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ASYMPTOTICALLY EFFICIENT RANK MANOVA TESTS FOR RESTRICTED ALTERNATIVES IN RANDOMIZED BLOCK DESIGNS

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Abstract. In a randomized block design MANOVA model, for intra-block as well as aligned rank tests for homogeneity of treatment effects against some restricted alternatives, asymptotic optimality is studied by reference to the corresponding restricted likelihood ratio tests. Tests based on aligned ranks are better than intra-block rank tests when the error distributions are homogeneous across the blocks.

Key words and phrases: Intra-block ranking, Kuhn-Tucker-Lagrange point formula, locally most powerful rank test, randomized blocks, ranking after alignment, restricted likelihood ratio tests, union-intersection (UI-) principle.

1. Introduction

Consider a (multi-response) randomized block design with n blocks of p plots each, and let $\boldsymbol{X}_{ij} = (X_{ij}^{(1)},\dots, X_{ij}^{(q)})'$ be the response (vector) of the j-th treatment in the i-th block, $j = 1,\dots,p$, and let $\operatorname{vec}\boldsymbol{X}_i = (\boldsymbol{X}_{i1}',\dots,\boldsymbol{X}_{ip}')$, for $i = 1,\dots,n$. Assume that $\operatorname{vec}\boldsymbol{X}_i$ has a pq-variate continuous distribution function (d.f.) $F_i^{(1)}$, where

$$(1.1)\qquad F_i^{(1)}(\boldsymbol{y}) = F_i(\boldsymbol{y} - \operatorname{vec}\boldsymbol{\beta}),\qquad \boldsymbol{y}\in E^{pq},\qquad \text{for}\quad i = 1,\dots,n\,,$$

$\operatorname{vec}\boldsymbol{\beta} = (\boldsymbol{\beta}_1',\dots,\boldsymbol{\beta}_p')$, $\boldsymbol{\beta}_j = (\beta_j^{(1)},\dots,\beta_j^{(q)})'$, $j = 1,\dots,p$, and for each i, the d.f. F_i is assumed to be symmetric in its p compartments (each being a q-vector). Thus, we consider independent blocks and exchangeable intra-block error vectors. In (1.1), the $\boldsymbol{\beta}_j$ stands for the treatment effects (vectors) while the

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block effects may not be additive (may even be stochastic). The null hypothesis of no treatment effect is framed as

$$H_0\colon \operatorname{vec} \boldsymbol{\beta} = \mathbf{0}\,, \tag{1.2}$$

i.e., $\boldsymbol{\beta}_1 = \cdots = \boldsymbol{\beta}_p = \mathbf{0}$. For this usual MANOVA model, a global alternative relates to the non-homogeneity of the $\boldsymbol{\beta}_j$, i.e., $\operatorname{vec} \boldsymbol{\beta} \neq \mathbf{0}$. In the current study, we shall be mainly interested in some restricted alternatives. For the univariate model (i.e., $q = 1$), the most common form of such a restricted alternative is the so-called ordered alternative

$$H^{<}\colon \beta_1 \le \beta_2 \le \cdots \le \beta_p\,, \tag{1.3}$$

with at least one strict inequality. A general account of rank tests for ordered alternatives in randomized blocks is given in Chapter 7 of Puri and Sen (1971). Later on, De (1976) extended the methodology of Sen (1968) by effectively incorporating the union-intersection (UI-) principle of Roy (1953) to form an aligned rank test for H_0 against $H^{<}$ when the X_{ij} have i.i.d. error components. Boyd and Sen (1984) used the concept of locally most powerful rank (LMPR) tests along with the UI-principle, although the issue of asymptotic optimality of their proposed tests has not been addressed properly. For the ordered alternative problem, Araki and Shirahata (1981) and Shiraishi (1984) constructed some rank tests (for $q = 1$) based on the usual likelihood principle, and these may also be characterized as UI-LMPR tests. The main objectives of the current study are the following:

(i) For a general $q \ge 1$ and a general form of restricted alternative

$$H^*\colon \boldsymbol{\beta} \in \Gamma = \{\boldsymbol{\beta} \in E^{pq}\colon \boldsymbol{A} \operatorname{vec} \boldsymbol{\beta} \ge \mathbf{0},\ \boldsymbol{A} \in C(a,pq)\}\,, \tag{1.4}$$

where $C(a,pq)$ is the set of $a \times pq$ matrices of rank a: $1 \le a \le pq$, we characterize that the UI-LMPR tests have the same asymptotic optimality properties as the restricted likelihood ratio tests (when the scores are chosen appropriately). Note that (1.3) is a special case of (1.4).

(ii) We establish the asymptotic power-superiority of the ranking after alignment procedure to the intra-block ranking procedure, for a general class of restricted alternatives when the errors are homogeneous across the blocks.

UI-LMPR tests based on intra-block and aligned rankings are considered in Sections 2 and 3, and a relative picture of their asymptotic power properties is presented in the concluding section.

2. UI-LMPR tests based on intra-block rankings

We consider here an extension of rank MANOVA tests proposed by Gerig (1969). Let $R_{ij}^{(k)}$ be the rank of $X_{ij}^{(k)}$ among $X_{i1}^{(k)},\ldots,X_{ip}^{(k)}$, for $j=1,\ldots,p$; $k=1,\ldots,q$ and $n=1,\ldots,n$, and let

$$\boldsymbol{R}_i = \begin{pmatrix} R_{i1}^{(1)} & \cdots & R_{ip}^{(1)} \\ \vdots & \vdots & \vdots \\ R_{i1}^{(q)} & \cdots & R_{ip}^{(q)} \end{pmatrix}, \quad \text{for} \quad i=1,\ldots,n\,. \tag{2.1}$$

Let $\boldsymbol{R}_i^*$ be the reduced rank matrix obtained from $\boldsymbol{R}_i$ by permuting its columns in such a way that the top row is in the natural order, for $i=1,\ldots,n$, and let $S(\boldsymbol{R}^{**})=S(\boldsymbol{R}_1^*,\ldots,\boldsymbol{R}_n^*)$ be the set of $[(p!)^n]$ matrices which are (column-) permutationally equivalent to $\boldsymbol{R}^{**}=(\boldsymbol{R}_1^*,\ldots,\boldsymbol{R}_n^*)$. Since the $\boldsymbol{X}_{i1},\ldots,\boldsymbol{X}_{ip}$ are interchangeable r.v.'s (under H_0), the conditional (permutational) distribution of $(\boldsymbol{R}_1,\ldots,\boldsymbol{R}_n)$ over $S(\boldsymbol{R}^{**})$ is (discrete) uniform over the $(p!)^n$ possible realizations, and the corresponding probability measure is denoted by $\mathscr{P}^{(1)}$. For each $N=np$, let us consider the linear rank statistics

$$T_{Njl}^0 = N^{-1/2} \sum_{i=1}^{n} a_l(R_{ij}^{(l)})\,, \tag{2.2}$$

for $j=1,\ldots,p$; $l=1,\ldots,q$, where

$$a_l(r) = \mathscr{E}\{f_{[\cdot]l}^*(U_{(r)})\}, \qquad r=1,\ldots,p, \quad l=1,\ldots,q\,, \tag{2.3}$$

$U_{(1)},\ldots,U_{(p)}$ are the ordered r.v.'s of a sample of size p from the d.f. $F_{[l]}$ with the corresponding p.d.f. $f_{[l]}$, and $f_{[\cdot]l}^*(\cdot)$ is the usual log-derivative of $f_{[l]}$ and is defined as in (4.6) of Tsai and Sen (1987). Then, we may easily verify that

$$E\{T_{Njl}^0 \,|\, \mathscr{P}_n^{(1)}\} = 0, \qquad j=1,\ldots,p; \quad l=1,\ldots,q\,; \tag{2.4}$$

$$\operatorname{Cov}\{T_{Njl}^0, T_{Nj'l'}^0 \,|\, \mathscr{P}_n^{(1)}\} = (\delta_{jj'} - p^{-1})v_{Nll'}\,, \tag{2.5}$$

$j,j'=1,\ldots,p$; $l,l'=1,\ldots,q$, where $\delta_{jj'}$ is the usual Kronecker delta and $V_{N1}=((v_{Nll'}))$ is defined by

$$v_{Nll'} = \left[\sum_{i=1}^{n}\sum_{j=1}^{p} a_l(R_{ij}^{(l)})a_{l'}(R_{ij}^{(l')})\right] \Big/ \{n(p-1)\}\,, \tag{2.6}$$

$l,l'=1,\ldots,q$. Thus, writing $\boldsymbol{T}_{N1}=\operatorname{vec}\boldsymbol{T}_N^{0\prime}$, we have $E\{\boldsymbol{T}_{N1}\,|\,\mathscr{P}_n^{(1)}\}=\boldsymbol{0}$ and

$$E\{\boldsymbol{T}_{N1}\boldsymbol{T}'_{N1}|\mathscr{P}_n^{(1)}\} = \boldsymbol{V}_{N1} \otimes (\boldsymbol{I}_p - p^{-1}\boldsymbol{l}_p\boldsymbol{l}'_p) = \boldsymbol{\Sigma}_{N1}, \quad \text{say}, \tag{2.7}$$

where $\otimes$ denotes the usual Kronecker product.

We consider now a sequence $\{K_N: \boldsymbol{\beta} = N^{-1/2}\gamma, \gamma \text{ fixed}\}$ of alternative hypotheses, so that under K_N we have for each i (= 1,..., n),

$$F_{i[j]l}(x) = F_{i[\cdot]l}(x - N^{-1/2}\gamma_j^{(l)}), \quad j = 1,\ldots,p; \quad l = 1,\ldots,q; \tag{2.8}$$

$$F_{i[j]ll'}(x, y) = F_{i[\cdot]ll'}(x - N^{-1/2}\gamma_j^{(l)}, y - N^{-1/2}\gamma_j^{(l')}), \quad \forall j, l. \tag{2.9}$$

Moreover, we assume that the following limits exist:

$$F_{[\cdot]l}(x) = \lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} F_{i[\cdot]l}(x), \tag{2.10}$$

$$F_{[\cdot]ll'}(x, y) = \lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} F_{i[\cdot]ll'}(x, y). \tag{2.11}$$

We let

$$\nu_{ll'} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [-\dot{f}_{[\cdot]l}(x)/f_{[\cdot]l}(x)] \cdot [-\dot{f}_{[\cdot]l'}(y)/f_{[\cdot]l'}(y)]\, dF_{[\cdot]ll'}(x, y), \tag{2.12}$$

where $\dot{f}_{[\cdot]l}(x)$ denotes the derivative of p.d.f. $f_{[\cdot]l}(x)$. Then, under H_0 as well as $\{K_N\}$, following Lemma 7.3.10 of Puri and Sen (1971), we have

$$\boldsymbol{V}_{N1} \to \boldsymbol{V}_1 = ((\nu_{ll'}^{(1)})), \quad \text{in probability}, \quad \text{as} \quad N \to \infty, \tag{2.13}$$

$$\boldsymbol{T}_{N1} \xrightarrow[\mathscr{D}]{} \Phi_{pq}(\cdot\,; \boldsymbol{\Sigma}_1\boldsymbol{\lambda}, \boldsymbol{\Sigma}_1), \quad \boldsymbol{\lambda} = \text{vec}\,\gamma, \quad (\text{under } \{K_N\}), \tag{2.14}$$

where $\Phi_{pq}(\cdot\,; \boldsymbol{\Sigma}_1\boldsymbol{\lambda}, \boldsymbol{\Sigma}_1)$ denotes the normal distribution with mean vector $\boldsymbol{\Sigma}_1\boldsymbol{\lambda}$ and covariance matrix $\boldsymbol{\Sigma}_1$, with

$$\boldsymbol{\Sigma}_1 = \boldsymbol{V}_1 \otimes \left(\boldsymbol{I}_p - \frac{1}{p}\mathbf{1}_p\mathbf{1}'_p\right). \tag{2.15}$$

Based on $\boldsymbol{T}_{N1}$, we then can derive a suitable test statistic for testing H_0 against H^* defined in (1.4). First, we note that the set $\Gamma^* = \{\boldsymbol{\beta}; \boldsymbol{\beta} \geq 0\}$ is positively homogeneous in the sense that for every $\gamma \in \Gamma^*$ and $\delta > 0$, $\delta\gamma \in \Gamma^*$. So for a given $\gamma > \mathbf{0}$, by mimicking the proof of Theorem 4.2 of Tsai and Sen (1987), the LMPR test for testing H_0 against $H_\gamma: \boldsymbol{\beta} = \delta\gamma$, γ being fixed, is based on $\boldsymbol{\lambda}'\boldsymbol{B}(\boldsymbol{\Sigma}_{N1})\boldsymbol{\Sigma}_{N1}^{-1}\boldsymbol{T}_{N1}$, where $\boldsymbol{B}(\boldsymbol{\Sigma}_{N1})$ denotes the block diagonal matrix of $\boldsymbol{\Sigma}_{N1}$. Namely, we have

$$(2.16) \qquad \boldsymbol{B}(\boldsymbol{\Sigma}_{N1}) = \text{Diag}\, \boldsymbol{V}_{N1} \otimes \left(\boldsymbol{I}_p - \frac{1}{p} \mathbf{1}_p \mathbf{1}_p' \right),$$

where Diag $\boldsymbol{V}_{N1}$ is the diagonal matrix of $\boldsymbol{V}_{N1}$, and $\boldsymbol{\Sigma}_{N1}^{-1}$ stands for the generalized inverse matrix of $\boldsymbol{\Sigma}_{N1}$. Furthermore, for every $\gamma \in \Gamma$, we may write

$$(2.17) \qquad T_N(\gamma) = \boldsymbol{\lambda}'\boldsymbol{B}(\boldsymbol{\Sigma}_{N1})\boldsymbol{\Sigma}_{N1}^{-1}\boldsymbol{T}_{N1} / \{\boldsymbol{\lambda}'\boldsymbol{B}(\boldsymbol{\Sigma}_{N1})\boldsymbol{\Sigma}_{N1}^{-1}\boldsymbol{B}(\boldsymbol{\Sigma}_{N1})\boldsymbol{\lambda}\}^{1/2}.$$

By noting that $H^* = \bigcup_{\gamma \in \Gamma} H_\gamma$ and making the use of the UI-principle, the overall test statistic for H_0 versus H^* is granted as

$$(2.18) \qquad Q_{N1} = \sup \{T_N(\gamma), \gamma \in \Gamma\}.$$

For the computation of Q_{N1} in (2.18), we need to maximize $\boldsymbol{\lambda}'\boldsymbol{B}(\boldsymbol{\Sigma}_{N1}) \cdot \boldsymbol{\Sigma}_{N1}^{-1}\boldsymbol{T}_{N1}$ subject to $\boldsymbol{A}\boldsymbol{\lambda} \geq \mathbf{0}$ and $\boldsymbol{\lambda}'\boldsymbol{B}(\boldsymbol{\Sigma}_{N1})\boldsymbol{\Sigma}_{N1}^{-1}\boldsymbol{B}(\boldsymbol{\Sigma}_{N1})\boldsymbol{\lambda} = 1$. If we let $h(\boldsymbol{\lambda}) = -\boldsymbol{\lambda}'\boldsymbol{B}(\boldsymbol{\Sigma}_{N1})\boldsymbol{\Sigma}_{N1}^{-1}\boldsymbol{T}_{N1}$, $\boldsymbol{h}_1(\boldsymbol{\lambda}) = -\boldsymbol{A}\boldsymbol{\lambda}$ and $h_2(\boldsymbol{\lambda}) = \boldsymbol{\lambda}'\boldsymbol{B}(\boldsymbol{\Sigma}_{N1})\boldsymbol{\Sigma}_{N1}^{-1}\boldsymbol{B}(\boldsymbol{\Sigma}_{N1})\boldsymbol{\lambda} - 1$, then for this non-linear programming problem, the Kuhn-Tucker-Lagrange (K.T.L.) point formula theorem can be used to arrive at the following result: Let

$$(2.19) \qquad \boldsymbol{U}_{N1} = \boldsymbol{A}\boldsymbol{B}^{-1}(\boldsymbol{\Sigma}_{N1})\boldsymbol{T}_{N1}$$

and

$$(2.20) \qquad \boldsymbol{\Delta}_{N1} = \boldsymbol{A}\boldsymbol{B}^{-1}(\boldsymbol{\Sigma}_{N1})\boldsymbol{\Sigma}_{N1}\boldsymbol{B}^{-1}(\boldsymbol{\Sigma}_{N1})\boldsymbol{A}'.$$

Also, let J be a subset of $\mathscr{A}_0 = \{1, \ldots, a\}$ and J' be its complement. For each of the 2^a sets J, we partition $\boldsymbol{U}_{N1}$ and $\boldsymbol{\Delta}_{N1}$ (following rearrangements, if necessary) as

$$(2.21) \qquad \boldsymbol{U}_{N1} = \begin{bmatrix} \boldsymbol{U}_{N1(J)} \\ \boldsymbol{U}_{N1(J')} \end{bmatrix} \begin{matrix} k(J) \\ k(J') \end{matrix} \quad \text{and}$$

$$\boldsymbol{\Delta}_{N1} = \begin{bmatrix} \boldsymbol{\Delta}_{N1(JJ)} & \boldsymbol{\Delta}_{N1(JJ')} \\ \boldsymbol{\Delta}_{N1(J'J)} & \boldsymbol{\Delta}_{N1(J'J')} \end{bmatrix}$$

where $k(J)$ denotes the cardinality of set J. Also, for each J ($\varnothing \subseteq J \subseteq A_0$), we let

$$(2.22) \qquad \boldsymbol{U}_{N1(J:J')} = \boldsymbol{U}_{N1(J)} - \boldsymbol{\Delta}_{N1(JJ')}\boldsymbol{\Delta}_{N1(J'J')}^{-1}\boldsymbol{U}_{N1(J')},$$

$$(2.23) \qquad \boldsymbol{\Delta}_{N1(JJ:J')} = \boldsymbol{\Delta}_{N1(JJ)} - \boldsymbol{\Delta}_{N1(JJ')}\boldsymbol{\Delta}_{N1(J'J')}^{-1}\boldsymbol{\Delta}_{N1(J'J)}.$$

Then, we have

$$Q_{N1}^2 = T'_{N1}\Sigma_{N1}^{-1}T_{N1} - U'_{N1}\Delta_{N1}^{-1}U_{N1} + \sum_{\varnothing \subseteq J \subseteq A_0} \{U'_{N1(J:J')}\Delta_{N1(JJ:J')}^{-1} U_{N1(J:J')}\} \cdot 1\{U_{N1(J:J')} > \mathbf{0}, \Delta_{N1(J'J')}^{-1}U_{N1(J')} \le \mathbf{0}\}, \tag{2.24}$$

where $1(B)$ stands for the indicator function of the set B.

It is well-known (viz., Robertson *et al.* (1988)) that for restricted alternatives, optimal tests are hard to construct (in general); the likelihood ratio test (for restricted alternatives) in general has good power properties, although tests which are asymptotically most stringent (and somewhere most powerful) may not agree with such a likelihood ratio test. By restricted likelihood ratio test (asymptotic) optimality (RLRTAO), we mean the optimality properties enjoyed by the restricted likelihood ratio test in such a testing problem. Then, following the general results in Tsai and Sen (1987), it can be shown that within the class of intra-block rank tests, with the scores defined in (2.3), Q_{N1} in (2.18) has the RLRTAO property for the general class of (contiguous) restricted alternatives of the type in (1.4). This RLRTAO property also applies to the ordered alternative problem as a special case.

3. UI-LMPR tests and ranking after alignment

In intra-block ranking, because of the lack of information from the inter-block comparisons, the tests are generally less efficient (when the errors are homogeneous) than the aligned rank tests which incorporate the inter-block information through the alignment procedure. For the standard MANOVA model in two-way layouts, Sen (1968) formulated aligned rank tests based on general scores and studied their asymptotic efficiency in a unified manner. Here, we extend the results to test against general forms of restricted alternatives (as in (1.4)) and show that under fairly general regularity conditions, aligned rankings lead to more efficient tests for such alternatives too. To do this, we need to eliminate the (nuisance) block effects by simple alignment procedures, namely, we substract suitable estimates of the block effects (vectors) from the respective $X_{ij}^{(l)}$ and on the residuals, we make an overall ranking (ignoring blocks) of all the treatments (in a coordinatewise manner). We may use any translation-equivariant estimator of the block effects; for simplicity, we take them as the block averages. Thus, we define the aligned random vectors as

$$Y_{ij} = X_{ij} - \frac{1}{p}\sum_{j=1}^{p} X_{ij}, \quad i = 1,\ldots,n; \quad j = 1,\ldots,p. \tag{3.1}$$

For convenience, we assume that $\boldsymbol{Y}_{ij}$ has a q-variate continuous c.d.f. F_{ij}^0, $\forall\, i = 1,\dots,n,\ j = 1,\dots,p$. Let $S_{ij}^{(l)}$ be the rank of $Y_{ij}^{(l)}$ among the N ($= np$) observations $Y_{11}^{(l)}, Y_{12}^{(l)},\dots, Y_{np}^{(l)}$ for $j = 1,\dots,p$, $i = 1,\dots,n$ and $l = 1,\dots,q$. Thus, corresponding to the aligned observation $\boldsymbol{Y}_{ij}$, we have a rank vector $\boldsymbol{S}_{ij} = (S_{ij}^{(1)},\dots, S_{ij}^{(q)})'$, $i = 1,\dots,n, j = 1,\dots,p$. We also define the rank collection matrix $\boldsymbol{S}_N$ by $(\boldsymbol{S}_{11},\dots,\boldsymbol{S}_{np})$. Note that under H_0, $\boldsymbol{Y}_{i1}, \boldsymbol{Y}_{i2},\dots, \boldsymbol{Y}_{ip}$ are interchangeable random vectors, so the joint distribution of $\boldsymbol{Y}_N = (\boldsymbol{Y}_{11}',\dots, \boldsymbol{Y}_{1p}',\dots, \boldsymbol{Y}_{np}')$ remains invariant under the finite group $\mathscr{G}_n$ of transformation $\{g_n^0\}$ (which maps the sample space onto itself). Thus, for any $g_n^0 \in \mathscr{G}_n$, there exists $\boldsymbol{Y}_n^* = g_n^0 \boldsymbol{Y}_N$ which is permutationally equivalent to $\boldsymbol{Y}_N$. If we denote $\boldsymbol{S}_N^*$ the rank collection matrix corresponding to $\boldsymbol{Y}_N^*$, then $\boldsymbol{S}_N^* = g_n^0 \boldsymbol{S}_N$ and is permutationally equivalent to $\boldsymbol{S}_N$. Thus, under H_0, the conditional distribution of $\boldsymbol{S}_N$ over the $(p!)^n$ realizations $\{\boldsymbol{S}_N^* = g_n^0 \boldsymbol{S}_N;\ g_n^0 \in \mathscr{G}_n\}$ is uniform, each realization having the conditional probability $(p!)^{-n}$; we denote this conditional probability measure by $\mathscr{P}_n^{(2)}$. Also, as in (2.3), we define the scores $a_{Nl}^0(r)$, $r = 1,\dots, N$, $l = 1,\dots, q$ with the only change that here $U_{(1)},\dots, U_{(N)}$ stand for the order statistics of a sample of size N from the d.f. $F_{[j]l}^0$. Then, the tests to be considered are based on statistic $\boldsymbol{T}_{N2} = \operatorname{vec} \boldsymbol{T}_N^*$ with

$$\boldsymbol{T}_N^* = \left(N^{-1/2} \sum_{\alpha=1}^{N} Z_{N\alpha j}^{(l)} a_{Nl}^0(\alpha) \right), \tag{3.2}$$

where $Z_{N\alpha j}^{(l)} = 1$ if the α-th smallest observation among the N values of $Y_{ij}^{(l)}$ is from the j-th treatment and l-th variate and $Z_{N\alpha j}^{(l)} = 0$, otherwise, for $\alpha = 1,\dots, N$, $j = 1,\dots,p$ and $l = 1,\dots, q$. Define $\boldsymbol{V}_{N2}$, $\boldsymbol{\Sigma}_{N2}$, $\boldsymbol{U}_{N2}$, $\boldsymbol{\Delta}_{N2}$ and Q_{N2}^2 the same as in $\boldsymbol{V}_{N1}$, $\boldsymbol{\Sigma}_{N1}$, $\boldsymbol{U}_{N1}$, $\boldsymbol{\Delta}_{N1}$ and Q_{N1}^2, respectively, with $\boldsymbol{T}_{N1}$ being replaced by $\boldsymbol{T}_{N2}$. Furthermore, consider a restricted (contiguous) alternative $\{K_N\}$, then we have

$$F_{i[j]l}^0(x) = F_{i[\cdot]l}^0(x - N^{-1/2}\gamma_j^{(l)}) \tag{3.3}$$

and

$$F_{i[j]ll'}^0(x, y) = F_{i[\cdot]ll'}^0(x - N^{-1/2}\gamma_j^{(l)}, y - N^{-1/2}\gamma_j^{(l')}) . \tag{3.4}$$

Finally, we define $\boldsymbol{V}_2$ and $\boldsymbol{\Sigma}_2$ the same as in $\boldsymbol{V}_1$ and $\boldsymbol{\Sigma}_1$ with $F_{i[\cdot]l}(x)$ and $F_{i[\cdot]ll'}(x, y)$ being replaced by $F_{i[\cdot]l}^0$ and $F_{i[\cdot]ll'}^0(x, y)$, respectively. Then under parallel arguments as in the previous section, we have

$$\boldsymbol{T}_{N2} \xrightarrow[\{K_N\}]{\mathscr{D}} \Phi_{pq}(\,\cdot\,, \boldsymbol{\Sigma}_2\boldsymbol{\lambda}, \boldsymbol{\Sigma}_2) . \tag{3.5}$$

Then, for testing (1.2) against (1.4) (under $\{K_N\}$), the UI-LMPR test Q_{N2}^2 is

RLRTAO within the class of aligned rank tests at the respective level of significance. As a result, this RLRTAO property remains ascribable for the special case of the ordered alternatives in (1.3).

4. Asymptotic power comparison of Q_{N1}^2 and Q_{N2}^2

Note that the optimal properties of Q_{N1}^2 relate to intra-block ranking methods which sacrifice the inter-block informations to a greater extent. The test based on Q_{N2}^2 is based on the aligned ranks, and hence is expected to perform better than Q_{N1}^2. For testing against global alternatives, the usual power superiority of aligned rank tests to the intra-block rank tests has been studied by a host of workers (viz. Sen (1968) where other references are cited). In the current context, we have a general class of restricted alternatives (which are not describable in terms of linear restraints) where the conventional techniques (described in Chapter 7 of Puri and Sen (1971)) may not work out neatly. In the following theorem, the power superiority of the Q_{N2}^2 test to the Q_{N1}^2 test is established for a subspace of the restricted parameter space under $\{K_N\}$. Note that over the complementary subspace, although the same picture is likely to hold, the current method of attack may not work out.

THEOREM 4.1. *Suppose that the critical levels $x_\alpha^{(k)}$, $k = 1, 2$, satisfy*

$$\lim_{N\to\infty} P\{Q_{Nk}^2 > x_\alpha^{(k)} \mid H_0\} = \alpha \quad (0<\alpha<1), \quad \textit{for} \quad k = 1, 2 . \tag{4.1}$$

Define $\Delta_k = \lim_{N\to\infty} E\{\Delta_{Nk} \mid H_0\}$, $\boldsymbol{\eta}_k = \boldsymbol{A}\boldsymbol{B}^{-1}(\boldsymbol{\Sigma}_k)\boldsymbol{\Sigma}_k\boldsymbol{\lambda}$, $k = 1, 2$, *and let* $\boldsymbol{\mu}^{(k)} = \Delta_k^{-1/2}\boldsymbol{\eta}_k$ *and* $\beta_k(\gamma, \boldsymbol{\Sigma}_k) = \lim_{N\to\infty} P\{Q_{Nk}^2 \geq x_\alpha^{(k)} \mid K_N\}$, *for* $k = 1, 2$. *Then*

$$\beta_1(\gamma, \boldsymbol{\Sigma}_1) \leq \beta_2(\gamma, \boldsymbol{\Sigma}_2) \tag{4.2}$$

whenever $\boldsymbol{\mu}^{(2)} \geq \boldsymbol{\mu}^{(1)}$ *and* $\gamma \in \Omega_0 \cap \Gamma$, *where*

$$\Omega_0 = \left\{ \boldsymbol{\mu}^{(1)} \in E^{+a} \colon \mu_j^{(1)} \geq \frac{1}{2} (x_\alpha^{(1)})^{1/2}, j = 1, \ldots, a \right\} . \tag{4.3}$$

OUTLINE OF THE PROOF. For every i ($= 1, \ldots, n$), let us define

$$\Lambda_{1i} = \operatorname{Var}(X_{i1} \mid H_0) \quad \text{and} \quad \Lambda_{2i} = \operatorname{Cov}(X_{i1}, X_{i2} \mid H_0) . \tag{4.4}$$

Note that by (2.10) and (2.11), the following limits exist:

$$\Lambda_1 = \lim_{n\to\infty} \left\{ n^{-1} \sum_{i=1}^{n} \Lambda_{1i} \right\} \quad \text{and} \quad \Lambda_2 = \lim_{n\to\infty} \left\{ n^{-1} \sum_{i=1}^{n} \Lambda_{2i} \right\} . \tag{4.5}$$

From (2.14) and (3.5), we have then

$$V_1^{-1} = \Lambda_1 - \Lambda_2 \quad \text{and} \qquad (4.6)$$
$$V_2^{-1} = p^{-1}(p-1)(\Lambda_1 - \Lambda_2) ;$$

$$\Sigma_1 = p^{-1}(p-1)\Sigma_2, \qquad \Delta_1 = (p-1)^{-1} p \Delta_2 \quad \text{and} \qquad (4.7)$$
$$\eta_1 = \eta_2 = \eta, \quad \text{say} .$$

Next, we define

$$(4.8) \quad Q_k^2 = \sum_{\varnothing \subseteq J \subseteq A_0} \{Z'_{k(J:J')} \Delta_{k(JJ:J')}^{-1} Z_{k(J:J')}\} 1(Z_{k(J:J')} > 0, \Delta_{k(J'J')}^{-1} Z_{k(J')} \leq 0) ,$$

for $k = 1,2$, where $Z_k \sim \Phi_a(\cdot\,; \eta, \Delta_k)$, $k = 1,2$.

The first two terms on the right hand side of (2.24) cancel whenever Rank $(A) = pq$ (i.e., $a = pq$). For $a < pq$, this difference is asymptotically independent of the last term (sum), and under the sequence of local alternatives, it has asymptotically a noncentral chi square distribution with $pq - a$ degrees of freedom and a noncentrality parameter, say Θ_1; let us denote this asymptotic r.v. by Z_1^*. We define Z_2^* similarly for the same component of Q_{N2}^2 (the aligned rank statistic), so that Z_2^* has asymptotically a noncentral chi-square distribution with $pq - a$ degrees of freedom and noncentrality parameter Θ_2. It is easy to show that $\Theta_1 = p^{-1}(p-1)\Theta_2$, so that $\Theta_1 \leq \Theta_2$. Hence, Z_2^* is stochastically larger than Z_1^*. Thus, to establish (4.2), we need to show that

$$(4.9) \qquad P\{Q_2^2 + Z_2^* \geq x_\alpha\} \geq P\{Q_1^2 + Z_1^* \geq x_\alpha\} ,$$

where Z_k^* is independent of Q_k^2, for $k = 1,2$. Note that the Z_k^* and Q_k^2 are all nonnegative r.v.'s and further, Z_2^* is stochastically larger than Z_1^*. Hence, to verify (4.9), it suffices to show that if $P\{Q_k^2 \geq x_\alpha^{(k)} | H_0\} = \alpha$, $k = 1,2$, then $P\{Q_1^2 \geq x_\alpha^{(1)} | K_N\} \leq P\{Q_2^2 \geq x_\alpha^{(2)} | K_N\}$. Without any loss of generality, we assume that $\Delta_1 = I$, and note that

$$(4.10) \quad P\{Q_2^2 \geq x_\alpha^{(2)} | K_N\} - P\{Q_1^2 \geq x_\alpha^{(1)} | K_N\}$$
$$= \sum_{\varnothing \subseteq J \subseteq A_0} \left[\int_{B_2(J)} d\Phi_a\left(z; \left(\frac{p}{p-1} \right)^{1/2} \eta, I \right) - \int_{B_1(J)} d\Phi_a(z; \eta, I) \right]$$

where

$$(4.11) \qquad B_k(J) = \{z \in E^a; z_{J'} \leq 0, z_J > 0, \|z_J\|^2 \geq x_\alpha^{(k)}\}$$

$\forall\, k = 1, 2$ with $\|\cdot\|$ denoting the Euclidean norm. Since $P\{Q_k^2 \geq x_\alpha^{(k)} | H_0\} = \alpha$, $k = 1, 2$, it is clear from (4.10) and (4.11) that $x_\alpha^{(1)} = x_\alpha^{(2)}$. Furthermore, we write $x_\alpha^{(1)} = x_\alpha$ and $\boldsymbol{\eta}^\delta = \boldsymbol{\eta} + \boldsymbol{\delta}$, where $\boldsymbol{\delta} > \mathbf{0}$. For $\boldsymbol{\eta} \in \Omega_0 \cap \Gamma$,

(i) if $a = 1$, then we have

$$\text{(4.12)} \quad \sum_{\varnothing \subseteq J \subseteq A_0} \int_{B_1(J)} [d\Phi_a(\boldsymbol{z}; \boldsymbol{\eta}^\delta, \boldsymbol{I}) - d\Phi_a(\boldsymbol{z}; \boldsymbol{\eta}, \boldsymbol{I})]$$
$$= \left(\frac{1}{2\pi}\right)^{1/2} \left\{ \int_{z \leq 0} [e^{-(z-\eta^\delta)^2/2} - e^{-(z-\eta)^2/2}] dz + \int_{z > \sqrt{x_\alpha}} [e^{-(z-\eta^\delta)^2/2} - e^{-(z-\eta)^2/2}] dz \right\} \geq 0 \, ;$$

(ii) if $a = 2$, by regrouping the sums and symmetric arguments, we have

$$\text{(4.13)} \quad \sum_{\varnothing \subseteq J \subseteq A_0} \int_{B_1(J)} [d\Phi_a(\boldsymbol{z}; \boldsymbol{\eta}^\delta, \boldsymbol{I}) - d\Phi_a(\boldsymbol{z}; \boldsymbol{\eta}, \boldsymbol{I})]$$
$$= \frac{1}{2\pi} \left\{ \int_{\sqrt{x_\alpha} - \eta_1 - \delta_1}^{\sqrt{x_\alpha} - \eta_1} \int_{-\infty}^{-\eta_2 - \delta_2} e^{-\|z\|^2/2} dz_2 dz_1 - \int_{-\eta_1 - \delta_1}^{-\eta_1} \int_{-\infty}^{-\eta_2 - \delta_2} e^{-\|z\|^2/2} dz_2 dz_1 \right.$$
$$\left. + \int_{\sqrt{x_\alpha} - \delta_2 - \eta_2}^{\sqrt{x_\alpha} - \eta_2} \int_{-\infty}^{-\delta_1 - \eta_1} e^{-\|z\|^2/2} dz_1 dz_2 - \int_{-\eta_2 - \delta_2}^{-\eta_2} \int_{-\infty}^{-\eta_1 - \delta_1} e^{-\|z\|^2/2} dz_1 dz_2 \right\}$$
$$+ \text{ a nonnegative quantity}$$
$$\geq 0 \, ,$$

where the nonnegative quantity corresponds to $J = \{1, 2\}$. Thus, by induction, we obtain that for every $a \geq 1$, (4.10) is nonnegative for $\gamma \in \Omega_0 \cap \Gamma$, and hence, the proof of the theorem is complete.

Remark. For $a = 2$, let $\Omega = \{\boldsymbol{\mu}^{(1)} \in E^2; \mu_1^{(1)} \geq (x_\alpha^{(1)})^{1/2}/2, \mu_2^{(1)} \leq 0\} \cup \{\boldsymbol{\mu}^{(1)} \in E^2; \mu_1^{(1)} \leq 0, \mu_2^{(1)} \geq (x_\alpha^{(1)})^{1/2}/2\} \cup \{\boldsymbol{\mu}^{(1)} \in E^2; \mu_1^{(1)} \leq 0, \mu_2^{(1)} \leq 0\} \cup \Omega_0$, then under parallel arguments as in (ii) of Theorem 4.1, we have

$$\text{(4.14)} \quad \beta_1(\gamma, \boldsymbol{\Sigma}_1) \leq \beta_2(\gamma, \boldsymbol{\Sigma}_2) \, ,$$

whenever $\boldsymbol{\mu}^{(2)} \geq \boldsymbol{\mu}^{(1)}$ and $\gamma \in \Omega \cap \Gamma$. A generalization of (4.14) for the case of $a \geq 3$ constitutes an open problem.

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