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ON TESTS FOR THE MEAN DIRECTION OF THE LANGEVIN DISTRIBUTION

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Abstract. The asymptotic expansions of the distribution of a sum of independent random vectors with Langevin distribution are given. The power functions of the likelihood ratio criterion, Watson statistic, Rao statistic and the modified Wald statistic for testing the hypothesis of the mean direction are obtained asymptotically and a numerical comparison is made.

Key words and phrases: Asymptotic expansion, central limit theorem, Langevin distribution, likelihood ratio criterion, Watson statistic, Rao statistic, modified Wald statistic.

1. Introduction

A p-dimensional random vector x is said to have the Langevin (or von Mises-Fisher) distribution if its density function on the sphere $S_p = \{x | x \in R^p, x'x = \|x\|^2 = 1\}$ is given by

$$\exp(\kappa\mu'x)/a_p(\kappa) \tag{1.1}$$

where $\mu'\mu = 1$ and $\kappa > 0$, and

$$a_p(\kappa) = (2\pi)^{p/2} I_{p/2-1}(\kappa)/\kappa^{p/2-1}$$

with $I_\nu(\kappa)$ denoting the modified Bessel function of the first kind and order ν.

Watson (1983) proposed some test statistics for testing parameters κ and μ and considered the limiting behaviors of these under the null hypothesis and Pitman's alternatives, respectively. Chou (1986) studied the asymptotic expansions of the distribution of the Watson statistic for testing the mean vector and its power function.

In this paper we consider the asymptotic expansions of the distributions of the likelihood ratio criterion, Watson statistic, Rao statistic and a modified Wald statistic for testing the mean direction vector, and some numerical comparisons for the powers of these statistics are obtained.

2. Central limit theorem

Let x be a random vector with the probability density function (1.1), then the characteristic function of x is given by

$$\psi(t) = E[\exp(it'x)] = a_p(\omega)/a_p(\kappa)\,,$$
$$\omega = (\kappa^2 - t't + 2it'\mu\kappa)^{1/2}\,.$$

This implies that the mean vector and the covariance matrix are given as

$$E[x] = A_p(\kappa)\mu\,,$$
$$\Omega = E[xx'] - E[x]E[x'] = A_p'(\kappa)\mu\mu' + \frac{A_p(\kappa)}{\kappa}(I - \mu\mu')\,,$$

where

$$A_p(\kappa) = \frac{d}{d\kappa}\log a_p(\kappa) \quad \text{and} \quad A_p'(\kappa) = \frac{d}{d\kappa}A_p(\kappa)\,.$$

For simplicity we denote $A_p(\kappa)$ as A.

Now let $x_1, x_2, \ldots, x_n$ be a random sample from a population with the Langevin distribution. Then the characteristic function of $y = \sqrt{n}(\bar{x} - A\mu)$, $\bar{x} = \sum x_i/n$, is expressed as

$$\phi_y(t) = \exp(-in^{1/2}At'\mu)[a_p(\tilde{\omega})/a_p(\kappa)]^n$$

where

$$\tilde{\omega} = (\kappa^2 - n^{-1}t't + 2in^{-1/2}\kappa t'\mu)^{1/2}\,.$$

Expanding $\phi_y(t)$ and inverting the characteristic function with lengthy algebra, we have the asymptotic expansion of the joint density function of y formally as

$$f(y) = (2\pi)^{-p/2}|\Omega|^{-1/2}\exp\left(-\frac{1}{2}y'\Omega^{-1}y\right) \cdot\{1 + n^{-1/2}F_1(\mu) + n^{-1}F_2(\mu) + o(n^{-1})\}\,, \tag{2.1}$$

where

$$F_1(\mu) = b_{10}\mu'y + b_{30}(\mu'y)^3 + b_{11}\mu'y \cdot y'y\,,$$

$$F_2(\mu) = b_{00} + b_{20}(\mu'y)^2 + b_{40}(\mu'y)^4 + b_{60}(\mu'y)^6 + b_{01}(y'y) + b_{02}(y'y)^2 + b_{21}(\mu'y)^2 y'y + b_{41}(\mu'y)^4 y'y + b_{22}(\mu'y)^2(y'y)^2$$

and

$$b_{10} = \frac{1}{2}(p-1)\left(\frac{1}{A'\kappa} - \frac{1}{A}\right) - \frac{1}{2}\frac{A''}{(A')^2}\,,$$

$$b_{30} = \frac{1}{2}\frac{1}{AA'} - \frac{1}{2}\frac{\kappa}{A^2} + \frac{1}{6}\frac{A''}{(A')^3}\,,$$

$$b_{11} = -\frac{1}{2}\frac{1}{A'A} + \frac{1}{2}\frac{\kappa}{A^2}\,,$$

$$b_{00} = \frac{1}{8}\frac{A'''}{(A')^2} - \frac{5}{24}\frac{(A'')^2}{(A')^3} - \frac{1}{8}(p-1)(p-3)\left(\frac{1}{A'\kappa} - \frac{1}{A\kappa}\right) + \frac{1}{4}(p-1)\frac{A''}{(A')^2\kappa}\,,$$

$$b_{20} = -\frac{1}{4}\frac{A'''}{(A')^3} + \frac{5}{8}\frac{(A'')^2}{(A')^4} + \frac{1}{8}(p-1)(p+3)\frac{1}{A^2} + \frac{1}{4}p\frac{A''}{(A')^2A} - \frac{1}{4}p(p-1)\frac{1}{A'A\kappa} - \frac{1}{2}(p-1)\frac{A''}{(A')^3\kappa} + \frac{1}{8}(p-1)(p-3)\frac{1}{(A')^2\kappa^2}\,,$$

$$b_{40} = \frac{1}{24}\frac{A'''}{(A')^4} + \frac{1}{4}(p-1)\frac{1}{(A')^2A\kappa} - \frac{1}{8}(4p+1)\frac{1}{A'A^2} + \frac{1}{4}\frac{A''\kappa}{(A')^2A^2} + \frac{1}{8}(2p+3)\frac{\kappa}{A^3} - \frac{5}{24}\frac{(A'')^2}{(A')^5} + \frac{1}{12}(p-1)\frac{A''}{(A')^4\kappa} - \frac{1}{12}(p+5)\frac{A''}{(A')^3A}\,,$$

$$b_{60} = \frac{1}{72}\frac{(A'')^2}{(A')^6} + \frac{1}{12}\frac{A''}{(A')^4A} + \frac{1}{8}\frac{1}{(A')^2A^2} - \frac{1}{12}\frac{A''\kappa}{(A')^3A^2} - \frac{1}{4}\frac{\kappa}{A'A^3} + \frac{1}{8}\frac{\kappa^2}{A^4}\,,$$

$$b_{01} = -\frac{1}{4}\frac{A''}{(A')^2A} + \frac{1}{4}(p-1)\left(\frac{1}{A'A\kappa} - \frac{1}{A^2}\right),$$

$$b_{02} = \frac{1}{8}\left(\frac{\kappa}{A^3} - \frac{1}{A'A^2}\right),$$

$$b_{21} = \frac{1}{4}(2p+1)\frac{1}{A'A^2} + \frac{1}{2}\frac{A''}{(A')^3A} - \frac{1}{4}(p-1)\frac{1}{(A')^2A\kappa} - \frac{1}{4}\frac{A''\kappa}{(A')^2A^2},$$

$$b_{41} = -\frac{1}{12}\frac{A''}{(A')^4A} - \frac{1}{4}\frac{1}{(A')^2A^2} + \frac{1}{12}\frac{A''\kappa}{(A')^3A^2} + \frac{1}{2}\frac{\kappa}{A'A^3} - \frac{1}{4}\frac{\kappa^2}{A^4},$$

$$b_{22} = \frac{1}{8}\frac{1}{(A')^2A^2} - \frac{1}{4}\frac{\kappa}{A'A^3} + \frac{1}{8}\frac{\kappa^2}{A^4}.$$

Chou (1986) considered a similar problem and gave an asymptotic expression for the normalized vector as $z = \Omega^{-1/2}y = u + vn^{-1/2} + wn^{-1} + o_p(n^{-1})$, where u is a p-dimensional normal random vector with mean 0 and covariance matrix I_p, and v and w are functions of u, respectively. For the case $\mu = (\mu_0 + n^{-1/2}\delta)/\|\mu_0 + n^{-1/2}\delta\|$, $\mu_0'\delta = 0$, (2.1) is expressed as follows.

$$f_n(y) = (2\pi)^{-p/2}|\Omega_0|^{-1/2}\exp\left(-\frac{1}{2}y'\Omega_0^{-1}y\right) \cdot\{1 + n^{-1/2}\tilde{F}_1(\mu_0) + n^{-1}\tilde{F}_2(\mu_0) + o(n^{-1})\} \tag{2.2}$$

where

$$\Omega_0 = A'\mu_0\mu_0' + \frac{A}{\kappa}(I - \mu_0\mu_0')$$

and

$$\tilde{F}_1(\mu_0) = \left(\frac{\kappa}{A} - \frac{1}{A'}\right)\mu_0'y\cdot\delta'y + F_1(\mu_0),$$

$$\tilde{F}_2(\mu_0) = \frac{1}{2}\left(\frac{\kappa}{A} - \frac{1}{A'}\right)^2(\mu_0'y)^2(\delta'y)^2$$

$$-\frac{1}{2}\left(\frac{\kappa}{A}-\frac{1}{A'}\right)\{\|\delta\|^2(\mu_0'y)^2-(\delta'y)^2\}$$

$$+\left(\frac{\kappa}{A}-\frac{1}{A'}\right)\mu_0'y\cdot\delta'yF_1(\mu_0)$$

$$+\delta'y\{b_{10}+3b_{30}(\mu_0'y)^2+b_{11}y'y\}+F_2(\mu_0)\,.$$

3. Asymptotic expansions of some test statistics

Here we consider testing hypothesis H: $\mu=\mu_0$ (a given vector, $\|\mu_0\|=1$) against K: $\mu\neq\mu_0$. Let $x_1, x_2,\ldots, x_n$ be a random sample from a Langevin distribution with parameters μ and κ. When κ is unknown, the likelihood ratio criterion is given by

$$\lambda=\{a_p(\hat{\kappa})/a_p(\tilde{\kappa})\}^n\exp\{n(\tilde{\kappa}\mu_0'\bar{x}-\hat{\kappa}\|\bar{x}\|)\}$$

where $\tilde{\kappa}$ is the maximum likelihood estimator of κ under H and it satisfies $A_p(\tilde{\kappa})=\mu_0'\bar{x}$, and $\hat{\kappa}$ is the maximum likelihood estimator of κ under K and it satisfies $A_p(\hat{\kappa})=\|\bar{x}\|$. Then $L1=-2\log\lambda$ is expressed asymptotically as

$$L1=\frac{\kappa}{A}\{y'y-(\mu_0'y)^2\}+\frac{1}{\sqrt{n}}\left(\frac{1}{A'A}-\frac{\kappa}{A^2}\right)\mu_0'y\{y'y-(\mu_0'y)^2\}$$

$$+\frac{1}{n}\left\{\frac{1}{4}\left(\frac{1}{A'A^2}-\frac{\kappa}{A^3}\right)(y'y)^2\right.$$

$$-\frac{1}{2}\left(\frac{3}{A'A^2}-\frac{3\kappa}{A^3}+\frac{A''}{(A')^3A}\right)(\mu_0'y)^2y'y$$

$$\left.+\left(\frac{5}{4}\frac{1}{A'A^2}-\frac{5}{4}\frac{\kappa}{A^3}+\frac{1}{2}\frac{A''}{(A')^3A}\right)(\mu_0'y)^4\right\}+o_p\left(\frac{1}{n}\right).$$

Calculating a characteristic function of $L1$ by use of (2.1) for $\mu=\mu_0$ and inverting it, we have the asymptotic expansion of the distribution of $L1$ under H as follows.

(3.1) $$\Pr(L1\leq x)=\Pr(\chi^2_{p-1}\leq x)+n^{-1}d\{\Pr(\chi^2_{p+1}\leq x)-\Pr(\chi^2_{p-1}\leq x)\}+o(n^{-1})$$

where

$$d=\frac{1}{8}(p-1)(p-3)\left(\frac{1}{A'\kappa^2}-\frac{1}{A\kappa}\right)-\frac{1}{4}(p-1)\frac{A''}{(A')^2\kappa}$$

and χ_f^2 is a chi-squared random variable with f degrees of freedom. This implies that the Bartlett correction factor for $L1$ is given by

$$\rho = 1 - \frac{1}{n}\left\{\frac{1}{4}(p-3)\left(\frac{1}{A'\kappa^2} - \frac{1}{A\kappa}\right) - \frac{1}{2}\frac{A''}{(A')^2\kappa}\right\}.$$

When κ is known as $\kappa = \kappa_0$, the likelihood ratio criterion for testing hypothesis H: $\mu = \mu_0$, against the alternative K: $\mu \neq \mu_0$ is given as

$$\lambda_0 = \exp\{\kappa_0 n(\mu_0'\bar{x} - \|\bar{x}\|)\}\ .$$

The distribution of $L2 = -2\log\lambda_0$ is expressed asymptotically as

$$\begin{aligned}(3.2)\qquad \Pr(L2 \leq x) = {} & \Pr(\chi^2_{p-1} \leq x) \\ & - \frac{1}{8}(p-1)(p-3)\frac{1}{A_0\kappa_0}\frac{1}{n} \\ & \cdot\{\Pr(\chi^2_{p+1} \leq x) - \Pr(\chi^2_{p-1} \leq x)\} \\ & + o(n^{-1})\ ,\end{aligned}$$

where A_0 stands for $A_p(\kappa_0)$.

Watson (1983) proposed other test statistics for testing this hypothesis:

$$W1 = \frac{n\hat{\kappa}}{A(\hat{\kappa})}\|(I - \mu_0\mu_0')\bar{x}\|^2 \quad : \quad \kappa \text{ is unknown}\ ,$$

$$W2 = \frac{n\kappa_0}{A_0}\|(I - \mu_0\mu_0')\bar{x}\|^2 \quad : \quad \kappa = \kappa_0 \text{ is known}\ .$$

Chou (1986) gave the asymptotic expansions of the distributions of $W1$ and $W2$ up to the order n^{-1} under the null hypothesis.

Rao (1948) proposed a test statistic under a more general set up, which is known as the Rao statistic or score statistic. Hayakawa and Puri (1985) also proposed a modified Wald statistic. For a Langevin population, a Rao statistic R and a modified Wald statistic MW for unknown κ are expressed as follows:

$$(3.3)\qquad R = \frac{n\tilde{\kappa}}{A(\tilde{\kappa})}\|(I - \mu_0\mu_0')\bar{x}\|^2\ ,$$

$$(3.4)\qquad MW = nA_p(\tilde{\kappa})\tilde{\kappa}(\hat{\mu}_2 - \mu_{20})'(I_{p-1} - \mu_{20}\mu_{20}')^{-1}(\hat{\mu}_2 - \mu_{20})\ ,$$

where $\mu_0' = (\mu_{10}, \mu_{20}')$, $\bar{x}' = (\bar{x}_1,\ \bar{x}_2')$ and $\hat{\mu}' = (\hat{\mu}_1, \hat{\mu}_2') = \bar{x}'/\|\bar{x}\|$, respectively.

These have the following asymptotic expansions of distributions.

(3.5) $$\Pr(R \le x) = \Pr(\chi^2_{p-1} \le x) + n^{-1} \sum_{\alpha=0}^{2} R_\alpha \Pr(\chi^2_{p-1+2\alpha} \le x) + o(n^{-1}),$$

$$R_0 = \frac{1}{8}(p-1)(p-3)\left(\frac{1}{A\kappa} - \frac{1}{A'\kappa^2}\right) + \frac{1}{4}(p-1)\frac{A''}{(A')^2\kappa},$$

$$R_1 = -\frac{1}{4}(p-1)^2\left(\frac{1}{A\kappa} - \frac{1}{A'\kappa^2}\right) - \frac{1}{4}(p-1)\frac{A''}{(A')^2\kappa},$$

$$R_2 = \frac{1}{8}(p^2-1)\left(\frac{1}{A\kappa} - \frac{1}{A'\kappa^2}\right),$$

(3.6) $$\Pr(MW \le x) = \Pr(\chi^2_{p-1} \le x) + n^{-1} \sum_{\alpha=0}^{2} MW_\alpha \Pr(\chi^2_{p-1+2\alpha} \le x) + o(n^{-1}),$$

$$MW_0 = \frac{1}{8}(p-1)(p-3)\left(\frac{1}{A\kappa} - \frac{1}{A'\kappa^2}\right) + \frac{1}{4}(p-1)\frac{A''}{(A')^2\kappa},$$

$$MW_1 = \frac{1}{4}(p-1)(p+3)\frac{1}{A\kappa} + \frac{1}{4}(p-1)^2\frac{1}{A'\kappa^2} - \frac{1}{4}(p-1)\frac{A''}{(A')^2\kappa},$$

$$MW_2 = -\frac{1}{8}(p^2-1)\left(\frac{1}{A'\kappa^2} + \frac{3}{A\kappa}\right).$$

Using these expansions we can find the percentile points of these statistics.

4. Power comparison

To compare the powers of these statistics, we give the asymptotic expansions of their power functions under a sequence of Pitman's alternatives K_n: $\mu = (\mu_0 + n^{-1/2}\delta)/\|\mu_0 + n^{-1/2}\delta\|$, $\mu_0'\delta = 0$. The power functions are obtained in a similar way as in the case of the null hypothesis by the use of the probability density function (2.2).

(4.1) $$\Pr(L1 \ge x \mid K_n) = \Pr\{\chi^2_{p-1}(\Delta) \ge x\} + n^{-1} \sum_{\alpha=0}^{3} L_\alpha^{(1)}(\nu) \Pr\{\chi^2_{p-1+2\alpha}(\Delta) \ge x\} + o(n^{-1}),$$

where

$$L_0^{(1)}(v) = \frac{1}{8}(A'\kappa^2 + 3A\kappa)v^2 + \frac{1}{8}(p-1)(p-3)\left(\frac{1}{A\kappa} - \frac{1}{A'\kappa^2}\right)$$
$$+ \frac{1}{4}(p-1)\frac{A''}{(A')^2\kappa},$$

$$L_1^{(1)}(v) = -\frac{1}{4}(A'\kappa^2 + A\kappa)v^2 + \left\{\frac{1}{2}\frac{A}{A'\kappa} - \frac{1}{2} + \frac{1}{4}\frac{A''A}{(A')^2}\right\}v$$
$$- \frac{1}{8}(p-1)(p-3)\left(\frac{1}{A\kappa} - \frac{1}{A'\kappa^2}\right) - \frac{1}{4}(p-1)\frac{A''}{(A')^2\kappa},$$

$$L_2^{(1)}(v) = \frac{1}{8}\left(A'\kappa^2 - 2A\kappa + \frac{A^2}{A'}\right)v^2 + \left\{-\frac{1}{2}\frac{A}{A'\kappa} + \frac{1}{2} - \frac{1}{4}\frac{A''A}{(A')^2}\right\}v,$$

$$L_3^{(1)}(v) = -\frac{1}{8}\left(\frac{A^2}{A'} - A\kappa\right)v^2,$$

and $\chi_f^2(\Delta)$ is a non-central chi-squared random variable with f degrees of freedom and the non-centrality parameter $\Delta = A\kappa v/2$, and $v = \|\delta\|^2$. The power function of $L2$ for known κ_0 is given by

$$\text{(4.2)} \quad \Pr(L2 \geq x | K_n)$$
$$= \Pr\{\chi_{p-1}^2(\Delta_0) \geq x\} + n^{-1}\sum_{\alpha=0}^{3} L_\alpha^{(2)}(v)\Pr\{\chi_{p-1+2\alpha}^2(\Delta_0) \geq x\}$$
$$+ o(n^{-1}),$$

where

$$L_0^{(2)}(v) = \frac{1}{8}(A_0'\kappa_0^2 + 3A_0\kappa_0)v^2 - \frac{1}{4}(p-1)v$$
$$+ \frac{1}{8}(p-1)(p-3)\frac{1}{A_0\kappa_0},$$

$$L_1^{(2)}(v) = -\frac{1}{4}(A_0'\kappa_0^2 + 2A_0\kappa_0)v^2 + \frac{1}{2}(p-1)v$$
$$- \frac{1}{8}(p-1)(p-3)\frac{1}{A_0\kappa_0},$$

$$L_2^{(2)}(v) = \frac{1}{8}(A_0'\kappa_0^2 + 2A_0\kappa_0)v^2 - \frac{1}{4}(p-1)v,$$

$$L_2^{(3)}(v) = -\frac{1}{8} A_0\kappa_0 v^2$$

and $\Delta_0 = A_0\kappa_0 v/2$. The power function of the Watson statistic is given as follows.

(4.3) $\Pr(W1 \geq x | K_n)$

$$= \Pr\{\chi^2_{p-1}(\Delta) \geq x\} + n^{-1} \sum_{\alpha=0}^{4} W_\alpha^{(1)}(v) \Pr\{\chi^2_{p-1+2\alpha}(\Delta) \geq x\} + o(n^{-1}),$$

where

$$W_0^{(1)}(v) = \frac{1}{8}(A'\kappa^2 + 3A\kappa)v^2 + \frac{1}{8}(p-1)(p-3)\left(\frac{1}{A\kappa} - \frac{1}{A'\kappa^2}\right) + \frac{1}{4}(p-1)\frac{A''}{(A')^2\kappa},$$

$$W_1^{(1)}(v) = -\frac{1}{4}(A'\kappa^2 + A\kappa)v^2 + \left\{\frac{1}{2}\frac{A}{A'\kappa} - \frac{1}{2} + \frac{1}{4}\frac{A''A}{(A')^2}\right\}v + \frac{1}{2}(p-1)\left(\frac{1}{A\kappa} - \frac{1}{A'\kappa^2}\right) - \frac{1}{4}(p-1)\frac{A''}{(A')^2\kappa},$$

$$W_2^{(1)}(v) = \frac{1}{8}\left(A'\kappa^2 - 2A\kappa + \frac{A^2}{A'}\right)v^2 - \left\{\frac{1}{4}(p+3)\frac{A}{A'\kappa} + \frac{1}{4}\frac{A''A}{(A')^2} - \frac{1}{4}(p+3)\right\}v + \frac{1}{8}(p^2-1)\left(\frac{1}{A'\kappa^2} - \frac{1}{A\kappa}\right),$$

$$W_3^{(1)}(v) = -\frac{1}{4}\left(\frac{A^2}{A'} - A\kappa\right)v^2 + \frac{1}{4}(p+1)\left(\frac{A}{A'\kappa} - 1\right)v,$$

$$W_4^{(1)}(v) = \frac{1}{8}\left(\frac{A^2}{A'} - A\kappa\right)v^2.$$

(4.4) $\Pr(W2 \geq x | K_n)$

$$= \Pr\{\chi^2_{p-1}(\Delta_0) \geq x\} + n^{-1} \sum_{\alpha=0}^{4} W_\alpha^{(2)}(v) \Pr\{\chi^2_{p-1+2\alpha}(\Delta_0) \geq x\} + o(n^{-1}),$$

where

$$W_0^{(2)}(v) = \frac{1}{8}(A_0'\kappa_0^2 + 3A_0\kappa_0)v^2 + \frac{1}{4}(p-1)\left(\frac{A_0'\kappa_0}{A_0} - 1\right)v$$

$$+ \frac{1}{8}(p^2-1)\left(\frac{A_0'}{A_0^2} - \frac{1}{A_0\kappa_0}\right),$$

$$W_1^{(2)}(v) = -\frac{1}{2}A_0\kappa_0 v^2 - \frac{1}{4}(p-1)\left(\frac{A_0'\kappa_0}{A_0} - 1\right)v$$

$$- \frac{1}{4}(p^2-1)\left(\frac{A_0'}{A_0^2} - \frac{1}{A_0\kappa_0}\right),$$

$$W_2^{(2)}(v) = -\frac{1}{4}(A_0'\kappa_0^2 - A_0\kappa_0)v^2 - \frac{1}{4}(p+1)\left(\frac{A_0'\kappa_0}{A_0} - 1\right)v$$

$$+ \frac{1}{8}(p^2-1)\left(\frac{A_0'}{A_0^2} - \frac{1}{A_0\kappa_0}\right),$$

$$W_3^{(2)}(v) = \frac{1}{4}(p+1)\left(\frac{A_0'\kappa_0}{A_0} - 1\right)v,$$

$$W_4^{(2)}(v) = \frac{1}{8}(A_0'\kappa_0^2 - A_0\kappa_0)v^2 .$$

Chou (1986) considered the asymptotic expansion of Watson statistic under a sequence of alternatives K_n': $\mu_0 = (\mu + n^{-1/2}\delta)/\|\mu + n^{-1/2}\delta\|$, $\mu'\delta = 0$. Her expression is slightly different from ours, because of the difference in the alternatives. Rao statistic has a similar expression of the power function.

(4.5) $$\Pr(R \geq x \mid K_n) = \Pr\{\chi^2_{p-1}(\Delta) \geq x\} + n^{-1}\sum_{\alpha=0}^{4} R_\alpha(v)\Pr\{\chi^2_{p-1+2\alpha}(\Delta) \geq x\} + o(n^{-1})$$

where

$$R_0(v) = \frac{1}{8}(A'\kappa^2 + 3A\kappa)v^2 + \frac{1}{8}(p-1)(p-3)\left(\frac{1}{A\kappa} - \frac{1}{A'\kappa^2}\right)$$

$$+ \frac{1}{4}(p-1)\frac{A''}{(A')^2\kappa},$$

$$R_1(v) = -\frac{1}{4}(A'\kappa^2 + A\kappa)v^2 + \left\{\frac{1}{2}\frac{A}{A'\kappa} + \frac{1}{4}\frac{A''A}{(A')^2} - \frac{1}{2}\right\}v$$
$$-\frac{1}{4}(p-1)^2\left(\frac{1}{A\kappa} - \frac{1}{A'\kappa^2}\right) - \frac{1}{4}(p-1)\frac{A''}{(A')^2\kappa},$$
$$R_2(v) = \frac{1}{8}\left(A'\kappa^2 - 2A\kappa + \frac{A^2}{A'}\right)v^2$$
$$+\left\{\frac{1}{4}(p-1)\frac{A}{A'\kappa} - \frac{1}{4}\frac{A''A}{(A')^2} - \frac{1}{4}(p-1)\right\}v$$
$$+\frac{1}{8}(p^2-1)\left(\frac{1}{A\kappa} - \frac{1}{A'\kappa^2}\right),$$
$$R_3(v) = -\frac{1}{4}(p+1)\left(\frac{A}{A'\kappa} - 1\right)v,$$
$$R_4(v) = \frac{1}{8}\left(A\kappa - \frac{A^2}{A'}\right)v^2.$$

The power function of the modified Wald statistic is expressed as follows

$$\text{(4.6)} \quad \Pr(MW \geq x \mid K_n)$$
$$= \Pr\{\chi^2_{p-1}(\Delta) \geq x\} + n^{-1/2}\sum_{\alpha=1}^{3} M_\alpha^{(1)} \Pr\{\chi^2_{p-1+2\alpha}(\Delta) \geq x\}$$
$$+ n^{-1}\sum_{\alpha=0}^{6} M_\alpha^{(2)} \Pr\{\chi^2_{p-1+2\alpha}(\Delta) \geq x\} + o(n^{-1}),$$

where

$$M_1^{(1)} = \frac{1}{2}(p+1)v_1,$$
$$M_2^{(1)} = \frac{1}{2}A\kappa v v_1 - \frac{1}{2}(p+1)v_1,$$
$$M_3^{(1)} = -\frac{1}{2}A\kappa v v_1,$$
$$M_0^{(2)} = \frac{1}{8}(A'\kappa^2 + 3A\kappa)v^2 + \frac{1}{4}(p-1)\left(\frac{1}{A} - 1\right)v$$
$$+\frac{1}{8}(p-1)(p-3)\left(\frac{1}{A\kappa} - \frac{1}{A'\kappa^2}\right) + \frac{1}{4}(p-1)\frac{A''}{(A')^2\kappa},$$

$$M_1^{(2)} = -\frac{1}{4}(A'\kappa^2 - \kappa + 2A\kappa)v^2$$

$$+\left\{\frac{1}{2}\frac{A}{A'\kappa} + \frac{1}{4}\frac{A''A}{(A')^2} - \frac{1}{4}(p-1)\frac{1}{A} + \frac{1}{4}(p-3)\right\}v$$

$$+\frac{1}{2}(p+1)\frac{1}{A\kappa}v_2 + \frac{1}{4}(p-1)(p+3)\frac{1}{A\kappa}$$

$$+\frac{1}{4}(p-1)^2\frac{1}{A'\kappa^2} - \frac{1}{4}(p-1)\frac{A''}{(A')^2\kappa},$$

$$M_2^{(2)} = \frac{1}{8}\left(A'\kappa^2 - 2\kappa + \frac{A^2}{A'}\right)v^2$$

$$+\left\{\frac{1}{2}v_2 + \frac{1}{4}(p-1)\frac{A}{A'\kappa} - \frac{1}{4}\frac{A''A}{(A')^2} + \frac{1}{4}(3p+5)\right\}v$$

$$-\frac{1}{8}(p+1)(p+7)\frac{1}{A\kappa}v_2 + \frac{1}{8}(p+3)(p+5)v_1^2$$

$$-\frac{1}{8}(p^2-1)\left(\frac{1}{A'\kappa^2} + \frac{3}{A\kappa}\right),$$

$$M_3^{(2)} = \frac{1}{2}A\kappa v^2 + \left\{\frac{1}{4}(p+5)A\kappa v_1^2 - \frac{1}{4}(p+5)v_2\right.$$

$$\left. -\frac{1}{4}(p+1)\frac{A}{A'\kappa} - \frac{3}{4}(p+1)\right\}v$$

$$-\frac{1}{4}(p+3)(p+5)v_1^2 + \frac{1}{8}(p+1)(p+3)\frac{1}{A\kappa}v_2,$$

$$M_4^{(2)} = \frac{1}{8}\left(A^2\kappa^2v_1^2 - A\kappa v_2 - 3A\kappa - \frac{A^2}{A'}\right)v^2$$

$$+\left\{-\frac{1}{2}(p+5)A\kappa v_1^2 + \frac{1}{4}(p+3)v_2\right\}v$$

$$+\frac{1}{8}(p+3)(p+5)v_1^2,$$

$$M_5^{(2)} = -\frac{1}{8}(2A^2\kappa^2v_1^2 - A\kappa v_2)v^2 + \frac{1}{4}(p+5)A\kappa v_1^2 v,$$

$$M_6^{(2)} = \frac{1}{8}A^2\kappa^2v_1^2v^2,$$

and $\nu_1 = \mu_{20}'\delta_2/\mu_{10}^2$, $\nu_2 = \mu_{20}'\mu_{20}/\mu_{10}^2$ and $\delta' = (\delta_1, \delta_2')$.

Let T be one of statistics $L1$, $L2$, $W1$, $W2$, R and MW. Table 1 gives the upper five percentile points of these for $p = 3$ and $n = 20$. I in Table 1 gives the numerical solution of x of $0.05 = \Pr\{T \geq x\}$, where the right-hand side is expressed as the formal asymptotic expansion in the preceding section. Using the method of Fisher *et al.* (1981), we generated one million values for each statistic and obtained the upper five percentile point. We repeated this procedure one hundred times. II in Table 1 shows the mean of these and the standard derivation in the parentheses. Cornish and Fisher's generalized inversion formula for the percentile point does not give enough approximation compared with the results of simulations.

It should be noted that the percentile points of $L2$ are the same for all κ in I, because the asymptotic expansion (3.2) of the distribution is the same up to the order $1/n$ as the chi-squared distribution with 2 degrees of freedom for $p = 3$.

By use of a similar argument of Watson (1984), $L1$, $W1$ and R are approximated for large κ as

$$L1 = n(p-1)\log\left(1 + \frac{1}{n-1}F\right),$$

$$W1 = n(p-1)\frac{F}{n-1}\cdot\frac{1}{2}\left(1 + \frac{\mu_0'\bar{x}}{\|\bar{x}\|}\right),$$

$$R = n(p-1)\left\{\frac{F}{n-1}\bigg/\left(1 + \frac{F}{n-1}\right)\right\}\cdot\frac{1}{2}\left(1 + \frac{\|\bar{x}\|}{\mu_0'\bar{x}}\right),$$

Table 1. Upper five percentile points for $p = 3$ and $n = 20$.

κ		$L1$	$L2$	$W1$	$W2$	R	MW
1	I	6.182	5.991	6.376	5.936	5.992	3.134
	II	6.203	5.980	6.429	5.934	5.992	3.259
		(0.025)	(0.023)	(0.027)	(0.022)	(0.023)	(0.012)
5	I	6.273	5.991	6.613	5.963	5.948	5.541
	II	6.307	5.993	6.712	5.964	5.939	5.522
		(0.027)	(0.025)	(0.031)	(0.027)	(0.024)	(0.021)
10	I	6.281	5.991	6.684	5.976	5.897	5.713
	II	6.309	5.991	6.799	5.977	5.885	5.697
		(0.027)	(0.026)	(0.032)	(0.026)	(0.024)	(0.022)
15	I	6.282	5.991	6.704	5.981	5.881	5.763
	II	6.310	5.991	6.799	5.981	5.867	5.746
		(0.025)	(0.024)	(0.030)	(0.025)	(0.022)	(0.021)
20	I	6.283	5.991	6.712	5.984	5.873	5.785
	II	6.310	5.991	6.808	5.984	5.860	5.770
		(0.025)	(0.024)	(0.030)	(0.025)	(0.022)	(0.021)

where F is an F-random variable with $p-1$ and $(n-1)(p-1)$ degrees of freedom. Since $\bar{x}$ converges in probability as κ goes to infinitive, $W1$ and R are approximated as $n(p-1)F/(n-1)$ and $n(p-1)(F/(n-1))/\{1+F/(n-1)\}$, respectively. We have the following inequality.

$$W1 \geq L1 \geq R .$$

Table 1 shows this relation for large κ.

$L2$ and $W2$ are also approximated for large κ as

$$L2 = 2n\kappa(\|\bar{x}\| - \mu_0'\bar{x}) ,$$

$$W2 = 2n\kappa(\|\bar{x}\| - \mu_0'\bar{x})\frac{1}{2A(\kappa)}(\|\bar{x}\| + \mu_0'\bar{x}) .$$

By a similar argument these are distributed approximately as a chi-squared distribution with $p-1$ degrees of freedom. The approximation of MW to a chi-squared distribution is poor. Table 2 shows the powers of these statistics for $p=3$, $n=20$, $\alpha=\pi/2$, $\beta=0$ and $\tilde{\theta}=\pi/2$. By use of the method of Fisher *et al.* (1981), we generated one million values of these statistics under Pitman's alternatives

$$\mu = (\mu_0 + \gamma\tilde{\delta}/\sqrt{n})/\|\mu_0 + \gamma\tilde{\delta}/\sqrt{n}\|$$

where

$$\mu_0 = \begin{pmatrix} \sin\alpha\cos\beta \\ \sin\alpha\sin\beta \\ \cos\alpha \end{pmatrix} \quad \text{and} \quad \tilde{\delta} = \begin{pmatrix} \cos\alpha\cos\beta\cos\tilde{\theta} - \sin\beta\sin\tilde{\theta} \\ \cos\alpha\sin\beta\cos\tilde{\theta} + \cos\beta\sin\tilde{\theta} \\ -\sin\alpha\cos\tilde{\theta} \end{pmatrix} .$$

I in Table 2 shows the theoretical value calculated by the asymptotic expansion of its power function in the preceding section and II shows the power due to the simulation.

From Table 2 we note following points.

(i) The agreements between I and II are satisfactory for small γ.

(ii) Simulation shows that the powers of $L1$, $W1$, R and MW are almost the same, and $L2$ and $W2$ are more powerful than the others; that is, the knowledge of κ increases the power, as is to be expected.

Table 2. Powers (× 1000) for $p = 3$, $n = 20$, $\alpha = \pi/2$, $\beta = 0$ and $\tilde{\theta} = \pi/2$.

γ	κ		$L1$	$L2$	$W1$	$W2$	R	MW
0.15	1	I	51	51	51	51	51	51
		II	50	50	50	50	50	50
	5	I	57	57	57	57	56	57
		II	55	56	55	56	54	54
	10	I	64	66	65	66	64	64
		II	63	65	63	65	63	64
	15	I	72	74	72	74	71	71
		II	72	74	72	74	72	72
	20	I	80	84	82	84	79	79
		II	81	83	81	83	81	80
0.35	1	I	53	53	53	53	53	52
		II	53	52	53	53	53	53
	5	I	86	88	87	88	86	86
		II	87	89	87	89	87	86
	10	I	135	141	137	141	135	135
		II	136	142	136	142	136	135
	15	I	187	198	189	198	186	186
		II	186	198	186	198	186	186
	20	I	240	256	243	255	240	240
		II	240	256	240	256	240	239

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