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SOME OPTIMAL DESIGNS FOR COMPARING A SET OF TEST TREATMENTS WITH A SET OF CONTROLS*

MIKE JACROUX

Department of Pure and Applied Mathematics, Washington State University, Pullman, WA 99164-2930, U.S.A.

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Abstract. In this paper, we give a detailed study of the problem of optimally comparing a set of t test treatments to a set of s controls under a 0-way elimination of heterogeneity model. The relationships between designs that are A and MV-optimal for comparing the test treatments to the controls and those that are A and MV-optimal for comparing all treatments are also studied.

Key words and phrases: $A(c)$-optimal designs, $MV(c)$-optimal designs, A-optimal designs, MV-optimal designs, controls, test treatments.

1. Introduction

In this paper, we consider the problem of comparing s standard treatments or controls, denoted by $1, 2, \ldots, s$ to t test treatments, denoted by $s+1, \ldots, v$ where $v = s + t \geq 3$, $s \leq t$. We shall assume that there are n experimental units available for testing, and that a completely randomized design is to be used for comparing the treatments. Any allocation of treatments to experimental units is called a design, which we denote by d. Under a given design d, r_{di} is used to denote the number of experimental units to which treatment i is assigned, $i = 1, \ldots, v$. The model assumed for analyzing the data from a given design d is the 0-way elimination of heterogeneity model, which specifies that an observation y_{dij} (the j-th observation on treatment i under d) can be expressed as

$$(1.1) \qquad y_{dij} = t_i + \varepsilon_{ij}, \quad i = 1, \ldots, v, \quad j = 1, \ldots, r_{di},$$

where t_i represents the effect of treatment i and the ε_{ij}'s are independent random error terms having expectation zero and constant variance σ^2. Since the value of σ^2 does not have an effect on any of the computations or

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results given in this paper, we shall henceforth assume $\sigma^2 = 1$. Under these assumptions, the method of least squares yields best linear unbiased estimators

$$\hat{t}_i - \hat{t}_j = \sum_{p=1}^{r_{di}} y_{dip}/r_{di} - \sum_{p=1}^{r_{dj}} y_{djp}/r_{dj}\,, \tag{1.2}$$

for treatment differences of the form $t_i - t_j$. For the problem of comparing s standard treatments to t test treatments, there will clearly be a number of designs available in almost any given experimental setting. Let D denote the class of available designs. The criteria we consider here for choosing a best design in D are the following:

$MV(c)$-optimality-A design $d* \in D$ is said to be $MV(c)$-optimal in D if for any other $d \in D$,

$$\max_{1 \le i \le s,\, s+1 \le j \le v} \mathrm{Var}_{d*}(\hat{t}_i - \hat{t}_j) \le \max_{1 \le i \le s,\, s+1 \le j \le v} \mathrm{Var}_d(\hat{t}_i - \hat{t}_j)\,, \tag{1.3}$$

where $\mathrm{Var}_d(\hat{t}_i - \hat{t}_j)$ denotes the variance of the least squares estimate $\hat{t}_i - \hat{t}_j$ derived under d.

$A(c)$-optimality-A design $d* \in D$ is said to be $A(c)$-optimal in D if for any other $d \in D$,

$$\sum_{i=1}^{s} \sum_{j=s+1}^{v} \mathrm{Var}_{d*}(\hat{t}_i - \hat{t}_j) \le \sum_{i=1}^{s} \sum_{j=s+1}^{v} \mathrm{Var}_d(\hat{t}_i - \hat{t}_j)\,. \tag{1.4}$$

We note that the $MV(c)$- and $A(c)$-optimality criteria defined above have been the most widely studied with respect to comparing a set of test treatments to a set of controls. In fact, a good many results have been obtained in recent years for comparing a set of test treatments to a single control (see e.g., Bechhofer and Tamhane (1981), Majumdar and Notz (1983), Hedayat and Majumdar (1984, 1985), Notz (1985), Jacroux (1986, 1987*a*, 1987*b* and 1989), Stufken (1987) and Cheng *et al.* (1988)). For a more detailed summary of the known results on this topic, the reader is referred to Hedayat *et al.* (1988). The only results known to the author for comparing $s > 1$ controls to t test treatments are those given in Majumdar (1986). In particular, Majumdar (1986) derives some sufficient conditions for a design to be $MV(c)$- and $A(c)$-optimal in a block design setting. He then shows how these sufficient conditions can be applied to establish the $MV(c)$- and $A(c)$-optimality of several infinite families of block designs and row-column designs. In this paper, we further consider the problem of comparing a set of s controls to a set of t test treatments. In Section 2, we find sufficient conditions for a design to be $MV(c)$- and $A(c)$-optimal under model (1.1). The relationship between the $MV(c)$- and $A(c)$-optimality criteria is also studied in this section, as well as the relationship between these criteria and more "global" criteria that can be used for selecting a design that is optimal for estimating all possible treatment differences $t_i - t_j$, $i, j = 1, \ldots, v$, $i \neq j$.

2. Results on optimality

To begin with, we note that under a given design d and model (1.1),

$$\mathrm{Var}_d(\hat{t}_i - \hat{t}_j) = 1/r_{di} + 1/r_{dj}\,. \tag{2.1}$$

We shall henceforth use $D(s, t; n)$ to denote the class of all possible allocations of v treatments to n experimental units. Using $[x]$ to denote the integral part of the decimal expansion for a real number $x > 0$, we shall also use the following notation throughout the sequel:

(2.2)

$$r = [n/v]\,,$$
$$n = vr + q, \qquad 0 \le q \le v - 1\,,$$
$$n_{dc} = \sum_{i=1}^{s} r_{di} = \text{the number of experimental units allocated to the controls under a given design } d\,,$$
$$r_{dc} = [n_{dc}/s]\,,$$
$$n_{dc} = sr_{dc} + q_{dc}, \qquad 0 \le q_{dc} \le s - 1\,,$$
$$n_{dt} = \sum_{i=s+1}^{v} r_{di} = \text{the number of experimental units allocated to the test treatments under a given design } d\,,$$
$$r_{dt} = [n_{dt}/t]\,,$$
$$n_{dt} = tr_{dt} + q_{dt}, \qquad 0 \le q_{dt} \le t - 1\,,$$
$$x = [t/s]\,,$$
$$t = sx + q_1, \qquad 0 \le q_1 \le s - 1\,.$$

2.1 *MV(c)-optimal designs*

In view of (2.1), we see that finding an $MV(c)$-optimal design in $D(s, t; n)$ is equivalent to finding an allocation of treatments to experimental units which minimizes

$$\max_{1 \le i \le s,\, s+1 \le j \le v} 1/r_{di} + 1/r_{dj}\,. \tag{2.3}$$

However, if we assume that $r_{d1} \le \cdots \le r_{ds}$ and $r_{d,s+1} \le \cdots \le r_{dv}$ for a given design d, then it is easily seen that a design $\hat{d}$ having $r_{\hat{d}i} = r_{dt}$ for $i = s + 1, \ldots, v$ and $r_{\hat{d}i} = [(n - tr_{dt})/s]$ or $[(n - tr_{dt})/s] + 1$ for $i = 1, \ldots, s$ will have

$$\max_{1\le i\le s,\, s+1\le j\le v} \mathrm{Var}_{\bar{d}}\,(\hat{t}_i - \hat{t}_j) \le \max_{1\le i\le s,\, s+1\le j\le v} \mathrm{Var}_d\,(\hat{t}_i - \hat{t}_j)\,.$$

Thus the problem reduces to that of finding an integer r_2^* that minimizes the expression

$$h(r_2) = 1/r_2 + 1/[(n - tr_2)/s], \qquad t \le r_2 \le [n/t]\,. \tag{2.4}$$

This last expression is easily minimized using a calculator. While minimizing the function $h(r_2)$ defined in (2.4) is a relatively simple thing to do, one can also find an optimal design via the following lemma, which can be proved using simple algebraic manipulation of (2.4).

LEMMA 2.1. *Let $h(r_2)$ be as defined in* (2.4). *Then the following facts hold for $h(r_2)$.*

(i) *$h(r_2 - 1) \le h(r_2)$ implies $h(r_2) \le h(r_2 + 1)$ for $r_2 = 1,\dots,[n/t]$.*

(ii) *$h(r_2) \ge h(r_2 + 1)$ implies $h(r_2 - 1) \ge h(r_2)$ for $r_2 = 1,\dots,[n/t]$.*

Comment. Observe that because $r = [n/v]$, there must always exist at least one treatment having r or fewer replications assigned to it under any design d, and that $[(n - tr)/s] \ge r$. If $[(n - tr)/s] = r$, then since $s \le t$, it follows that

$$\begin{aligned} h(r) &= 1/r + 1/[(n - tr)/s] = 1/r + 1/r \le 1/(r + 1) + 1/(r - 1) \\ &\le 1/(r + 1) + 1/[(n - t(r + 1))/s] = h(r + 1)\,. \end{aligned}$$

Also, if $[(n - tr)/s] \ge r + 1$, then

$$\begin{aligned} h(r) &= 1/r + 1/[(n - tr)/s] \le 1/(r + 1) + 1/r \\ &\le 1/(r + 1) + 1/[(n - t(r + 1))/s] = h(r + 1)\,. \end{aligned}$$

Thus, in all cases we see that $h(r) \le h(r + 1)$. Hence, from Lemma 2.1, we see that to find an $MV(c)$-optimal design, one need only minimize $h(r_2)$ given in (2.4) over all $r_2, t \le r_2 \le [n/v] = r$.

From Lemma 2.1, we get the following theorem.

THEOREM 2.1. *Let $h(r_2)$ be as defined in* (2.4) *and suppose r_2^* satisfies*

$$h(r_2^*) \le \min\,\{h(r_2^* - 1),\, h(r_2^* + 1)\}\,.$$

Then any design $d*$ having $r_{d*i} \ge r_2^*$ for $i = s + 1,\dots, v$ and $r_{d*i} \ge [(n - tr_2^*)/s]$

for $i = 1,\dots,s$ *is* $MV(c)$*-optimal in* $D(s,t;n)$.

Example 2.1. Suppose $n = 41$, $s = 2$ and $t = 7$. Using Theorem 2.1, we find that $r_2^* = 4$. Thus one $MV(c)$-optimal design $d*$ has $r_{d*1} = 6$, $r_{d*2} = 7$ and $r_{d*3} = \cdots = r_{d*9} = 4$. Another $MV(c)$-optimal design $\bar{d}$ has $r_{\bar{d}1} = r_{\bar{d}2} = 6$ and $r_{\bar{d}3} = \cdots = r_{\bar{d}8} = 4$ and $r_{\bar{d}9} = 5$.

In the case that $t = xs$ and $n = r_1 s + r_2 xs$ for integers $r_1, r_2, x \geq 1$, the problem of minimizing (2.4) is equivalent to minimizing

$$h_1(r_2) = 1/r_2 + s/(n - sxr_2) \,. \tag{2.5}$$

Since $h_1(r_2)$ is continuous and convex, we can find the integral value of r_2 that minimizes (2.5) by finding the value of r_2 such that

$$h_1(r_2) \leq h_1(r_2 - 1) \quad \text{and} \quad h_1(r_2) \leq h_1(r_2 + 1) \,. \tag{2.6}$$

Algebraic manipulation of (2.6) yields the following theorem.

THEOREM 2.2. *Suppose* $t = sx$ *and* $n = r_1 s + r_2 xs$ *for integers* $r_1, r_2, x \geq 1$. *Then the design* $d*$ *in* $D(s,t;n)$ *having* $r_{d*i} = r_2^*$ *for* $i = s + 1,\dots,v$ *and* $r_{d*i} = (n - tr_2^*)/s$ *for* $i = 1,\dots,s$ *is* $MV(c)$*-optimal in* $D(s,t;n)$ *provided* r_2^* *satisfies*

$$\begin{aligned} &\frac{\{2nx - sx(x-1) - (s^2x^2(x-1)^2 + 4n^2x)^{1/2}\}}{2sx(x-1)} \\ &\leq r_2^* \\ &\leq \frac{\{2nx + sx(x-1) - (s^2x^2(x-1)^2 + 4n^2x)^{1/2}\}}{2sx(x-1)} \,. \end{aligned} \tag{2.7}$$

Example 2.2. Let $s = 2$, $t = 6$ and $n = 40$. Then according to Theorem 2.2, the design $d*$ in $D(2,6;40)$ having $r_{d*1} = r_{d*2} = 8$ and $r_{d*3} = \cdots = r_{d*8} = 4$ is $MV(c)$-optimal in $D(2,6;40)$.

A special situation which will be of particular interest later occurs when an $MV(c)$-optimal design assigns r experimental units to each test treatment.

THEOREM 2.3. *Suppose* $t = xs$ *and* $n = r_1 s + r_2 xs$ *for integers* $r_1, r_2, x \geq 1$. *Then the design* $d*$ *having* $r_{d*i} = r$ *for* $i = s + 1,\dots,v$ *and* $r_{d*i} = (sr + q)/s$ *for* $i = 1,\dots,s$ *is* $MV(c)$*-optimal in* $D(s,t;n)$ *provided* r *satisfies*

$$(2.8) \qquad r \leq \frac{\{(q+sx) + ((q+sx)x(q+s))^{1/2}\}}{s(x-1)} .$$

PROOF. By the comment following Lemma 2.1, it follows that $h_1(r+1) \geq h_1(r)$. Thus, for $d*$ to be $MV(c)$-optimal, the following inequality must be satisfied:

$$1/r + s/(sr+q) \leq 1/(r-1) + s/(sr+q+sx) .$$

Simple algebraic manipulation of this last inequality yields (2.8).

Example 2.3. Consider $D(1,6;34)$. Then the design $d*$ having $r_{d*1} = 10$ and $r_{d*2} = \cdots = r_{d*7} = 4$ is $MV(c)$-optimal in $D(1,6;34)$ by Theorem 2.3 and has treatments 2, 3, 4, 5, 6 and 7 each replicated $r = [34/7] = 4$ times.

2.2 *$A(c)$-optimal designs*

To find an $A(c)$-optimal design in $D(s,t;n)$, it follows from (2.1) that we must find an allocation of treatments to experimental units that minimizes

$$(2.9) \qquad \sum_{i=1}^{s} \sum_{j=s+1}^{v} \{1/r_{di} + 1/r_{dj}\}$$

subject to the restriction that $\sum_{i=1}^{v} = r_{di} = n$. Using the notation given in (2.2) and elementary algebra, it is easily seen that if d has treatment replication numbers $r_{d1},\ldots,r_{dv}$, then a design $\hat{d}$ having $r_{\hat{d}i} = r_{dc}$ or $r_{dc}+1$ for $i = 1,\ldots,s$ and $r_{\hat{d}i} = r_{dt}$ or $r_{dt}+1$ for $i = s+1,\ldots,v$ has

$$\begin{aligned}(2.10) \qquad \sum_{i=1}^{s} \sum_{j=s+1}^{v} \mathrm{Var}_{\hat{d}}\,(\hat{t}_i - \hat{t}_j) &= (s-q_{dc})(t-q_{dc})(1/r_{dc} + 1/r_{dt}) \\ &\quad + (s-q_{dc})q_{dt}(1/r_{dc} + 1/(r_{dt}+1)) \\ &\quad + q_{dc}(t-q_{dt})(1/(r_{dc}+1) + 1/r_{dt}) \\ &\quad + q_{dc}q_{dt}(1/(r_{dc}+1) + 1/(r_{dt}+1)) \\ &\leq \sum_{i=1}^{s} \sum_{j=s+1}^{v} \mathrm{Var}_d\,(\hat{t}_i - \hat{t}_j) \\ &= \sum_{i=1}^{s} \sum_{j=s+1}^{v} (1/r_{di} + 1/r_{dj}) .\end{aligned}$$

From the above argument, we see that finding an $A(c)$-optimal design in $D(s,t;n)$ is equivalent to finding a design $\tilde{d}$ in $D(s,t;n)$ that has $r_{\tilde{d}i} = \tilde{r}_2$ for $i = s+1,\ldots,s+(t-\tilde{z})$, $r_{\tilde{d}i} = \tilde{r}_2 + 1$ for $i = s+(t-\tilde{z})+1,\ldots,v$, $r_{\tilde{d}i} = r_1(\tilde{r}_2,\tilde{z})$

for $i = 1,\dots,s - n + t\tilde{r}_2 + \tilde{z} + sr_1(\tilde{r}_2,\tilde{z})$ and $r_{\tilde{d}i} = r_1(\tilde{r}_2,\tilde{z}) + 1$ for $i = s - n + t\tilde{r}_2 + \tilde{z} + sr_1(\tilde{r}_2,\tilde{z}) + 1,\dots,s$, where $r_1(\tilde{r}_2,\tilde{z}) = [(n - t\tilde{r}_2 - \tilde{z})/s]$ and $\tilde{r}_2$ and $\tilde{z}$ minimize

$$(2.11)\quad \begin{aligned} g(r_2,z) = {} & (t-z)(s-n+tr_2+z+sr_1(r_2,z))\{1/r_2 + 1/r_1(r_2,z)\} \\ & + (t-z)(n-tr_2-z-sr_1(r_2,z))\{1/r_2 + 1/(r_1(r_2,z)+1)\} \\ & + z(s-n+tr_2+z+sr_1(r_2,z))\{1/(r_2+1) + 1/r_1(r_2,z)\} \\ & + z(n-tr_2-z-sr_1(r_2,z))\{1/(r_2+1) + 1/(r_1(r_2,z)) + 1)\} \end{aligned}$$

over all $(r_2,z) \in A = \{r_2 = 1,\dots,[n/t], z = 0,\dots,t\}$. We note that if we let $L = \{n_2 | \; n_2$ is an integer, $t \le n_2 \le n\}$, then the function $g(r_2,z)$ defined in (2.11) can also be expressed for $n_2 \in L$ as:

$$(2.12)\qquad G(n_2) = g(r_2,z)$$

where $n_2 = tr_2 + z$ with $r_2 = [n_2/t]$ and $z = n_2 - t[n_2/t]$. Using elementary algebra, the following lemma is easily proven.

LEMMA 2.2. *Let $G(n_2)$ be as defined in* (2.12). *Then the following facts hold for $G(n_2)$ and $n_2 \in L$:*

(i) $G(n_2 - 1) \le G(n_2)$ *implies* $G(n_2) \le G(n_2 + 1)$.

(ii) $G(n_2) \ge G(n_2 + 1)$ *implies* $G(n_2 - 1) \ge G(n_2)$.

As an immediate consequence of Lemma 2.2, we obtain the following theorem.

THEOREM 2.4. *Let $G(n_2)$ be as defined in* (2.12). *Then* $G(\tilde{n}_2) = \min_{n_2 \in L} G(n_2)$ *if and only if*

$$(2.13)\qquad G(\tilde{n}_2) \le \min\{G(\tilde{n}_2 - 1),\ G(\tilde{n}_2 + 1)\}.$$

Thus, a design $\tilde{d}$ is $A(c)$-optimal in $D(s,t;n)$ if $r_{\tilde{d}i} = [\tilde{n}_2/t]$ or $[\tilde{n}_2/t] + 1$ for $i = s+1,\dots,v$ and $r_{\tilde{d}i} = [(n - \tilde{n}_2)/s]$ or $[(n - \tilde{n}_2)/s] + 1$ for $i = 1,\dots,s$ where $\tilde{n}_2$ satisfies (2.13).

Example 2.4. Consider the class of designs $D(2,7;41)$. In this case, the value of n_2 that satisfies (2.13) is $\tilde{n}_2 = 27$. Thus, an $A(c)$-optimal design $\tilde{d}$ in $D(2,7;41)$ has $r_{\tilde{d}1} = r_{\tilde{d}2} = 7$, $r_{\tilde{d}3} = 3$ and $r_{\tilde{d}4} = \cdots = r_{\tilde{d}9} = 4$. It can also be seen that $G(27) = G(28)$, hence another $A(c)$-optimal design $\bar{d}$ in $D(2,7;41)$ has $r_{\bar{d}1} = 6$, $r_{\bar{d}2} = 7$ and $r_{\bar{d}3} = \cdots r_{\bar{d}9} = 4$.

One case of the previous results which will be of interest occurs later, when the number of replications assigned to the controls is a multiple of the number of replications assigned to the test treatments.

COROLLARY 2.1. *Suppose* $t = xs$ *and* $n = r_1x + r_2xs$ *for integers* $r_1, r_2, x \geq 1$. *Then the design* $\tilde{d}$ *having* $r_{\tilde{d}i} = \tilde{r}_2$ *for* $i = s+1,\ldots,v$ *and* $r_{\tilde{d}i} = m\tilde{r}_2$ *for* $i = 1,\ldots,s$ *and some integer* $m \geq 1$ *is* $A(c)$*-optimal in* $D(s,t;n)$ *provided* $\tilde{r}_2$ *satisfies*

$$\max\{(x - m^2)\tilde{r}_2, (m^2 - x)\tilde{r}_2\} \leq m + x\,. \tag{2.14}$$

PROOF. This result follows directly from Theorem 2.4.

Example 2.5. Consider the class of designs $D(1,5;49)$. Then the design $\tilde{d}$ having $r_{\tilde{d}1} = 14$ and $r_{\tilde{d}i} = 7$ for $i = 2,\ldots,6$ is $A(c)$-optimal in $D(1,5;49)$ by Corollary 2.1.

In Examples 2.1 and 2.4, we see that the optimal allocations of treatments to experimental units are almost the same with respect to the $MV(c)$- and $A(c)$-optimality criteria. However, as we shall see in the next example, this is not always the case.

Example 2.6. Consider the class of designs $D(1,15;30)$. For this class of designs the $MV(c)$-optimal design $d*$ has $r_{d*1} = 15$ and $r_{d*i} = 1$ for $i = 2,\ldots,16$ whereas the $A(c)$-optimal design $\tilde{d}$ has $r_{\tilde{d}1} = 5$ and $r_{\tilde{d}i} = 1$ or 2 for $i = 2,\ldots,16$.

In the next section, we shall study the relationship between the $MV(c)$- and $A(c)$-optimality criteria.

2.3 *Relationship between the* $MV(c)$*- and* $A(c)$*-optimality criteria*

In Section 2, we saw that in some classes $D(s,t;n)$, the $A(c)$- and $MV(c)$-optimal allocations of treatments to experimental units are approximately the same, while in other cases, they may be somewhat different. In this section, we derive conditions under which $A(c)$- and $MV(c)$-optimal designs in $D(s,t;n)$ are the same. With this in mind, let $\tilde{d} \in D(s,t;n)$ be $A(c)$-optimal with $r_{\tilde{d}i} = \tilde{r}_1$ or $\tilde{r}_1 + 1$ for $i = 1,\ldots,s$ and $r_{\tilde{d}j} = \tilde{r}_2$ or $\tilde{r}_2 + 1$ for $j = s+1,\ldots,v$. Also let

$$\begin{aligned} &\sum_{i=1}^{s} r_{\tilde{d}i} = \tilde{n}_1 = s\tilde{r}_1 + \tilde{q}_1, && 0 \leq \tilde{q}_1 \leq s-1\,,\\ &\sum_{j=s+1}^{v} r_{\tilde{d}j} = \tilde{n}_2 = t\tilde{r}_2 + \tilde{q}_2, && 0 \leq \tilde{q}_2 \leq t-1\,,\\ &t = xs + q_1, && 0 \leq q_1 \leq s-1\,. \end{aligned} \tag{2.15}$$

THEOREM 2.5. *Consider the class of designs* $D(s,t;n)$ *and let* $\tilde{d} \in D(s,t;n)$ *be* $A(c)$-*optimal.*

(a) *If* $\tilde{q}_1 + \tilde{q}_2 \geq s$, *then* $\tilde{d}$ *is not* $MV(c)$-*optimal in* $D(s,t;n)$.

(b) *Suppose* $\tilde{q}_1 + \tilde{q}_2 < s$, $\tilde{q}_1 + \tilde{q}_2 + q_1 < s$ *and* $\tilde{q}_1 + \tilde{q}_2 - q_1 \geq 0$. *If*

$$(2.16)\qquad \begin{aligned} -\tilde{r}_1^2 &= x(\tilde{r}_1 + \tilde{r}_2 - \tilde{r}_2^2) \quad \textit{and} \\ \tilde{r}_1^2 &\leq x(\tilde{r}_1 + \tilde{r}_2 + \tilde{r}_2^2)\,, \end{aligned}$$

then $\tilde{d}$ *is also* $MV(c)$-*optimal in* $D(s,t;n)$. *If either inequality in* (2.16) *is not satisfied, then* $\tilde{d}$ *is not* $MV(c)$-*optimal.*

(c) *Suppose* $\tilde{q}_1 + \tilde{q}_2 < s$, $\tilde{q}_1 + \tilde{q}_2 + q_1 < s$ *and* $\tilde{q}_1 + \tilde{q}_2 - q_1 < 0$. *If*

$$(2.17)\qquad \begin{aligned} -\tilde{r}_1^2 &\leq x(\tilde{r}_1 + \tilde{r}_2 - \tilde{r}_2^2) \quad \textit{and} \\ (\tilde{r}_1 + \tilde{r}_2)(\tilde{r}_1 - \tilde{r}_2 - 1) &\leq x(\tilde{r}_1 + \tilde{r}_2 + \tilde{r}_2^2)\,, \end{aligned}$$

then $\tilde{d}$ *is also* $MV(c)$-*optimal in* $D(s,t;n)$. *If either inequality in* (2.17) *is not satisfied, then* $\tilde{d}$ *is not* $MV(c)$-*optimal.*

(d) *Suppose* $\tilde{q}_1 + \tilde{q}_2 < s$, $\tilde{q}_1 + \tilde{q}_2 + q_1 \geq s$ *and* $\tilde{q}_1 + \tilde{q}_2 - q_1 \geq 0$. *If*

$$(2.18)\qquad \begin{aligned} (\tilde{r}_1 + \tilde{r}_2)(\tilde{r}_2 - \tilde{r}_1 - 1) &\leq x(\tilde{r}_1 + \tilde{r}_2 - \tilde{r}_2^2) \quad \textit{and} \\ \tilde{r}_2^2 &\leq x(\tilde{r}_1 + \tilde{r}_2 + \tilde{r}_2^2)\,, \end{aligned}$$

then $\tilde{d}$ *is also* $MV(c)$-*optimal in* $D(s,t;n)$. *If either inequality in* (2.18) *is not satisfied, then* $\tilde{d}$ *is not* $MV(c)$-*optimal.*

(e) *Suppose* $\tilde{q}_1 + \tilde{q}_2 < s$, $\tilde{q}_1 + \tilde{q}_2 + q_1 \geq s$ *and* $\tilde{q}_1 + \tilde{q}_2 - q_1 < 0$. *If*

$$(2.19)\qquad \begin{aligned} (\tilde{r}_1 + \tilde{r}_2)(\tilde{r}_2 - \tilde{r}_1 - 1) &\leq x(\tilde{r}_1 + \tilde{r}_2 - \tilde{r}_2^2) \quad \textit{and} \\ (\tilde{r}_1 + \tilde{r}_2)(\tilde{r}_1 - \tilde{r}_2 - 1) &\leq x(\tilde{r}_1 + \tilde{r}_2 + \tilde{r}_2^2)\,, \end{aligned}$$

then $\tilde{d}$ *is also* $MV(c)$-*optimal in* $D(s,t;n)$. *If either inequality in* (2.19) *is not satisfied, then* $\tilde{d}$ *is not* $MV(c)$-*optimal.*

PROOF. (a): For $\tilde{d}$, clearly

$$(2.20)\qquad \max_{1 \leq i \leq s,\, s+1 \leq j \leq v} \operatorname{Var}_{\tilde{d}}(\hat{t}_i - \hat{t}_j) = 1/\tilde{r}_1 + 1/\tilde{r}_2\,.$$

If $\tilde{q}_1 + \tilde{q}_2 \geq s$, then a design $\hat{d}$ having treatment replications

$$r_{\hat{d}i} \geq \tilde{r}_1 + 1\,,$$

for $i = 1,\ldots,s$ and $r_{\hat{d},s+1} = \cdots = r_{\hat{d},v} = \tilde{r}_2$ can be constructed. But then

$$\max_{1\le i\le s,\, s+1\le j\le v} \operatorname{Var}_{\tilde d}(\hat t_i - \hat t_j) = 1/(\tilde r_1+1) + 1/\tilde r_2 < 1/\tilde r_1 + 1/\tilde r_2$$

and $\tilde d$ is not $MV(c)$-optimal in $D(s,t;n)$.

(b): Because $\tilde q_1 + \tilde q_2 + q_1 < s$ and $\tilde q_1 + \tilde q_2 - q_1 \ge 0$, it follows that in order for $\tilde d$ to be $MV(c)$-optimal also, the following two inequalities must be satisfied:

$$1/\tilde r_1 + 1/\tilde r_2 \le 1/(\tilde r_1 + x) + 1/(\tilde r_2 - 1) \quad \text{and}$$
$$1/\tilde r_1 + 1/\tilde r_2 \le 1/(\tilde r_1 - x) + 1/(\tilde r_2 + 1)\,.$$

But these last two inequalities are satisfied if and only if the inequalities given in (2.16) are satisfied, which yields the desired result. The proofs for (c), (d) and (e) are similar to the proof of (b).

THEOREM 2.6. *Suppose* $t = xs$ *and* $n = r_1 s + r_2 xs$ *for integers* $r_1, r_2, x \ge 1$ *and suppose* $d* \in D(s,t;n)$ *has* $r_{d*1} = \cdots = r_{d*s} = r_1^* = (n - r_2^* xs)/s$ *and* $r_{d*,s+1} = \cdots = r_{d*v} = r_2^*$ *where* r_2^* *satisfies* (2.11). *If* r_2^* *also satisfies*

$$(2.21)\qquad \frac{\{(nx - (nx(n+xs-s))^{1/2}\}}{sx(x-1)} \le r_2^* \le \frac{\{(nx - (nx(n-xs+s))^{1/2}\}}{sx(x-1)}\,,$$

then $d*$ *is both* $MV(c)$*- and* $A(c)$*-optimal in* $D(s,t;n)$.

PROOF. This result follows directly from Theorems 2.2 and 2.4.

Example 2.7. Suppose $s = 2$, $t = 6$ and $n = 40$. *Then* $r_2^* = 4$ and r_2^* satisfies (2.11), thus $d*$ having $r_{d*1} = r_{d*2} = 8$ and $r_{d*3} = \cdots = r_{d*8} = 4$ is $MV(c)$-optimal in $D(2,6;40)$. It is easily verified that r_2^* also satisfies (2.21); hence $d*$ is also $A(c)$-optimal in $D(2,6;40)$ by Theorem 2.6.

2.4 *Relationship between* $MV(c)$, $A(c)$ *and other optimality criteria*

The $MV(c)$- and $A(c)$-optimality criteria defined earlier are specifically formulated to find "best" designs for making comparisons between a set of test treatments and a set of controls. Thus, these optimality criteria might be called local criteria, since they select designs that are optimal for estimating specific treatment differences of the form $t_i - t_j$, $1 \le i \le s$, $s+1 \le j \le v$. However, an experimenter may also be interested in eventually making comparisons between all treatments. Thus he would want to be able to "optimally" estimate all possible treatment differences of the form $t_i - t_j$. In this case, he would want to use a "global" criterion for selecting a best design. Two widely used global criteria are the MV- and A-optimality criteria.

DEFINITION 2.1. A design $d*$ is said to be MV-optimal in a given class of designs D if for any other $d \in D$, $\max_{i \neq j} \operatorname{Var}_{d*}(\hat{t}_i - \hat{t}_j) \leq \max_{i \neq j} \operatorname{Var}_d(\hat{t}_i - \hat{t}_j)$.

DEFINITION 2.2. A design $\hat{d}$ is said to be A-optimal in a given class of designs D if for any other $d \in D$, $\sum_{i=1}^{v} \sum_{\substack{j=1 \\ j \neq i}}^{v} \operatorname{Var}_{\tilde{d}}(\hat{t}_i - \hat{t}_j) \leq \sum_{i=1}^{v} \sum_{\substack{j=1 \\ j \neq i}}^{v} \operatorname{Var}_d(\hat{t}_i - \hat{t}_j)$.

In classes $D(s,t;n)$, it is easily seen that if $n = vr + q$, $0 \leq q \leq v-1$, then an MV-optimal design $d*$ is one which has

$$
(2.22) \qquad \begin{aligned} & r_{d*i} \geq r && \text{for} \quad i = 1,\ldots,v \quad \text{if} \quad 0 \leq q \leq v-2\,, \\ & r_{d*i} = r+1 && \text{for} \quad i = 1,\ldots,v-1 \quad \text{and} \quad r_{d*v} = r \quad \text{if} \quad q = v-1\,, \end{aligned}
$$

and an A-optimal design $\tilde{d}$ is one which has

$$
(2.23) \qquad r_{\tilde{d}i} = r \quad \text{or} \quad r+1 \qquad \text{for} \quad i = 1,\ldots,v\,.
$$

We shall now explore the various relationships that exist between the $MV(c)$-, $A(c)$-, MV- and A-optimality criteria.

THEOREM 2.7. *Let* $n = vr + q$, $0 \leq q \leq v-1$, *and let* $d* \in D(s,t;n)$ *be* $MV(c)$*-optimal with* $r_{d*1} \leq \cdots \leq r_{d*s}$, $r_{d*s+1} \leq \cdots \leq r_{d*v}$. *Also, let* $r_1^* = r_{d*1}$ *and* $r_2^* = r_{d*,s+1}$.

(a) *If* $r_2^* = r, r_1^* \geq r$ *and* $0 \leq q \leq v-2$, *then* $d*$ *is* MV*-optimal in* $D(s,t;n)$.

(b) *If* $r_2^* = r, r_1^* \geq r$ *and* $q = v-1$, *then* $d*$ *is not* MV*-optimal in* $D(s,t;n)$.

PROOF. (a): This follows directly from (2.22) and the fact that if $d \in D(s,t;n)$ is any design, since $q \leq v-2$, then d must have at least two treatments, i and j, such that $r_{di} + r_{dj} \leq 2r$.

(b): In this case, $r_{d*s+1} = \cdots = r_{d*v} = r$ and $r_{d*1} \geq r_1^*$ where $r_1^* = [(n-tr)/s] \geq r$. Since $t \geq s$ and $s + t \geq 3$, it follows that $\max_{i \neq j} \operatorname{Var}_{d*}(\hat{t}_i - \hat{t}_j) = 2/r$. But an MV-optimal design $\bar{d}$ has $r_{\bar{d}1} = \cdots = r_{\bar{d}s} = r+1$, $r_{\bar{d},s+1} = r$ and $r_{\bar{d},s+2} = \cdots = r_{\bar{d},v} = r+1$. Hence $\max_{i \neq j} \operatorname{Var}_{\bar{d}}(\hat{t}_i - \hat{t}_j) = 1/r + 1/(r+1)$. Thus, when $q = v-1$, the $MV(c)$ and MV-optimal designs in $D(s,t;n)$ are different.

COROLLARY 2.2. *Let* $n = vr + q$, $0 \leq q \leq v-2$, $n - tr = sr_1^* + q_1^*$, $0 \leq q_1^* \leq s-1$ *and* $t = xs + q_1$, $0 \leq q_1 \leq s-1$.

(a) *If* $q_1^* + q_1 \leq s - 1$ *and* $x(r^2 - r - r_1^*) \leq r_1^{*2}$, *then a design* $d*$ *which is* $MV(c)$*-optimal is also* MV*-optimal in* $D(s, t; n)$.

(b) *If* $q_1^* + q_1 \geq s$ *and* $x(r^2 - r - r_1^*) \leq r_1^{*2} + r_1^* + r - r^2$, *then a design* $d*$ *which is* $MV(c)$*-optimal is also* MV*-optimal in* $D(s, t; n)$.

PROOF. (a): Suppose $d* \in D(s, t; n)$. Then for $d*$ to be MV-optimal, it follows from (2.22) that $d*$ must have $r_{d*i} \geq r$ for all i. But for this to happen and for $d*$ to also be $MV(c)$-optimal, we see as in Subsection 2.1 that the following inequality must hold:

$$1/r + 1/[(n - tr)/s] \leq 1/(r - 1) + 1/[(n - t(r - 1))/s] .$$

Using the appropriate expressions given and the assumption that $q_1^* + q_1 \leq s - 1$, we see that this last inequality is equivalent to $1/r + 1/r_1^* \leq 1/(r - 1) + 1/(r_1^* + x)$ which will hold if $x(r^2 - r - r_1^*) \leq r_1^{*2}$.

(b): As in the proof of (a), $d*$ must have $r_{d*i} \geq r$ for $i = 1, \ldots, v$ to be $MV(c)$-optimal in $D(s, t; n)$. But for this to happen and for $d*$ to be also $MV(c)$-optimal, we see as in Subsection 2.1 and because $q_1^* + q_1 \geq s$ that the following inequality must hold:

$$1/r + 1/r_1^* \leq 1/(r - 1) + 1/(r_1^* + x + 1) .$$

But this latter inequality will be true as long as $x(r^2 - r - r_1^*) \leq r_1^{*2} + r_1^* + r - r^2$.

Example 2.8. Consider the class of designs $D(1, 7; 45)$. Then $r = 5$, $q = 5$, $r_1^* = 10$, $q_1^* = 0$, $x = 7$ and $q_1 = 0$. It now follows from Corollary 2.2(a) that the design $d*$ having $r_{d*1} = 10$ and $r_{d*2} = \cdots = r_{d*8} = 5$ is both MV-and $MV(c)$-optimal in $D(1, 7; 45)$.

Using arguments similar to those used to prove Theorem 2.7, we can also easily obtain the following result.

THEOREM 2.8. (a) *If* $s < t$, *then an* A*-optimal design in* $D(s, t; n)$ *is never* $A(c)$*-optimal in* $D(s, t; n)$.

(b) *If* $s = t$, *then any design* $\tilde{d}$ *having* $r_{\tilde{d}i} = r$ *or* $r + 1$ *for* $i = 1, \ldots, v$ *is both* A*- and* $A(c)$*-optimal in* $D(s, t; n)$.

Comment. Many of the results given in this section can be extended to experimental situations requiring usage of a block design or row-column design using techniques and results such as those given in Jacroux (1986) and Majumdar (1986).

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