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ON MITTAG-LEFFLER FUNCTIONS AND RELATED DISTRIBUTIONS

R. N. PILLAI

Department of Statistics, University of Kerala, Trivandrum-695 581, India

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Abstract. The distribution $F_\alpha(x) = 1 - E_\alpha(-x^\alpha)$, $0 < \alpha \leq 1$; $x \geq 0$, where $E_\alpha(x)$ is the Mittag-Leffler function is studied here with respect to its Laplace transform. Its infinite divisibility and geometric infinite divisibility are proved, along with many other properties. Its relation with stable distribution is established. The Mittag-Leffler process is defined and some of its properties are deduced.

Key words and phrases: Completely monotone function, Laplace transform, infinite divisibility, geometric infinite divisibility, stable process.

1. Introduction

The function $E_\alpha(z) = \sum_{k=0}^{\infty} [z^k / \Gamma(1 + \alpha k)]$, was first introduced by Mittag-Leffler in 1903 (Erdelyi (1955)). It was subsequently investigated by Wiman, Pollard, Humbert, Aggarwal and Feller. Many properties of the function follow from the Mittag-Leffler integral representation

$$E_\alpha(z) = \frac{1}{2\pi i} \int_C \frac{t^{\alpha-1} e^t}{t^\alpha - z} dt ,$$

where the path of integration C is a loop which starts and ends at $-\infty$ and encircles the circular disc $|t| \leq z^{1/\alpha}$. Feller conjectured and Pollard proved in 1948 that $E_\alpha(-x)$ is completely monotone for $x \geq 0$, if $0 < \alpha \leq 1$. It is proved in Feller (1966) that $E_\alpha(z)$ is an entire function of order $1/\alpha$ for $\alpha > 0$.

In Feller (1966), the Laplace transform of $E_\alpha(-x^\alpha)$ with $0 < \alpha \leq 1$ is shown to be $u^{\alpha-1}/(1 + u^\alpha)$, $u \geq 0$. But $E_\alpha(-x^\alpha)$ is not a probability distribution. But we have shown in Theorem 2.1 that $F_\alpha(x) = 1 - E_\alpha(-x^\alpha)$ has the Laplace transform

$$f_\alpha(u) = (1 + u^\alpha)^{-1}, \quad u \geq 0$$

which is completely monotone for $0 < \alpha \leq 1$, and therefore it is a (probability) distribution function. We call $F_\alpha(x)$, for $0 < \alpha \leq 1$, a Mittag-Leffler distribution. $F_1(x)$ is the exponential distribution. The function $F_\alpha(x) = 1 - E_\alpha(-x^\alpha)$, $x \geq 0$ is investigated for its many properties.

Sections 2 and 3 deal with infinite divisibility and geometric infinite divisibility, respectively, of $F_\alpha(x)$. Section 4 examines its relation with stable distributions and also develops the stochastic process associated with $F_\alpha(x)$.

2. Infinite divisibility of $F_\alpha(x)$

THEOREM 2.1. $F_\alpha(x) = \sum_{k=1}^{\infty} [(-1)^{k-1} x^{k\alpha} / \Gamma(1 + k\alpha)]$, $x \geq 0$, $0 < \alpha \leq 1$ *is a probability distribution with the Laplace transform* $(1 + u^\alpha)^{-1}$, $u \geq 0$.

PROOF. By Feller (1966), the Mittag-Leffler function $E_\alpha(-x^\alpha) = \sum_{k=0}^{\infty} [(-1)^k x^{k\alpha} / \Gamma(1 + k\alpha)]$, $x \geq 0$ has the Laplace transform $u^{\alpha-1}/(1 + u^\alpha)$, $0 < \alpha \leq 1$, $u \geq 0$. For any distribution $F(x)$ with Laplace transform $f(u)$, we have the relation

$$\int_0^\infty e^{-ux}(1 - F(x))\,dx = (1 - f(u))/u\,.$$

Equating $(1 - f_\alpha(u))/u$ with $u^{\alpha-1}/(1 + u^\alpha)$, we see that the Laplace transform $f_\alpha(u)$ of the function $1 - E_\alpha(-x^\alpha)$ is $(1 + u^\alpha)^{-1}$. But $f_\alpha(u) = (1 + u^\alpha)^{-1}$ is completely monotone and $f_\alpha(0) = 1$ and therefore $F_\alpha(x) = 1 - E_\alpha(-x^\alpha)$, $x \geq 0$ is a probability distribution.

THEOREM 2.2. *For* $0 < \rho < \alpha \leq 1$, $\int_0^\infty x^\rho dF_\alpha(x) = [\Gamma(1 - \rho/\alpha) \cdot \Gamma(1 + \rho/\alpha)/\Gamma(1 - \rho)]$.

PROOF. The proof follows by computation; see Wolfe (1975).

THEOREM 2.3. $F_\alpha(x)$ *is infinitely divisible.*

PROOF. By Feller (1966) a distribution $F(x)$, $x \geq 0$ is infinitely divisible iff its Laplace transform is of the form $f(u) = e^{-g(u)}$, $u \geq 0$ where $g(u)$ has a completely monotone derivative. Here for $F_\alpha(x)$, $f_\alpha(u) = e^{-g_\alpha(u)}$, where $g_\alpha(u) = \log(1 + u^\alpha)$ and $g'_\alpha(u) = \alpha u^{\alpha-1}/(1 + u^\alpha)$ for $\alpha \leq 1$. But $u^{\alpha-1}$ is completely monotone, $(1 + u^\alpha)^{-1}$ is completely monotone. Therefore the product is completely monotone and hence the result.

Remark 2.1. $f_\alpha(u) = (1+u^\alpha)^{-1}$ is the Laplace transform of a distribution with positive support iff $0 < \alpha \leq 1$.

PROOF. If $0 < \alpha \leq 1$, u^α has a completely monotone derivative. $(1+u)^{-1}$ is completely monotone. Therefore by Feller ((1966), p. 417), $(1+u^\alpha)^{-1}$ is completely monotone, $f_\alpha(0) = 1$ and thus $f_\alpha(u)$ is the Laplace transform of a distribution on R^+. Suppose $\alpha > 1$, then $f_\alpha''(u) = -\alpha u^{\alpha-2} \cdot [(\alpha-1)-(\alpha+1)u^\alpha]/(1+u^\alpha)^3$. Then $f_\alpha''(u)$ is positive for $u < [(\alpha-1)/(\alpha+1)]^{1/\alpha}$ and negative for $u > [(\alpha-1)/(\alpha+1)]^{1/\alpha}$. Therefore $f_\alpha(u)$ is not completely monotone. Therefore by Feller ((1966), p. 415), $f_\alpha(u)$ is not the Laplace transform of a distribution on R^+.

Remark 2.2. The canonical representation of the infinitely divisible distribution with Laplace transform $f_\alpha(u) = (1+u^\alpha)^{-1}$, $0 < \alpha \leq 1$ is given by

$$-\log f_\alpha(u) = \int_0^\infty \frac{1-e^{-ux}}{x} f(x)\,dx$$

where

$$f(x) = \alpha \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{\Gamma(1+k\alpha)}.$$

PROOF. $f_\alpha(u) = e^{-g_\alpha(u)}$, where $g_\alpha(u) = \log(1+u^\alpha)$ and $g_\alpha'(u) = \alpha u^{\alpha-1}/(1+u^\alpha)$ which is the ordinary Laplace transform of $\alpha \sum_{k=0}^{\infty} [(-1)^k x^k / \Gamma(1+k\alpha)]$ (Feller (1966), p. 429).

3. Geometric infinite divisibility of $F_\alpha(x)$

Klebanov *et al.* (1984) has introduced the concept of geometric infinite divisibility of a random variable or its distribution. A random variable X is geometrically infinitely divisible if for every p, $0 < p < 1$

$$X \stackrel{d}{=} \sum_{j=1}^{N_p} X_p^{(j)},$$

where $\{X_p^{(n)}, n \geq 1\}$ is a sequence of independent and identically distributed random variables, N_p is a geometric random variable with mean $1/p$ and it is independent of the sequence $\{X_p^{(n)}, n \geq 1\}$.

THEOREM 3.1. *$F_\alpha(x)$ is geometrically infinitely divisible.*

PROOF. It can be easily seen that a random variable X with characteristic function $\phi(u)$ is geometrically infinitely divisible iff $\phi(u)\,(p+q\phi(u))^{-1}$ is a characteristic function of a random variable for every p, $0<p<1$. In our case since $\phi(u)=(1+(-iu)^{\alpha})^{-1}$, this last statement is equivalent to checking whether or not $(1+p(-iu)^{\alpha})^{-1}$ is a characteristic function for every p, $0<p<1$. And this obviously is, since it is the characteristic function of $p^{1/\alpha}X_\alpha$ where X_α has the distribution $F_\alpha(x)$.

4. Relations with stable laws

Here we establish the relationship between $F_\alpha(x)$ and stable distributions with exponent α, for $0<\alpha<1$.

THEOREM 4.1. *The Mittag-Leffler distribution with parameter α is attracted to the stable distribution with exponent α, $0<\alpha<1$.*

PROOF. Let T_n denote the sum of n independent random variables, each with distribution $F_\alpha(x)$. Then the Laplace transform of $n^{-1/\alpha}T_n$ is $(1+u^{\alpha}/n)^{-n}$, which tends to $e^{-u^{\alpha}}$ as n tends to infinity.

The infinite divisibility of the Mittag-Leffler distribution enables us to develop a corresponding stochastic process. The stochastic process $\{X(t), t\ge 0\}$ with $X(0)=0$ and having stationary and independent increments, where $X(1)$ has the Laplace transform $(1+u^{\alpha})^{-1}$, $0<\alpha<1$ will be called Mittag-Leffler process.

THEOREM 4.2. *The Mittag-Leffler process $X(t)$ has the distribution function, for $t>0$,*

$$F_{\alpha,t}(x)=\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(t+k)x^{\alpha(t+k)}}{\Gamma(t)k!\Gamma(1+\alpha(t+k))}\,.$$

PROOF. $X(t)$ has the Laplace transform $(1+u^{\alpha})^{-t}$. For $u>1$, $(1+u^{\alpha})^{-t}=\sum_{k=0}^{\infty}\binom{-t}{k}u^{-\alpha(t+k)}$. But

$$u^{-\alpha(t+k)}=\frac{1}{\Gamma(\alpha(t+k))}\int_0^{\infty}x^{\alpha(t+k)-1}e^{-ux}dx\,.$$

Since the distribution is uniquely determined by the Laplace transform for any interval (a,∞), $a>0$, we have proved the result.

The following theorem brings out a connection between a positive stable process and a Mittag-Leffler process with parameters α, $0<\alpha<1$.

The proof follows by the Laplace transform technique and is not presented here.

THEOREM 4.3. *$F_{\alpha,t}(x)$ in Theorem 4.2 has the following property. For $0<\alpha<1$,*

$$F_{\alpha,t}(x)=\int_0^\infty S_{\alpha,s}(x)G_t\{ds\}\ ,$$

where $S_{\alpha,s}(x)$ is the distribution of the stable process with the Laplace transform $e^{-tu^{\alpha}}$ and

$$G_t(x)=\frac{1}{\Gamma(t)}\int_0^x y^{t-1}e^{-y}dy\ .$$

Remark 4.1. Theorem 4.3 is equivalent to stating that $F_{\alpha,t}(x)$ is obtained by randomizing the parameter s in $S_{\alpha,s}(x)$ with gamma distribution. Another way of saying is that the Mittag-Leffler process is subordinated to a stable process by the directing gamma process.

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