for the equation $f(x) + g(x) = 0$ in a Banach space, under some conditions. Although the results there have had a rather theoretical character, our analysis in this paper will bear a practical character.

In Section 2, we shall first give a local convergence theorem and, under an additional assumption, derive a superlinear convergence property. Next, we shall use majorant techniques to obtain a semilocal convergence theorem under Kantorovich-type assumptions. Finally, we shall show that the iteration (1.4) is globally convergent if the operator $f(x)$ is linear. The results generalize those of Broyden *et al.* (1973). In Section 3, we shall give some numerical examples to illustrate our results.

2. Convergence analysis

Throughout this paper, we shall use the l_2 vector and matrix norms, and denote $\| \ \|_2$ by $\| \ \|$. Let $S(x_0, r)$ be the open ball with center x_0 and radius r in R^n and let $\bar{S}(x_0, r)$ denote its closure.

LEMMA 2.1. *Let $s \in R^n$ with $s^t s = 1$, then*

$$\|I - ss^t\| = 1 .$$

PROOF. See Dennis and Moré (1977).

THEOREM 2.1. *Let* $x^* \in D$ *be a solution of* (1.3), *and* $f'(x^*)$ *be nonsingular. Suppose that for any* $x \in D$ *the following hold*

$$\|f'(x^*)^{-1}(f'(x) - f'(x^*))\| \le K\|x - x^*\|, \quad 0 \le K,$$
$$\|f'(x^*)^{-1}(g(x) - g(x^*))\| \le e\|x - x^*\|, \quad 0 \le e \le 1/3.$$

Then for any matrix B_0 *such that*

$$\|f'(x^*)^{-1}(B_0 - f'(x^*))\| \le b = 4(1-3e)/27,$$

any

$$\bar{S}(x^*, r) \subset D, \quad r \le 2(1-3e)/27K$$

is an attraction ball of iteration (1.4) *and*

$$\|x_{k+1} - x^*\| \le \frac{1}{2}\|x_k - x^*\|, \quad k = 0, 1, \dots .$$

PROOF. From the Banach Perturbation Lemma, B_0^{-1} exists and

$$\|B_0^{-1}f'(x^*)\| \le 1/(1-b) \le 27/23 \le 1 + q,$$

where $q = 1/2$. From $x_0 \in \bar{S}(x^*, r)$, we have

$$\begin{aligned}
\|x_1 - x^*\| &= \|x_0 - B_0^{-1}(F(x_0) - F(x^*)) - x^*\| \\
&\le \|B_0^{-1}f'(x^*)\| \Big\| f'(x^*)^{-1}\Big((B_0 - f'(x^*))(x_0 - x^*) \\
&\qquad - \int_0^1 (f'(x^* + t(x_0 - x^*)) - f'(x^*))dt(x_0 - x^*) \\
&\qquad - (g(x_0) - g(x^*))\Big)\Big\| \\
&\le (1+q)(b + e + Kr/2)\|x_0 - x^*\| \\
&\le 3(5 + 12e)/54\|x_0 - x^*\| \le q\|x_0 - x^*\| .
\end{aligned}$$

Hence, $x_1 \in \bar{S}(x^*, r)$. From Lemma 2.1 and the relation

$$\begin{aligned}
B_1 - f'(x^*) &= B_0 - f'(x^*) + (t_0 - B_0 s_0)s_0^t/s_0^t s_0 \\
&= (B_0 - f'(x^*))(I - s_0 s_0^t/s_0^t s_0) \\
&\quad + (t_0 - f'(x^*)s_0)s_0^t/s_0^t s_0 ,
\end{aligned}$$

we have

$$\begin{aligned}&\|f'(x^*)^{-1}(B_1 - f'(x^*))\| \\ &\quad \le \|f'(x^*)^{-1}(B_0 - f'(x^*))\| + K\|x^* - (x_0 + t(x_1 - x_0))\| \\ &\quad \le b + K\|x^* - x_0\| \le b + Kr \le 2b ,\end{aligned}$$

where $t \in [0, 1]$.

We shall now prove that for any $k \ge 0$,

$$\text{(2.1)} \quad \|f'(x^*)^{-1}(B_k - f'(x^*))\| \le 2b \quad \text{and} \quad \|x_{k+1} - x^*\| \le q\|x_k - x^*\| .$$

The proof is done by induction on k. Suppose that (2.1) holds for $k \le j - 1$. By the same technique as above, we obtain

$$\begin{aligned}\|f'(x^*)^{-1}(B_j - f'(x^*))\| &\le \|f'(x^*)^{-1}(B_{j-1} - f'(x^*))\| + K\|x^* - x_{j-1}\| \\ &\le b + K\sum_{i=0}^{j-1}\|x^* - x_i\| \\ &\le b + Kr/(1 - q) = b + 2Kr \le 2b\end{aligned}$$

and

$$\begin{aligned}\|x_{j+1} - x^*\| &= \|x_j - B_j^{-1}(F(x_j) - F(x^*)) - x^*\| \\ &\le \|B_j^{-1}f'(x^*)\| \Big\| f'(x^*)^{-1}\Big((B_j - f'(x^*))(x_j - x^*) \\ &\qquad - \int_0^1 (f'(x^* + t(x_j - x^*)) \\ &\qquad - f'(x^*))dt(x_j - x^*) - (g(x_j) - g(x^*))\Big)\Big\| \\ &\le (1 + q)\,(2b + Kr/2 + e)\|x_j - x^*\| \le q\|x_j - x^*\| .\end{aligned}$$

Hence, (2.1) holds. □

Remark 1. If $g(x) = 0$ and $e = 0$, then Theorem 2.1 implies a local convergence theorem for Broyden's method obtained by Broyden *et al.* (1973).

LEMMA 2.2. *If E is an $n \times n$ matrix and $s \in R^n$, then*

$$\left\| E\left(I - \frac{ss^t}{s^ts}\right)\right\|_F = \|E\|_F^2 - \left(\frac{\|Es\|}{\|s\|}\right)^2 ,$$

where $\| \; \|_F$ *denotes the Frobenius norm.*

PROOF. See Broyden (1970).

COROLLARY 2.1. *Suppose that the hypotheses of Theorem* 2.1 *hold and*

$$\lim_{k\to\infty} \frac{\|f'(x^*)^{-1}(g(x_k)-g(x^*))\|}{\|x_{k+1}-x_k\|} = 0 \,. \tag{2.2}$$

Then $\{x_k\}$ *converges superlinearly at* x^*.

PROOF. Let $E_k = f'(x^*)^{-1}(B_k - f'(x^*))$. From Lemma 2.2 and the inequality

$$\sqrt{\alpha^2-\beta^2} \le \alpha - (2\alpha)^{-1}\beta^2, \qquad (\alpha \ge \beta \ge 0, \quad \alpha \ne 0)\,,$$

we have

$$\begin{aligned} \|E_{k+1}\|_F &\le \|E_k(I - s_k s_k^t / s_k^t s_k)\|_F + K\|x^* - x_k\| \\ &\le \|E_k\|_F - (2\|E_k\|_F)^{-1}(\|E_k s_k\|/\|s_k\|)^2 + K\|x^* - x_k\| \,. \end{aligned}$$

From $\|E_k\| \le 2b$, we have that there exists a constant c such that $c \le (2\|E_k\|_F)^{-1}$. Hence, we obtain

$$c(\|E_k s_k\|/\|s_k\|)^2 \le \|E_k\|_F - \|E_{k+1}\|_F + K\|x^* - x_k\| \,.$$

This implies

$$c\sum_{k=0}^{\infty}(\|E_k s_k\|/\|s_k\|)^2 \le \|E_0\|_F + K\sum_{k=0}^{\infty} q^k\|x^* - x_0\| \le \|E_0\|_F + 2Kr$$

and

$$\lim_{k\to\infty} \frac{\|E_k s_k\|}{\|s_k\|} = 0 \,.$$

Furthermore, we have

$$\lim_{k\to\infty} \frac{\|f'(x^*)^{-1}(f(x_{k+1}) + g(x^*))\|}{\|x_{k+1} - k_k\|} = 0 \,,$$

since

$$\begin{aligned}E_k s_k &= f'(x^*)^{-1}(B_k - f'(x^*))\,(x_{k+1} - x_k)\\ &= f'(x^*)^{-1}(-f(x_k) - g(x_k) - f'(x^*)\,(x_{k+1} - x_k))\\ &= f'(x^*)^{-1}(f(x_{k+1}) - f(x_k) - f'(x^*)\,(x_{k+1} - x_k))\\ &\quad - f'(x^*)^{-1}(f(x_{k+1}) + g(x^*))\\ &\quad - f'(x^*)^{-1}(g(x_k) - g(x^*))\,.\end{aligned}$$

From Theorem 2.1, we have that for any $\varepsilon \in (0, 1)$, there is a $k_0 \geq 0$, such that if $k \geq k_0$, then

$$\begin{aligned}&\left|\frac{\|f'(x^*)^{-1}(f(x_{k+1}) + g(x^*))\|}{\|x_{k+1} - x^*\|} - 1\right|\\ &\leq \|x_{k+1} - x^*\|^{-1}\|f'(x^*)^{-1}(f(x_{k+1}) + g(x^*)\\ &\qquad\qquad - f(x^*) - g(x^*) - f'(x^*)(x_{k+1} - x^*))\|\\ &\leq \frac{K}{2}\|x_{k+1} - x^*\| < \varepsilon\,.\end{aligned}$$

Hence,

$$\begin{aligned}0 &= \lim_{k\to\infty} \frac{\|f'(x^*)^{-1}(f(x_{k+1}) + g(x^*))\|}{\|x_{k+1} - x_k\|}\\ &\geq \lim_{k\to\infty} \frac{(1-\varepsilon)\|x_{k+1} - x^*\|}{\|x_{k+1} - x^*\| + \|x_k - x^*\|}\\ &= \lim_{k\to\infty} (1-\varepsilon)\rho_k/(\rho_k + 1)\,,\end{aligned}$$

where $\rho_k = \|x_{k+1} - x^*\| / \|x_k - x^*\|$. Therefore $\lim_{k\to\infty} \rho_k = 0$. □

Remark 2. If $\|f'(x^*)^{-1}(g(x) - g(x^*))\| \leq e(r)\|x - x^*\|$, $r = \|x - x^*\|$ and $\lim_{r\to 0} e(r) = 0$, then the assumption (2.2) is satisfied.

In fact, we have

$$\begin{aligned}\|x_{k+1} - x_k\| &= \|B_k^{-1}(f(x_k) + g(x_k) - f(x^*) - g(x^*))\|\\ &\geq \left\|B_k^{-1}\int_0^1(B_k + f'(x^* + t(x_k - x^*))\right.\\ &\qquad \left. - f'(x^*) + f'(x^*) - B_k)dt(x_k - x^*)\right\|\\ &\qquad - \|B_k^{-1}f'(x^*)\|\,\|f'(x^*)^{-1}(g(x_k) - g(x^*))\|\end{aligned}$$

$$\geq \|x_k - x^*\| - (1+q)\frac{K}{2}\|x_k - x^*\|^2 - 2b(1+q)\|x_k - x^*\|$$

$$- (1+q)e(r_k)\|x_k - x^*\|$$

$$= \Delta_k \quad \text{(say)},$$

where $r_k = \|x_k - x^*\|$. Hence, for sufficiently large k,

$$\frac{\|f'(x^*)^{-1}(g(x_k) - g(x^*))\|}{\|x_{k+1} - x_k\|}$$

$$\leq \frac{e(r_k)\|x_k - x^*\|}{\Delta_k}$$

$$= \frac{e(r_k)}{1 - (1+q)2b - (1+q)\frac{K}{2}\|x_k - x^*\| - (1+q)e(r_k)}.$$

It follows from this that

$$\lim_{k\to\infty} \frac{\|f'(x^*)^{-1}(g(x_k) - g(x^*))\|}{\|x_{k+1} - x_k\|} = 0.$$

A simple example satisfying the conditions of Corollary 2.1 will be given in Section 3.

We shall now give a Kantorovich-type semilocal convergence theorem for the Broyden-like method (1.4), on the basis of the techniques used in Dennis (1971), Chen and Yamamoto (1989) and Yamamoto and Chen (1990).

We assume that a matrix B_0 is nonsingular and for any $x, y \in D$ the following hold

$$\|B_0^{-1}(f'(x) - f'(y))\| \leq K\|x - y\|, \qquad K \geq 0,$$

and

$$\|B_0^{-1}(g(x) - g(y))\| \leq e\|x - y\|, \qquad e \geq 0.$$

We put

$$b = \|B_0^{-1}f'(x_0) - I\|, \qquad a = \|B_0^{-1}(f(x_0) + g(x_0))\|,$$

$$\chi(r) = a - (1 - 3b - e)r + 2Kr^2, \qquad w(r) = 1 - 2b - 5Kr/2.$$

Let

$$1-3b-e>0 \quad \text{and} \quad a>0\,.$$

If $\chi(R)\le 0$ for some positive number R, then $8Ka\le(1-3b-e)^2$, and $\chi(r)$ has a zero

$$t^*=(1-3b-e)(1-\sqrt{1-8Ka/(1-3b-e)^2}\,)/8K$$

in $(0,R]$, since $\chi(r)$ is strictly convex. We define a scalar sequence $\{r_k\}$ by

$$r_0=0\,,\qquad r_{k+1}=r_k+w(r_k)^{-1}\chi(r_k),\qquad k\ge 0\,. \tag{2.3}$$

LEMMA 2.3. *Let $x_0,\dots,x_{k+1}$, $B_0,\dots,B_{k+1}$ be generated by iteration* (1.4). *If $\{x_i\}_{i=0}^{k+1}\subset D$ are distinct, then*

$$\|B_0^{-1}(B_{k+1}-f'(x_{k+1}))\|\le b+\frac{3K}{2}\sum_{i=0}^{k}\|x_{i+1}-x_i\|\,.$$

PROOF. See Dennis (1971).

THEOREM 2.2. *Suppose that $\chi(R)\le 0$ and $\bar S(x_0,R)\subset D$. Then the equation* (1.3) *has a solution x^* in the ball $\bar S(x_0,t^*)$, which is unique in*

$$\tilde S=\begin{cases}\bar S(x_0,R) & \text{if}\ \ \chi(R)<0\ \ \text{or}\ \ \chi(R)=0\ \ \text{and}\ \ t^*=R\\ S(x_0,R) & \text{if}\ \ \chi(R)=0\ \ \text{and}\ \ t^*<R\,.\end{cases}$$

The iteration (1.4) *is well defined for all $k\ge 0$, $x_k\in S(x_0,t^*)$ and $\{x_k\}$ satisfies the estimates*

$$\|x_{k+1}-x_k\|\le r_{k+1}-r_k \tag{2.4}$$

and

$$\|x^*-x_k\|\le t^*-r_k\,,\qquad k\ge 0\,, \tag{2.5}$$

where $\{r_k\}$ is defined by (2.3).

PROOF. Since $\chi(0)=a>0$, $\chi(t^*)=0$ and $\chi(r)$ is strictly convex, the scalar sequence $\{r_k\}$ is monotonically increasing and converges to t^*.

We shall now prove that the sequence $\{x_k\}$ defined by (1.4) satisfies (2.4). The proof is done by induction on k: for $k=0$, we have

$$\|x_1 - x_0\| = a = r_1 - r_0 .$$

Suppose that (2.4) holds for $k \leq j-1$. From Lemma 2.3, we have

$$\begin{aligned}
&\|B_0^{-1}(B_j - B_0)\| \\
&\quad \leq \|B_0^{-1}(B_j - f'(x_j) + f'(x_j) - f'(x_0) + f'(x_0) - B_0)\| \\
&\quad = 2\|B_0^{-1}(B_0 - f'(x_0))\| + \frac{3K}{2}\sum_{i=0}^{j-1}\|x_{i+1} - x_i\| + K\|x_j - x_0\| \\
&\quad \leq 2b + 5Kr_j/2 \\
&\quad \leq 2b + 5Kt^*/2 \\
&\quad = 2b + 5(1-3b-e)\,(1 - \sqrt{1 - 8Ka/(1-3b-e)^2}\,)/8 \\
&\quad \leq 1 - b - e < 1 ,
\end{aligned}$$

since $5(1 - \sqrt{1 - 8Ka/(1-3b-e)^2}\,)/8 < 1$. Hence B_j^{-1} exists and

$$\|B_j^{-1}B_0\| \leq (1 - 2b - 5Kr_j/2)^{-1} = w_j^{-1} ,$$

where $w_j = w(r_j)$. Furthermore,

$$\begin{aligned}
&\|x_{j+1} - x_j\| \\
&\quad = \|B_j^{-1}F(x_j)\| \\
&\quad \leq w_j^{-1}\|B_0^{-1}\{F(x_j) - B_{j-1}(x_j - x_{j-1}) \\
&\qquad\qquad - F(x_{j-1}) - (f'(x_{j-1}) - f'(x_{j-1}))\,(x_j - x_{j-1})\}\| \\
&\quad \leq w_j^{-1}\left\{\frac{K}{2}\|x_j - x_{j-1}\|^2\right. \\
&\qquad\qquad \left. + \left(b + \frac{3K}{2}\sum_{i=0}^{j-2}\|x_{i+1} - x_i\|\right)\|x_j - x_{j-1}\| + e\|x_j - x_{j-1}\|\right\} \\
&\quad \leq w_j^{-1}\left\{\chi(r_j) - \chi(r_{j-1})\right. \\
&\qquad\qquad + (1-2b)\,(r_j - r_{j-1}) - Kr_jr_{j-1} + \frac{5K}{2}r_{j-1}^2 \\
&\qquad\qquad \left. + \frac{3K}{2}(r_{j-1}r_j - r_j^2 - r_{j-1}^2)\right\}
\end{aligned}$$

$$\leq w_j^{-1}\Big\{\chi(r_j) - \chi(r_{j-1}) + (1-2b)\,(r_j - r_{j-1})$$

$$- K r_j r_{j-1} + \frac{5K}{2} r_{j-1}^2 - \frac{3K}{2} r_{j-1} r_j\Big\}\,.$$

Therefore, we have

$$\|x_{j+1} - x_j\| \leq r_{j+1} - r_j \quad \text{and} \quad \|x_{j+1} - x_0\| \leq r_{j+1}\,.$$

This implies that the iteration (1.4) starting from x_0 is well defined for all $k \geq 0$, and $\{x_k\}$ converges to a solution $x^* \in \bar{S}(x_0, t^*)$.

To prove the uniqueness of the solution in $\tilde{S}$, let y^* be a solution in $\tilde{S}$. Then it is easy to see that

$$\begin{aligned}
&\|y^* - x_0\| - a\\
&\quad \leq \|y^* - x_1\|\\
&\quad = \|y^* - x_0 + B_0^{-1}(F(x_0) - F(x^*) + (f'(x_0) - f'(x_0))\,(y^* - x_0))\|\\
&\quad = \Big\|B_0^{-1}\Big(\Big(B_0 - f'(x_0) - \int_0^1 (f'(x_0 + t(y^* - x_0))\\
&\qquad\quad - f'(x_0))dt\Big)\,(y^* - x_0) - (g(y^*) - g(x_0))\Big)\Big\|\\
&\quad \leq \frac{K}{2}\,\|y^* - x_0\|^2 + b\|y^* - x_0\| + e\|y^* - x_0\|\\
&\quad \leq \frac{5K}{2}\,\|y^* - x_0\|^2 + 3b\|y^* - x_0\| + e\|y^* - x_0\|\,,
\end{aligned}$$

from which we obtain $\chi(\|y^* - x_0\|) \geq 0$. This, together with $y^* \in \tilde{S}$, implies $\|y^* - x_0\| \leq t^*$. Next we shall show that the inequality

$$\|y^* - x_k\| \leq t^* - r_k, \quad k \geq 0\,, \tag{2.6}$$

holds. The proof is again done by induction on k: For $k = 0$, (2.6) is obvious. Suppose that (2.6) holds for all $k \leq j$. Then

$$\begin{aligned}
\|y^* - x_{j+1}\| = \Big\|B_j^{-1}\Big(&(B_j - f'(x_j))(y^* - x_j)\\
&- \int_0^1 (f'(x_j + t(y^* - x_j)) - f'(x_j))dt(y^* - x_j)\\
&- (g(y^*) - g(x_j))\Big)\Big\|
\end{aligned}$$

$$\le w_j^{-1}\left\{\left(b + \frac{3K}{2}\sum_{i=0}^{j-1}\|x_{i+1} - x_i\|\right)\|y^* - x_j\| + \frac{K}{2}\|y^* - x_j\|^2 + e\|y^* - x_j\|\right\}$$

$$\le w_j^{-1}\{\chi(t^*) - \chi(r_j)\} + t^* - r_j = t^* - r_{j+1} .$$

This implies $y^* = x^*$, the uniqueness of the solution, so that $x^* \in \bar{S}(x_0, t^*)$ and (2.5) follows from (2.4). □

Remark 3. By using the techniques in Chen and Yamamoto (1989) and Yamamoto and Chen (1990), we can show that if instead of x_0 and B_0 we begin (1.4) with y_0 and B_{y_0} such that

$$\|y_0 - x_0\| \le \frac{a}{2(1-b)} + \frac{(1-3b-e)^2}{8K(1-b)}$$

and

$$\|B_0^{-1}(B_{y_0} - B_0)\| \le 5K\|y_0 - x_0\|/2 + 2b ,$$

then the generated sequence $\{y_k\}$ converges to x^*.

Remark 4. If we take $g(x) = 0$, $e = 0$, then Theorem 2.2 reduces to an affine invariant version of Dennis' theorem (Dennis (1971), Theorem 3). Furthermore, Remark 3 implies that

$$\bar{S} = \{x \mid \|x - x_0\| \le (a + (1-3b)^2/4K)/2(1-b)\}$$

is a convergence ball of Broyden's (1965, 1970) method, that is, starting from any point of $\bar{S}$, Broyden's method converges to a solution of the equation (1.2).

COROLLARY 2.2. *Suppose that $D = R^n$ and $f(x) = Ax$, where A is an $n \times n$ nonsingular matrix. Then, starting from any $x_0 \in R^n$, the iteration* (1.4) *converges to the unique solution x^* of the equation* (1.3).

This follows from Theorem 2.2 and Remark 3 by taking $\chi(r) = a - (1 - 3b - e)r$, $R \ge a/(1 - 3b - e)$.

We can also give a simple, straightforward proof as follows: Since $t_k = As_k$ and

$$B_{k+1} - A = B_k - A + (t_k - B_k s_k)s_k^t / s_k^t s_k$$
$$= (B_k - A)(I - s_k s_k^t)/s_k^t s_k + (t_k - As_k)s_k^t / s_k^t s_k ,$$

we have

$$\|B_0^{-1}(B_{k+1} - A)\| \le \|B_0^{-1}(B_k - A)\| \le b$$

and

$$\|B_0^{-1}(B_{k+1} - B_0)\| \le \|B_0^{-1}(B_{k+1} - A)\| + \|B_0^{-1}(B_0 - A)\| \le 2b < 1 .$$

Hence B_{k+1}^{-1} exists and

$$\|B_{k+1}^{-1}B_0\| \le 1/(1-2b) .$$

Therefore,

$$\|x_{k+2} - x_{k+1}\|$$
$$= \|B_{k+1}^{-1}(-Ax_{k+1} - g(x_{k+1}) + B_k(x_{k+1} - x_k) + Ax_k + g(x_k))\|$$
$$\le \|B_{k+1}^{-1}B_0\|(\|A - B_k\| \ \|x_{k+1} - x_k\| + e\|g(x_{k+1}) - g(x_k)\|)$$
$$\le (b+e)/(1-2b)\|x_{k+1} - x_k\| .$$

Since $3b + e < 1$, it follows that $(b+e)/(1-2b) < 1$. Hence, Corollary 2.2 holds.

3. Numerical examples

In this section we shall give numerical examples illustrating our results. We first consider an example satisfying the conditions of Theorem 2.1 and Corollary 2.1.

Example 1. Consider the single equation

$$F(x) = e^{x-0.5} + 0.2x|x-1| - 1.05 = 0 .$$

This problem has a solution $x^* = 0.5$. Let $f(x) = e^{x-0.5}$ and $g(x) = 0.2x|x-1| - 1.05$. Then we have $K = 1.65$ and $e(r) = 0.2r$ in [0.1], since $f'(x^*)^{-1} = 1$ and

$$|g(x) - g(x^*)| = 0.2|(x - x^* - (x^2 - x^{*2}))|$$
$$= 0.2|(1 - (x + x^*))(x - x^*)| = 0.2|x - x^*| \ |x - x^*| .$$

If we choose B_0 such that $\|B_0 - 1\| \le b = 4(1 - 3e(r))/27$, then $\bar{S}(x^*, \hat{r})$ is an attraction ball of iteration (1.4), where $\hat{r} = 2(1 - 3 * 0.2 * 0.1)/27 * e^{0.1} \doteq 0.063$. We solve the equation by iteration (1.4), and choose $B_0 = f'(x_0)$ and the stopping criterion $\|F(x_k)\| \le 10^{-7}$. The results of numerical computation are shown in Table 1, where x_0 is the initial value, k is the iteration number and $\hat{x}$ is the approximate value for $x^* = 0.5$. The superlinear convergence of (1.4) for $x_0 = 1.0$ is shown in Table 2, although it is almost equal to linear convergence.

Table 1.

x_0	2.0	1.0	0.6	0.4	0.0
k	62	16	6	200	20
$\hat{x}$	0.5000001	0.5000001	0.5000000	0.5000031*	0.5000001

*Stopping criteria were not satisfied.

Table 2.

k	x_k	$F(x_k)$	ρ_k
0	1.0000000	0.5987213	0.2737143
1	0.6368572	0.1429184	0.3660067
2	0.5501727	0.5094931E − 1	0.3840817
3	0.5192704	0.1938308E − 1	0.3899238
4	0.5075140	0.7531057E − 2	0.3920927
5	0.5029462	0.2948839E − 2	0.3929231
6	0.5011576	0.1158071E − 2	0.3932353
7	0.5004552	0.4553262E − 3	0.3933263
8	0.5001791	0.1791028E − 3	0.3932888
9	0.5000704	0.7046536E − 4	0.3930629
10	0.5000277	0.2772386E − 4	0.3924813
11	0.5000108	0.1090817E − 4	0.3909701
12	0.5000042	0.4295704E − 5	0.3865522
13	0.5000017	0.1682770E − 5	0.3783302
14	0.5000006	0.6658424E − 6	0.3498170
15	0.5000002	0.2619884E − 6	0.2686837
16	0.5000001	0.9935910E − 7	

Example 2. Consider the Dirichlet problem

$$-\frac{\partial}{\partial x}\left(p(x,y)\,\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(q(x,y)\,\frac{\partial u}{\partial y}\right) + 2|u| = f(x,y)\,, \qquad (x,y) \in \Omega \subset R^2\,,$$

$$u(x,y) = v(x,y), \qquad (x,y) \in \partial\Omega\,,$$

where

$$p(x,y) = x(1-y), \qquad q(x,y) = y(1-x)\,,$$
$$f(x,y) = (2-x-y)^2 - 2(|(1-x)(1-y) - 0.5| - (1-x)(1-y))\,,$$
$$v(t,0) = v(0,t) = 0.5 - t, \quad v(t,1) = v(1,t) = -0.5, \qquad 0 \le t \le 1\,,$$
$$\Omega = (0,1) \times (0,1)\,.$$

This problem has a solution $u(x,y) = (x-1)(y-1) - 0.5$.

We discretize the elliptic partial differential equation by the standard five-point difference formula, and obtain a system of nonlinear algebraic equations

$$F(u) = Au + g(u) = 0, \qquad u \in R^n\,.$$

We put $f(u) = Au$ and $B_0 = A$. Then we have $K = 0$, $b = 0$ and $e = 2h^2\|A^{-1}\| = 2h^2/8\sin^2(\pi/2)h \le 1$, for $h \le 0.5$ (see Gregory and Karney (1969)), where h is the square mesh size of the side. Hence the conditions of Theorem 2.2 are satisfied. We solve the system by the Broyden-like method (1.4). Iterations were stopped after the condition $\|F(x_k)\| \le 10^{-6}$ was satisfied. The results of computation starting from $\hat{u}_i^{(0)} = 30(-1)^i$, $i = 1, 2, \ldots, n$ are shown in Table 3.

Table 3.

n	9	49	81	225
h	0.25	0.125	0.1	0.0625
$\hat{u}$	− 0.2500029	− 0.2500142	− 0.2500189	− 0.2500582
t	0.13	2.78	7.50	57.30
k	18	17	17	16

n: interior mesh number ($h = 1/(\sqrt{n}+1)$).
h: square mesh size of side.
$\hat{u}$: approximate values of u.
t: total computing time (sec.).
k: iteration number.
Exact solution: $u(0.5, 0.5) = -0.25$.

Computations were carried out on the Apollo DOMAIN 3000 at Department of Mathematics, Ehime University, Japan.

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