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A MOMENT'S APPROACH TO SOME CHARACTERIZATION PROBLEMS

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Abstract. For a random sample from a population having a finite k-th moment, it is shown that the constancy of regression of polynomial statistics of order k in the mean implies that all higher moments exist and are uniquely determined by the first k moments. This result is utilized to give a moment's approach to some characterization results.

Key words and phrases: Regression, polynomial statistics, characterization, Meixner hypergeometric, Meixner class.

1. Introduction

Let $X_1, X_2, \ldots, X_n$ be n identically independent random variables with common distribution function F. Further, let

$$\Lambda = \sum_{i=1}^{n} X_i$$

and

$$S_2 = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} X_i X_j .$$

Laha and Lukacs (1960) established that the constancy of regression of S_2 on Λ implies that F is a member of the Meixner class which consists of the binomial, negative binomial, Poisson, normal, gamma and Meixner hypergeometric distributions and their linear transformation. All members of the Meixner class are well known except the Meixner hypergeometric distribution. The name was introduced by Lai (1982) in credit to the work of Meixner (1934) who formulated the distribution through its characteristic function. Shanbhag (1979) obtained the explicit form of its density as

$$f(x,\theta)=(\cos\alpha)^{\rho}\frac{2^{\rho-2}}{\pi\Gamma(\rho)}e^{\alpha x}\Gamma\left(\frac{\rho}{2}+\frac{ix}{2}\right)\Gamma\left(\frac{\rho}{2}-\frac{ix}{2}\right)$$

$$-\infty<x<\infty,$$

where $\alpha=\tan^{-1}\theta$ and $\rho>0$ is a real constant. More details about this class can be found in Lai (1982), Alzaid (1983) and Morris (1982, 1983) who called the class the linear exponential with quadratic variance.

Recently, Heller (1983, 1984) obtained characterizations of the binomial, negative binomial and gamma distributions based on the zero regression of two polynomial statistics of the form

$$S_m=\sum_{i=1}^{n}\sum_{j=1}^{n}[a_{ij}(m)X_i^{m-1}X_j+b_{ij}(m)X_i^{m-2}X_j+c_{ij}(m)X_i^{m-2}] \tag{1.1}$$

on Λ for $m=k,\ k+1$, with $\sum_{i=1}^{n}a_{ii}(m)\neq 0$. Similar characterizations for Poisson and normal distributions are also indicated. Heller's proofs are based on the use of recurrence relations of the corresponding polynomials (see Morris (1982) for polynomials corresponding to the Meixner class) and the characteristic functions.

The purpose of this article is to show that if the k-th moment of F exists, then the zero regression of any polynomial of order m ($m\leq k$) on $\bar{X}$ implies that all moments exist and are uniquely determined in terms of the first m moments. This result is utilized to obtain characterization for some distributions which are uniquely determined by their moments. The result of Laha and Lukacs (1960) follows as a special case.

2. Main results

Let X_1 and X_2 be two independent identically distributed random variables and let L_k be a polynomial statistic defined by

$$L_k=\sum_{0\leq i,j\leq k}a_{i,j}X_1^iX_2^j,$$

where $a_{ij}, i,j=0,1,2,\ldots,k$ are real constants such that $a_{ij}>0$ for some i and j. Then we have the following theorem:

THEOREM 2.1. *Suppose that X_1 has finite moments of order $k+1$ and*

$$E(X_1^{k+1}|X_1+X_2)=E(L_k|\ X_1+X_2) \tag{2.1}$$

for some $k\geq 0$. Then X_1 has moments of all orders and the moments are *uniquely determined in terms of the first k moments.*

PROOF. Denote by I_m the indicator function of the set $\{-m < X_1 + X_2 < m\}$, $m = 1, 2, \ldots$. We first suppose that $k+1$ is an odd number. Then it follows from (2.1) that

$$E(X_1^{k+1}(X_1 + X_2)I_m | X_1 + X_2) = E(L_k(X_1 + X_2)I_m | X_1 + X_2) . \tag{2.2}$$

Therefore, in view of the assumption $E(X_1^{k+1}) < \infty$

$$E(X_1^{k+2}I_m) = E[((X_1 + X_2)L_k - X_1^{k+1}X_2)I_m] .$$

Since $k+2$ is even, applying the monotone convergence theorem to the left-hand side and the Lebesgue dominated convergence theorem to the right-hand side of the above identity, we get

$$E(X_1^{k+2}) = E[(X_1 + X_2)L_k - X_1^{k+1}X_2] . \tag{2.3}$$

The fact that the right-hand side of (2.3) involves only terms of the form $E(X_1^i)E(X_2^j)$, $i,j \leq k+1$ implies that $E(X_1^{k+2}) < \infty$. We next suppose that $k+1$ is an even integer. Then, multiplying (2.1) by $|X_1 + X_2| I_m$ and using essentially the same argument, one could conclude that $E(X_1^{k+2}) < \infty$.

Equation (2.1) implies that $E(X_1^{k+1})$ is linearly determined in terms of $E(X_1^r)$, $r = 0, 1, \ldots, k$. Consequently, using (2.3), we have $E(X_1^{k+2})$ is also determined in terms of the first k moments.

Now using (2.2) without I_m in place of (2.1) and repeating the same argument with k replaced by $k+1$, we conclude that $E(X_1^{k+3})$ is linearly determined in terms of the lower moments. Therefore, an induction argument implies that the theorem is valid.

COROLLARY 2.1. *Let X_1 and X_2 be as in Theorem* 2.1 *with $E(X_1^2) < \infty$ and*

$$E(X_1^2 - aX_1X_2 - bX_1 - c | X_1 + X_2) = 0 . \tag{2.4}$$

Then one of the following holds:

(i) *$a = 1$, $b = 0$ and hence X_1 has a normal distribution with c as its variance.*

(ii) *$a \neq 1$, $b \neq 0$ and hence $b^{-1}X_1 + cb^{-2}$ has a Poisson distribution with mean λ (say).*

(iii) *$a \neq 1$, $b^2 < 4(a-1)c$ and hence $(X_1 - \rho\mu_2)/(\mu_1 - \mu_2)$ has either a binomial distribution or a negative binomial distribution with parameters ρ and p where ρ, μ_1 and μ_2 are such that $\rho = (1-a)^{-1}$, $\mu_1 + \mu_2 = b$ and $\mu_1\mu_2 = c$.*

(iv) $a \neq 1$, $b^2 = 4(a-1)c$ *and hence* $X_1 - \mu$ *has a gamma distribution with parameters* ρ *and* λ *where* ρ *and* μ *are such that* $\rho = 1/(a-1)$ *and* $\mu = -b/[2(a-1)]$.

(v) $a \neq 1$, $b^2 < 4(a-1)c$ *and hence* $X_1 - \mu$ *has the Meixner hypergeometric distribution with parameters* $\rho = (a-1)$ *and* $\theta = \sqrt{4(a-1)c - b^2}/2$ *where* $\mu = -b/[2(a-1)]$.

PROOF. By Theorem 2.1 all moments of X_1 are determined in terms of $E(X_1)$. Suppose that X_1' and X_2' are i.i.d. random variables having distribution F' uniquely determined by its moments and satisfying (2.4). Then, by Theorem 2.1, we should have $F = F'$. Now, Corollary 2.1 follows from this observation and the fact that all distributions in (i)–(v) are uniquely determined by their moments and satisfy (2.4).

Remark. Corollary 2.1 is similar to the result of Laha and Lukacs (1960). They arrived at their result using the characteristic function methods. A moment argument has been considered by Shanbhag (1978) for the case of the normal distribution.

The following theorem gives a unified moments approach to the characterization results of Heller (1984).

THEOREM 2.2. *Let* X_1 *and* X_2 *be two i.i.d. random variables such that the k-th moment of* X_1 *exists and is finite. Then*

$$(2.5)\quad E(X_1^k + a_1 X_1^{k-1} X_2 + b_1 X_1^{k-1} + c_1 X_1^{k-2} X_2 + d_1 X_1^{k-2} | X_1 + X_2) = 0$$

and

$$(2.6)\quad E(X_1^{k-1} + a_2 X_1^{k-2} X_2 + b_2 X_1^{k-2} + c_2 X_1^{k-3} X_2 + d_2 X_1^{k-3} | X_1 + X_2) = 0\,,$$

for some real constants a_i, b_i, c_i *and* d_i, $i = 1, 2$, *imply that all moments exist and are uniquely determined in terms of* $E(X_1^{k-1})$, $E(X_1^{k-2})$ *and* $E(X_1)$.

PROOF. In view of Theorem 2.1 it is sufficient to show that the moments $E(X_1^2), E(X_1^3), \ldots, E(X_1^{k-3})$ are uniquely determined in terms of $E(X_1^{k-1})$, $E(X_1^{k-2})$ and $E(X_1)$. It is obvious from (2.6) that $E(X_1^{k-3})$ can be found in terms of $E(X_1)$, $E(X_1^{k-2})$ and $E(X_1^{k-1})$. Now multiplying (2.6) by $X_1 + X_2$ and subtracting the result from (2.5), one gets

$$E\{(a_1 - a_2) X_1^{k-1} X_2 + (b_1 - b_2) X_1^{k-1} + (c_1 - c_2 - b_2) X_1^{k-2} X_2 \\ + (d_1 - d_2) X_1^{k-2} - (a_2 X_1^{k-2} + c_2 X_1^{k-3}) X_2^2 + d_2 X_1^{k-3} X_2 | X_1 + X_2\} = 0\,.$$

This, in turn, implies that $E(X_1^2)$ is also determined in terms of $E(X_1)$, $E(X_1^{k-2})$ and $E(X_1^{k-1})$. Now an induction argument will lead to the required result.

As the members of the Meixner class have at most two parameters, one can use the above result to arrive at characterization of these distributions based on the constancy of regression of two polynomial statistics of the form (2.5) and (2.6).

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