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NONPARAMETRIC TEST OF RESTRICTED INTERCHANGEABILITY

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Abstract. Exact and large sample distributions of the rank order test under the null hypothesis of restricted interchangeability are obtained. Under given regularity conditions and under Pitman's shift in location alternative, the asymptotic relative efficiency of this nonparametric test in comparison with Votaw's (1948, *Ann. Math. Statist.*, **19**, 447–473) likelihood ratio test is given.

Key words and phrases: Interchangeability, Pitman's shift in location alternative, asymptotic relative efficiency, likelihood ratio test, asymptotic normality, permutational distribution.

1. Introduction

Let $\boldsymbol{X}_1,\dots,\boldsymbol{X}_n$ be n random vectors, each with an unknown p-variate (continuous) distribution function F, where $p = p_1 + \cdots + p_r$, $r \geq 1$,

$$\boldsymbol{X}_i = (\boldsymbol{X}_{i1},\dots,\boldsymbol{X}_{ir}) \qquad i = 1,\dots,n \,, \tag{1.1}$$

and

$$\boldsymbol{X}_{ij} = (X_{ij}^{(1)},\dots,X_{ij}^{(p_j)}) \qquad j = 1,\dots,r; \quad p_j \geq 1 \,. \tag{1.2}$$

We are interested in testing the null hypothesis $\{H_0\}$ of interchangeability within $\{\boldsymbol{X}_{ij}\}$ (i.e., H_0: $F(\boldsymbol{X}_{ij}) = F(\boldsymbol{X})$ for all values of $\boldsymbol{X} \in S_{ij}(\boldsymbol{X}_{ij})$ the set of all possible permutations of $\boldsymbol{X}_{ij}$) for $j = 1,\dots,r$ simultaneously, and for each $i = 1,\dots,n$. This hypothesis was first considered by Votaw (1948) under the normality assumption. In his case, testing the null hypothesis was reduced to testing for "compound symmetry" in a normal multivariate population. Sen (1967*a*, 1967*b*) introduced distribution-free rank order tests to test the hypothesis of interchangeability of one set of variates from a multivariate population that has a continuous cumulative distribution function.

In this paper, we will extend the distribution-free rank order test to test the hypothesis of restricted interchangeability as defined above. Rank scores for each set are defined in Section 2; these scores are very much dependent on the alternative hypothesis. Subsection 3.1 deals with the exact permutational distribution and its first and second moments. In Section 3, we define the test statistics and their quadratic forms. Also, the rejection rule under the exact permutational distribution is given. In actual practice, however, n is usually large. In this case, the labor involved in finding the rejection region under the exact distribution increases tremendously. In Section 4, we give a solution for the case when n is large by considering the asymptotic permutational distribution under suitable regularity conditions. The permutational distribution of the test statistics converges (in probability) to a chi-squared distribution, in conformity with other cases of permutationally distribution-free rank order tests. The standardized form of the test is studied asymptotically in Section 5. Section 6 is devoted to studying the asymptotic relative efficiency (A.R.E.) of the proposed test in comparison with Votaw's L_{1m}, which is derived later for our case under the shift in location alternative. We also show that, for normally distributed data, the A.R.E. is close to unity when using normal scores in the rank order test. A real-life example on the applications of the proposed procedure is given in Section 7.

2. Preliminary notions

Let $R_{ij}^{(k)}$ be the rank of $X_{ij}^{(k)}$ among the N_j $(=np_j)$ observations $X_{ij}^{(1)},\dots,X_{ij}^{(p_j)}$ for $k=1,\dots,p_j$; $i=1,\dots,n$. Thus, a separate ranking is made for each subset j, $j=1,\dots,r$. Then, the collection (rank) matrix is defined as

$$\boldsymbol{R}_N=[\boldsymbol{R}_{N_1}^{(1)},\dots,\boldsymbol{R}_{N_r}^{(r)}] = \begin{bmatrix} R_{11}^{(1)}\cdots R_{11}^{(p_1)} & \cdots & R_{1r}^{(1)}\cdots R_{1r}^{(p_r)} \\ R_{21}^{(1)}\cdots R_{21}^{(p_1)} & \cdots & R_{2r}^{(1)}\cdots R_{2r}^{(p_r)} \\ \vdots & & \vdots \\ R_{n1}^{(1)}\cdots R_{n1}^{(p_1)} & \cdots & R_{nr}^{(1)}\cdots R_{nr}^{(p_r)} \end{bmatrix}, \tag{2.1}$$

and has dimensions $n\times p$, where $N=\sum_{j=1}^{r} N_j$ and $p=\sum_{j=1}^{r} p_j$.

Define a class of rank scores as

$$B^{(j)}_{N_j,\beta} = J^{(j)}_{N_j}\left(\frac{\beta}{N_j+1}\right), \tag{2.2}$$

where J_{N_j} needs to be defined only at $\beta/(N_j+1)$ for $\beta = 1,\ldots, N_j$. Also define

$$T^{(j)}_{N_j,k} = \frac{1}{n}\sum_{i=1}^{n} B^{(j)}_{N_j,R^{(k)}_{ij}}, \qquad k = 1,\ldots,p_j\,, \tag{2.3}$$

$$\boldsymbol{T}^{(j)}_{N_j} = (T^{(j)}_{N_j,1}, T^{(j)}_{N_j,2},\ldots, T^{(j)}_{N_j,p_j}), \qquad j = 1,\ldots, r\,, \tag{2.4}$$

and

$$\boldsymbol{T}_N = (\boldsymbol{T}^{(1)}_{N_1}, \boldsymbol{T}^{(2)}_{N_2},\ldots, \boldsymbol{T}^{(r)}_{N_r})\,. \tag{2.5}$$

From the above definition, $T^{(j)}_{N_j,k}$ is the average score of the k-th column in the j-th set. By virtue of the assumed continuity of $F(\boldsymbol{x})$, the possibility of ties among the observed values may be ignored in probability.

3. Rank permutation test for the case of restricted interchangeability

3.1 *Permutational distribution*

If $X^{(1)}_{ij},\ldots, X^{(p_j)}_{ij}$ are interchangeable for each $j = 1,\ldots, r$, then the joint distribution of $\boldsymbol{X}_N = (\boldsymbol{X}_1,\ldots,\boldsymbol{X}_n)$ remains invariant under the finite group $\boldsymbol{\Gamma}_n$ of transformations $\{g_n\}$ (which maps the sample into itself), where $g_n(\boldsymbol{Y}_N) = \boldsymbol{Y}^*_N = (\boldsymbol{Y}^*_1,\ldots, \boldsymbol{Y}^*_n)$, $\boldsymbol{Y}^*_i = (\boldsymbol{Y}^*_{i1},\ldots, \boldsymbol{Y}^*_{ir})$ and $\boldsymbol{Y}^*_{ij}$ is a column permutation of the matrix $\boldsymbol{Y}_{ij}$ for each $j = 1,\ldots, r$ and $i = 1,\ldots, n$. Thus $\boldsymbol{\Gamma}_n$ contains a set of $\left[\prod_{j=1}^{r} p_j!\right]^n$ points $\boldsymbol{Y}^*_N$ which are permutationally equivalent to $\boldsymbol{Y}_N$; this set will be denoted by $S(\boldsymbol{Y}_N)$. It follows from the above discussion that under the null hypothesis of restricted interchangeability, the conditional distribution of $\boldsymbol{Y}_N$, given a set $S(\boldsymbol{Y}_N)$, is uniform over the $\left[\prod_{j=1}^{r} p_j!\right]^n$ possible realizations, which implies that

$$p\{\boldsymbol{R}^{(j)}_{N_j} = \boldsymbol{R}^*_{N_j} \mid S(\boldsymbol{R}_{N_j})\} = (p_j!)^{-n} \qquad \text{for any} \quad \boldsymbol{R}^*_{N_j} \in S(\boldsymbol{R}_{N_j})\,, \tag{3.1}$$

$$p\{\boldsymbol{R}_N = \boldsymbol{R}^*_N \mid S(\boldsymbol{R}_N)\} = \left(\prod_{j=1}^{r} p_j!\right)^{-n} \qquad \text{for any} \quad \boldsymbol{R}^*_N \in S(\boldsymbol{R}_N)\,. \tag{3.2}$$

Denote the conditional distribution given in (3.2) by the probability measure P_n.

Next, let

$$\bar{B}_{N_j}^{(j)} = \frac{1}{N_j} \sum_{i=1}^{N_j} B_{N_j,i}^{(j)} , \tag{3.3}$$

$$\bar{B}_{N_j,R_{ij}^{(\cdot)}}^{(j)} = \frac{1}{p_j} \sum_{k=1}^{p_j} B_{N_j,R_{ij}^{(k)}}^{(j)} , \tag{3.4}$$

and

$$\sigma_{N_j}^2(\boldsymbol{R}_{N_j}^{(j)}) = \frac{p_j}{N_j(p_j-1)} \sum_{i=1}^{n} \sum_{k=1}^{p_j} [B_{N_j,R_{ij}^{(k)}}^{(j)} - \bar{B}_{N_j,R_{ij}^{(\cdot)}}^{(j)}]^2 ; \quad j = 1,\ldots,r . \tag{3.5}$$

Then, conditional on the probability measure P_n, it can be shown that

$$E(\boldsymbol{T}_N | P_n) = \boldsymbol{B}_N = (\bar{B}_{N_1}^{(1)},\ldots,\bar{B}_{N_1}^{(1)},\ldots,\bar{B}_{N_r}^{(r)},\ldots,\bar{B}_{N_r}^{(r)}) , \tag{3.6}$$

where

$$\bar{B}_{N_j}^{(j)} = E(T_{N_j,k}^{(j)} | P_n), \qquad k = 1,\ldots,p_j , \tag{3.7}$$

and

$$\boldsymbol{V}_N = \begin{pmatrix} \boldsymbol{V}_1 & & 0 \\ & \ddots & \\ 0 & & \boldsymbol{V}_r \end{pmatrix} \text{ is the (conditional) variance-covariance matrix of } \boldsymbol{T}_N , \tag{3.8}$$

where

$$\boldsymbol{V}_j = \left[\frac{\delta_{kl} p_j - 1}{n p_j} \sigma_{N_j}^2(\boldsymbol{R}_{N_j}^{(j)}) \right]_{k,l=1,\ldots,p_j}, \qquad j = 1,\ldots,r ; \tag{3.9}$$

δ_{kl} is the usual Kronecker delta.

3.2 *Proposed test*

From (3.5), $\sigma_{N_j}^2(\boldsymbol{R}_{N_j}^{(j)})$ depends on the collection rank matrix, but remains invariant under $S(\boldsymbol{R}_N)$. Thus, if we use the generalized inverse of the (permutational) covariance matrix of $\boldsymbol{T}_N$, the proposed test statistic is in the following quadratic form:

$$\begin{aligned} W_N &= (\boldsymbol{T}_N - \boldsymbol{B}_N)\boldsymbol{V}_N^{-}(\boldsymbol{T}_N - \boldsymbol{B}_N)' \\ &= n \sum_{j=1}^{r} \sum_{k=1}^{p_j} (T_{N_j,k}^{(j)} - \bar{B}_{N_j}^{(j)})^2 / \sigma_{N_j}^2(\boldsymbol{R}_{N_j}^{(j)}) . \end{aligned} \tag{3.10}$$

Hence it can be shown that, if $\sigma^2_{N_j}(\boldsymbol{R}^{(j)}_{N_j})$ is finite and non-zero for each j ($j = 1,\ldots,r$), then under the permutational probability measure P_n, W_N will have $\left(\prod_{j=1}^{r} p_j!\right)^n$ possible realizations, which are all conditionally equally likely. On the other hand, if H_0 does not hold and the p variates have locations which are not all equal, then at least one of $T^{(j)}_{N_j,k}$ will be different from $\bar{B}^{(j)}_{N_j}$ for $j = 1,\ldots,r$, and hence W_N, being a positive semi-definite quadratic form in T_N, will be stochastically larger. Thus it appears reasonable to base our permutation test on the following rejection rule:

$$(3.11)\qquad \Lambda(\boldsymbol{T}_N) = \begin{cases} 1, & \text{if } W_N > W_{N,\alpha}(\boldsymbol{R}_N)\,; \\ \gamma_N(\boldsymbol{R}_N), & \text{if } W_N = W_{N,\alpha}(\boldsymbol{R}_N)\,; \\ 0, & \text{if } W_N < W_{N,\alpha}(\boldsymbol{R}_N)\,; \end{cases}$$

where $W_{N,\alpha}(\boldsymbol{R}_N)$ and $\gamma_N(\boldsymbol{R}_N)$ are chosen so that $E\{\Lambda(\boldsymbol{T}_N)|P_n\} = \alpha$. Thus, if in actual practice n is not large, we can consider the set $T_N[S(\boldsymbol{R}_N)]$ of $\left(\prod_{j=1}^{r} p_j!\right)^n$ values of $\boldsymbol{T}_N$ (and hence of W_N), which will provide us with the permutational distribution function of W_N, and the same may be used to find $W_{N,\alpha}(\boldsymbol{R}_N)$. However, if n is not very small, the labor involved in this procedure increases tremendously. To avoid such labor we shall consider in the next section the asymptotic permutation test.

4. Asymptotic permutation distribution of W_N

As in the case of the study of asymptotic theory of rank order tests for various other problems of statistical inference, we shall impose certain regularity conditions on $B^{(j)}_{N_j,\beta}$ in (2.2) as well as on $F(x)$. Extending the idea of Chernoff and Savage (1958) to the multivariate case, we shall find it convenient to extend the domain of the definition of $J^{(j)}_{N_j}$ in (2.2) to (0, 1) by letting $J^{(j)}_{N_j}$ be constant on $[\beta/(N_j+1), (\beta+1)/(N_j+1))$, $\beta = 1,\ldots,N_j$; $j = 1,\ldots,r$.

Let us now define

$$(4.1)\qquad F^{(j)}_{N_j[k]}(x) = \frac{1}{n}\,[\text{Number of } X^{(k)}_{ij} \le x], \qquad k = 1,\ldots,p_j\,,$$

$$(4.2)\qquad H^{(j)}_{N_j}(x) = \frac{1}{p_j}\sum_{k=1}^{p_j} F^{(j)}_{N_j[k]}(x), \qquad j = 1,\ldots,r\,.$$

We denote the joint c.d.f. of $(X^{(k)}_{ij}, X^{(l)}_{ij})$ by $F_{j[k,l]}(x,y)$, and the c.d.f. of $X^{(k)}_{ij}$ by $F_{j[k]}(x)$, for $l \neq k = 1,\ldots,p_j$. Further, we define

(4.3) $$F^{(j)}_{N_j[k,l]}(x,y) = \frac{1}{n}\,[\text{Number of } (X^{(k)}_{ij}, X^{(l)}_{ij}) \le (x,y)]$$

$$l \ne k = 1,\ldots,p_j\,,$$

(4.4) $$H_j(x) = \frac{1}{p_j}\sum_{k=1}^{p_j} F_{j[k]}(x)\,.$$

Next we define the regularity conditions that will be used throughout this paper:

(4.5) $$\lim_{n\to\infty} J^{(j)}_{N_j}(H) = J_j(H)$$

exists for all $0 < H < 1$ and is not constant, (C.1)

(4.6) $$\frac{1}{N_j}\sum_{\beta=1}^{N_j}\left[J^{(j)}_{N_j}\left(\frac{\beta}{N_j+1}\right) - J_j\left(\frac{\beta}{N_j+1}\right)\right] = o(N_j^{-1/2})\,, \qquad \text{(C.2)}$$

and

$$\int_{-\infty}^{\infty}\left[J^{(j)}_{N_j}\left(\frac{N_j}{N_j+1}H^{(j)}_{N_j}(x)\right) - J_j\left(\frac{N_j}{N_j+1}H^{(j)}_{N_j}(x)\right)\right] dF^{(j)}_{N_j[k]}(x)$$
$$= o_p(N_j^{-1/2})\,, \qquad k = 1,\ldots,p\,.$$

(4.7) $J_j(H)$ is absolutely continuous in H: $0 < H < 1$, and

$$\left|\frac{d^s}{dH^s}J_j(H)\right| \le K[H(1-H)]^{-s-1/2+\delta} \qquad \text{(C.3)}$$

for $s = 0, 1$, and $\delta > 0$, where K is a constant and $j = 1,\ldots,r$.

For the permutational distribution theory, we require two more mild conditions for the existence and convergence of $\sigma^2_{N_j}(\boldsymbol{R}^{(j)}_{N_j})$. These we state below:

(4.8) $$\frac{1}{N_j}\sum_{\beta=1}^{N_j}\left[\left\{J^{(j)}_{N_j}\left(\frac{\beta}{N_j+1}\right)\right\}^2 - \left\{J_j\left(\frac{\beta}{N_j+1}\right)\right\}^2\right] = o(1)\,, \qquad \text{(C.4)}$$

and

(4.9) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[J^{(j)}_{N_j}\left(\frac{N_j}{N_j+1}H_{N_j}(x)\right)J^{(j)}_{N_j}\left(\frac{N_j}{N_j+1}H_{N_j}(y)\right)\right.$$
$$\left. - J_j\left(\frac{N_j}{N_j+1}H_{N_j}(x)\right)J_j\left(\frac{N_j}{N_j+1}H_{N_j}(y)\right)\right] dF^{(j)}_{N_j[k,l]}(x,y)$$

$$= o_p\left(\frac{N_j}{N_j+1}\right),$$

where $l \neq k = 1,\ldots,p_j$ and $j = 1,\ldots,r$.

Finally, we define

$$\nu_{k,l}^{(j)}(F_j) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J_j(H_j(x))J_j(H_j(y))\,dF_{j[k,l]}(x,y) \quad \text{for} \quad l \neq k = 1,\ldots,p_j. \tag{4.10}$$

$$\nu_{k,k}^{(j)}(F_j) = \int_{-\infty}^{\infty} [J_j(H_j(x))]^2 dF_{j[k]}(x), \qquad k = 1,\ldots,p_j, \tag{4.11}$$

$$\boldsymbol{\nu}_j(F_j) = (\nu_{k,l}^{(j)}(F_j))_{k,l=1,\ldots,p_j}, \qquad j = 1,\ldots,r, \tag{4.12}$$

$$\text{Rank of } \boldsymbol{\nu}_j(F_j) \geq 2. \qquad \text{(C.5)} \tag{4.13}$$

To show the asymptotic distribution of W_N, we need the following two lemmas.

LEMMA 4.1. *Define*

$$A_j^2 = \int_0^1 J_j^2(u)\,du, \tag{4.14}$$

and

$$\bar{\nu}_j = \binom{p_j}{2}^{-1} \sum\sum_{1\leq k<l\leq p_j} \nu_{k,l}^{(j)}(F_j), \tag{4.15}$$

then, if (C.5) *holds,*

$$A_j^2 - \bar{\nu}_j > 0, \qquad j = 1,\ldots,r. \tag{4.16}$$

LEMMA 4.2. *Under regularity conditions* (C.1) *through* (C.5), $\sigma_{N_j}^2(R_{N_j}^{(j)})$, *defined in* (3.5), *converges in probability to* $[A_j^2 - \bar{\nu}_j] > 0$, *where* A_j^2 *and* $\bar{\nu}_j$ *are defined in* (4.14) *and* (4.15), *respectively.*

THEOREM 4.1. *Under the regularity conditions* (C.1) *through* (C.5) *and the permutational probability measure* P_n, *the distribution of* $\{n^{1/2}(T_{N_j,k}^{(j)} - \bar{B}_{N_j}^{(j)}),\ k = 1,\ldots,p_j\}$, *is asymptotically in probability a multinormal distribution* [*of rank* $(p_j - 1)$] *with the null mean vector and covariance matrix given by* (3.9) *for each* j ($= 1,\ldots,r$).

The singularity of the above distribution comes from the fact that

$$\sum_{k=1}^{p_j} (T_{N_j,k}^{(j)} - \bar{B}_{N_j}^{(j)}) = 0, \quad j = 1, \ldots, r. \tag{4.17}$$

Then it follows that there are at most $(p_j - 1)$ linearly independent quantities $(T_{N_j,k}^{(j)} - \bar{B}_{N_j}^{(j)})$. Thus, the vector $\{n^{1/2}(T_{N_j,k}^{(j)} - \bar{B}_{N_j}^{(j)}),\ k = 1,\ldots,p_j;\ j = 1,\ldots,r\}$ has $\sum_{j=1}^{r} (p_j - 1)$ linearly independent quantities $(T_{N_j,k}^{(j)} - \bar{B}_{N_j}^{(j)})$.

Proofs of Theorem 4.1, Lemmas 4.1 and 4.2 can be found in Sen (1967*b*).

THEOREM 4.2. *Under the regularity conditions* (C.1) *through* (C.5) *and the permutational probability measure* P_n, *the statistic* W_N *defined in* (3.10) *has, asymptotically in probability, a chi-squared distribution with* $\sum_{j=1}^{r} (p_j - 1)$ *degrees of freedom.*

PROOF. First let us write

$$W_{N_j}^{(j)} = n \sum_{k=1}^{p_j} (T_{N_j,k}^{(j)} - \bar{B}_{N_j}^{(j)})^2 / \sigma_{N_j}^2(\boldsymbol{R}_{N_j}^{(j)}), \quad j = 1, \ldots, r. \tag{4.18}$$

Then, from (3.10), we can write W_N as

$$W_N = \sum_{j=1}^{r} W_{N_j}^{(j)}. \tag{4.19}$$

Using Theorem 4.1 and Cochran's Theorem it can be shown that, under the permutational (conditional) law P_n, $W_{N_j}^{(j)}$ has, asymptotically in probability, a chi-squared distribution with $(p_j - 1)$ degrees of freedom (Sen (1967*b*)). But we know that under P_n, the r subsets are permutationally independent. Hence, W_N is the sum of r conditionally independent and asymptotically chi-squared random variables. It follows that the distribution of W_N is asymptotically chi-squared with degrees of freedom equal to $\sum_{j=1}^{r} (p_j - 1)$.

5. Asymptotic multinormality of the standardized form of T_N

Using (2.2), (4.1) and (4.2) we can rewrite $T_{N_j,k}^{(j)}$ as

$$T_{N_j,k}^{(j)} = \int_{-\infty}^{\infty} J_{N_j}^{(j)} \left(\frac{N_j}{N_j + 1} H_{N_j}(x) \right) dF_{N_j[k]}^{(j)}(x), \quad k = 1, \ldots, p_j, \tag{5.1}$$

for each $j = 1, \ldots, r$. As in the case of Sen (1967*b*), (5.1) has the same form as that of Chernoff-Savage type of rank order statistics related to the multisample case. Next, we define the following:

$$\mu_{N_j,k}^{(j)} = \int_{-\infty}^{\infty} J_j(H_j(x))\, dF_{j[k]}(x), \qquad k = 1,\ldots,p_j; \quad j = 1,\ldots,r\,, \tag{5.2}$$

and

$$\begin{aligned}\delta_{kl,mq}^{(jh)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} &[F_{jh[k,l]}(x,y) - F_{j[k]}(x)F_{h[l]}(y)] \\ &\cdot J_j'(H_j(x))J_h'(H_h(y))\, dF_{j[m]}(x)\, dF_{h[q]}(y)\,,\end{aligned} \tag{5.3}$$

for $j \neq h$ or $j = h$ and $k \neq l$, where $F_{jh[k,l]}(x,y)$ is the c.d.f. of $(X_{ij}^{(k)}, X_{ih}^{(k)})$,

$$\begin{aligned}\delta_{kk,mq}^{(jj)} = &\iint_{-\infty<x<y<\infty} F_{j[k]}(x)[1 - F_{j[k]}(y)] \\ &\qquad \cdot J_j'(H_j(x))J_j'(H_j(y))\, dF_{j[m]}(x)\, dF_{j[q]}(y) \\ &+ \iint_{-\infty<x<y<\infty} F_{j[k]}(x)[1 - F_{j[k]}(y)] \\ &\qquad \cdot J_j'(H_j(x))J_j'(H_j(y))\, dF_{j[m]}(y)\, dF_{j[q]}(x)\,,\end{aligned} \tag{5.4}$$

for $k,m = 1,\ldots,p_j$; $l,q = 1,\ldots,p_h$; $j,h = 1,\ldots,r$. Finally, let

$$D_{jh,kl}^{*} = \frac{1}{p_j p_h} \sum_{m=1}^{p_j} \sum_{q=1}^{p_h} [\delta_{kl,mq}^{(jh)} + \delta_{mq,kl}^{(jh)} - \delta_{ml,kq}^{(jh)} - \delta_{kq,ml}^{(jh)}]\,, \tag{5.5}$$

$j,h = 1,\ldots,r$.

THEOREM 5.1. *If conditions* (C.1), (C.2) *and* (C.3) *of Section* 4 *hold, then the random vector* $[n^{1/2}(T_{N_j,k}^{(j)} - \mu_{N_j,k}^{(j)}),\ k = 1,\ldots,p_j; j = 1,\ldots,r]$ *has, asymptotically, a multinormal distribution with a null mean vector and a dispersion matrix* $\boldsymbol{\beta}^*$, *where*

$$\boldsymbol{\beta}^* = ((D_{jh,kl}^{*}))_{\substack{k=1,\ldots,p_j;\, j=1,\ldots,r \\ l=1,\ldots,p_h;\, h=1,\ldots,r}}\,. \tag{5.6}$$

The proof of Theorem 5.1 is similar to that given in Theorem 5.1 of Sen (1967*b*) with some modification. The details of the proof can be found in Jerdack (1987).

It has already been pointed out that the asymptotic multinormal distribution, derived in Theorem 4.1, is singular and is of rank at most equal to $\sum_{j=1}^{r} (p_j - 1)$. If the null hypothesis holds, then it follows from (5.3), (5.4) and the above theorem that, if $j \neq h$,

$$\text{cov}\,(T^{(j)}_{N_j,k},\, T^{(h)}_{N_h,l}/\,H_0) = o(N_j^{-1})\,, \tag{5.7}$$

and if $j = h$,

$$\lim_{n\to\infty} \{N_j \,\text{cov}\,(T^{(j)}_{N_j,k},\, T^{(j)}_{N_j,l}/\,H_0)\} = (\delta_{kl}p_j - 1)(A_j^2 - \bar{\nu}_j)\,, \tag{5.8}$$

$k = 1,\ldots,p_j$; $l = 1,\ldots,p_h$; $j, h = 1,\ldots,r$, where A_j^2 and $\bar{\nu}_j$ are defined in Lemma 4.1, and δ_{kl} is the usual Kronecker delta. Consequently, with the help of Lemma 4.1, we arrive at the following.

COROLLARY 5.1. *If H_0 holds and conditions* (C.1), (C.2) *and* (C.3) *of Section* 4 *hold, then under condition* (C.5), $[N_j^{1/2}(T^{(j)}_{N_j,k} - \mu_j)$, $k = 1,\ldots,p_j$; $j = 1,\ldots,r]$ *has a singular multinormal distribution of rank* $\sum_{j=1}^{r}(p_j - 1)$, *where* $\mu_j = \int_0^1 J_j(u)\,du$, $j = 1,\ldots,r$.

We shall now consider the usual type of Pitman's translation alternatives. For this we replace the parent c.d.f., $F(\mathbf{x})$, by a sequence of c.d.f.'s, $F_{[N]}(\mathbf{x})$, such that the marginal c.d.f.'s $F^{j}_{[k][N]}(\mathbf{x})$ of $\{F_{[N]}(\mathbf{x})\}$ satisfy the sequence of alternatives $\{H_N\}$, where

$$H_N\colon F^{(j)}_{[k][N]}(x) = G_j(x + N_j^{-1/2}\theta_{kj})\,, \qquad k = 1,\ldots,p_j; \quad j = 1,\ldots,r\,, \tag{5.9}$$

where $G_j(x)$ is assumed to be an absolutely continuous (univariate) c.d.f. having a continuous density function $g_j(x)$, and where the assumptions of equality of scales and symmetry in H_0 are also assumed for the sequence of c.d.f.'s $\{F_{[N]}(x)\}$. Let us then define

$$\zeta(G_j) = \int_{-\infty}^{\infty} \frac{d}{dx} J_j(G_j(x))\,dG_j(x), \qquad j = 1,\ldots,r\,. \tag{5.10}$$

Then,

$$\lim_{n\to\infty} [N_j^{1/2} E\{(T^{(j)}_{N_j,k} - \mu_j)/\,H_N\}] = \theta_{kj}\zeta(G_j), \qquad k = 1,\ldots,p_j\,, \tag{5.11}$$

$$\lim_{n\to\infty} [N_j \,\text{cov}\,(T^{(j)}_{N_j,k},\, T^{(j)}_{N_j,l}/\,H_N)] = (\delta_{kl}p_j - 1)(A_j^2 - \bar{\nu}_j)\,, \qquad k, l = 1,\ldots,p_j\,, \tag{5.12}$$

$$\text{cov}\,(T^{(j)}_{N_j,k},\, T^{(h)}_{N_h,l}/\,H_N)] = o(N_j^{-1})\,, \qquad k = 1,\ldots,p_j; \quad h = 1,\ldots,l; \quad j \neq h = 1,\ldots,r\,. \tag{5.13}$$

Consequently, it follows from Theorem 5.1 that, under $\{H_N\}$, $\{N_j^{1/2}(T_{N_j,k}^{(j)} - \mu_j), k = 1,\ldots,p_j^{-1}; j = 1,\ldots,r\}$ has an asymptotic $\left(\sum_{j=1}^{r}(p_j - 1)\right)$ variate normal distribution with mean vector $\boldsymbol{\theta}$ and dispersion matrix Σ^* where

$$\boldsymbol{\theta} = (\theta_{11}\zeta(G_1),\ldots,\theta_{p_11}\zeta(G_1),\ldots,\theta_{1r}\zeta(G_r),\ldots,\theta_{p_rr}\zeta(G_r)) , \tag{5.14}$$

$$\Sigma^* = \begin{pmatrix} \Sigma_1^* & & 0 \\ & \ddots & \\ 0 & & \Sigma_r^* \end{pmatrix}, \tag{5.15}$$

where

$$\Sigma_j^* = [(\delta_{kl}p_j - 1)(A_j^2 - \bar{\nu}_j)]_{k,l=1,\ldots,p_j}, \qquad j = 1,\ldots,r . \tag{5.16}$$

It readily follows that, under $\{H_N\}$,

$$W_N^* = n \sum_{j=1}^{r} \sum_{k=1}^{p_j} (T_{N_j,k}^{(j)} - \bar{B}_{N_j}^{(j)})^2/(A_j^2 - \bar{\nu}_j) \tag{5.17}$$

has an asymptotic noncentral chi-squared distribution with $\sum_{j=1}^{r}(p_j - 1)$ degrees of freedom and non centrality parameter

$$\begin{aligned} \Delta_W &= \sum_{j=1}^{r} [\{\zeta(G_j)\}^2/(A_j^2 - \bar{\nu}_j)]\left[\frac{1}{p_j}\sum_{k=1}^{p_j}(\theta_{kj} - \bar{\theta}_j)^2\right] \\ &= \sum_{j=1}^{r} \Delta_W^{(j)} , \end{aligned} \tag{5.18}$$

where $\bar{\theta}_j = (1/p_j)\sum_{k=1}^{p_j}\theta_{kj}$.

From (3.10), Lemma 4.2 and (5.17), we readily find that under $\{H_N\}$ in (5.9), W_N is asymptotically equivalent to W_N^* in probability. We express this by writing $W_N \overset{p}{\sim} W_N^*$. Hence, we arrive at the following:

THEOREM 5.2. *Under the sequence of alternative hypotheses* $\{H_N\}$ *in* (5.9), *the statistic* W_N *in* (3.10) *has, asymptotically, a noncentral chi-squared distribution with* $\sum_{j=1}^{r}(p_j - 1)$ *degrees of freedom and non-centrality parameter* Δ_W *defined in* (5.18).

At this stage, we may consider an asymptotically distribution-free test for H_0. This may be formulated as follows. Let S_j^2 be a consistent estimator of $A_j^2 - \bar{\nu}_j$, in the sense that

$$S_j^2 \xrightarrow{P} A_j^2 - \bar{\nu}_j, \quad j = 1, \ldots, r. \tag{5.19}$$

Then it follows from (5.17) and under $\{H_N\}$ that

$$\hat{W}_N = n \sum_{j=1}^{r} \sum_{k=1}^{p_j} (T_{N_j,k}^{(j)} - \bar{B}_{N_j}^{(j)})^2 / S_j^2 \xrightarrow{P} W_N^* . \tag{5.20}$$

Hence, the test based on $\hat{W}_N$ will be, asymptotically, a distribution-free test of H_0. It further follows from the last theorem that the test based on W_N will be asymptotically power-equivalent to the one based on $\hat{W}_N$, for any sequence of alternatives of the type $\{H_N\}$ defined in (5.9).

6. Asymptotic relative efficiency

We shall now consider the asymptotic relative efficiency (A.R.E.) of the proposed rank order tests in comparison to the likelihood ratio test L_{1m}, considered by Votaw (1948). Votaw showed that, under the null hypothesis,

$$L_{1m} = \frac{|U_1|}{|U_2|}, \tag{6.1}$$

where U_1 and U_2 are block symmetric matrices. Using (4.7), (7.1) and (3.3) of Votaw's paper, it can be shown that

$$|U_1| = \left[\prod_{j=\gamma+1}^{r} W_{1j}^{p_j - 1} \right] |U_3|, \tag{6.2}$$

$$|U_2| = \left[\prod_{j=\gamma+1}^{r} W_{2j}^{p_j - 1} \right] |U_3|, \tag{6.3}$$

where γ is the number of subsets of cardinality equal to one and

$$W_{1j} = \frac{1}{p_j} \left\{ \sum_{k=1}^{p_j} \sigma_{kk}^{(jj)} - \frac{1}{p_j - 1} \sum_{k \neq l}^{p_j} \sigma_{kl}^{(jj)} \right\} + \frac{n}{p_j} \sum_{k=1}^{p_j} (\bar{X}_{\cdot j}^{(k)})^2 - \frac{n}{p_j(p_j - 1)} \sum_{k \neq l}^{p_j} \bar{X}_{\cdot j}^{(k)} \bar{X}_{\cdot j}^{(l)}, \tag{6.4}$$

$$W_{2j} = \frac{1}{p_j} \left\{ \sum_{k=1}^{p_j} \sigma_{kk}^{(jj)} - \frac{1}{p_j - 1} \sum_{k \neq l}^{p_j} \sigma_{kl}^{(jj)} \right\}, \tag{6.5}$$

and the elements of U_3 are

$$(6.6)\qquad U_3^{jh} = \begin{cases} \sigma_{jh}^{(jh)} & \text{if } j,h \le \gamma\,, \\ \dfrac{1}{(p_h)^{1/2}} \sum_{k=1}^{p_h} \sigma_{jk}^{(jh)} & \text{if } j \le \gamma \text{ and } h > \gamma\,, \\ \dfrac{1}{p_j} \sum_{k=1}^{p_j} \sum_{l=1}^{p_h} \sigma_{kl}^{(jj)} & \text{if } j = h > \gamma\,, \\ \dfrac{1}{(p_j p_k)^{1/2}} \sum_{k=1}^{p_j} \sum_{l=1}^{p_h} \sigma_{kl}^{(jh)} & \text{if } \gamma < j \neq h > \gamma\,, \end{cases}$$

where

$$\sigma_{kl}^{(jh)} = \sum_{i=1}^{n} (X_{ik}^{(j)} - \bar{X}_{\cdot k}^{(j)})(X_{il}^{(h)} - \bar{X}_{\cdot l}^{(h)})\,,$$

$$(6.7)\qquad k = 1,\dots,p_j; \quad l = 1,\dots,p_h; \quad j,h = 1,\dots,r\,,$$

$$\bar{X}_{\cdot j}^{(k)} = \frac{1}{n} \sum_{i=1}^{n} X_{ij}^{(k)}, \quad k = 1,\dots,p_j; \quad j = 1,\dots,r\,.$$

Then from (6.1), (6.2), (6.3), (6.4) and (6.5), we have

$$(6.8)\qquad L_{1m} = \prod_{j=\gamma+1}^{r} \left[\frac{1}{1 + (A_j / B_j)} \right]^{p_j - 1}\,,$$

where

$$(6.9)\qquad A_j = \frac{n}{p_j} \sum_{k=1}^{p_j} (\bar{X}_{\cdot j}^{(k)})^2 - \frac{n}{p_j(p_j-1)} \sum_{k \neq l}^{p_j} \bar{X}_{\cdot j}^{(k)} \bar{X}_{\cdot j}^{(l)}$$

$$= \frac{n}{p_j - 1} \sum_{k=1}^{p_j} (\bar{X}_{\cdot j}^{(k)} - \bar{X}_{\cdot j}^{(\cdot)})^2\,,$$

$$(6.10)\qquad B_j = \frac{1}{p_j} \left\{ \sum_{k=1}^{p_j} \sigma_{kk}^{(jj)} - \frac{1}{p_j - 1} \sum_{k \neq l}^{p_j} \sigma_{kl}^{(jj)} \right\} = \sigma_j^2 (1 - \bar{\rho}_j)$$

and

$$(6.11)\qquad \sigma_j^2 = \frac{1}{p_j} \sum_{k=1}^{p_j} \sigma_{kk}^{(jj)} \quad \text{and} \quad \sigma_j^2 \bar{\rho}_j = \frac{1}{p_j(p_j - 1)} \sum_{k \neq l}^{p_j} \sigma_{kl}^{(jj)}\,,$$

Then under the alternative hypothesis that is given in Section 5, $-n \log L_{1m}$ has, asymptotically, a chi-squared distribution with $\sum_{j=1}^{r} p_j - r - \gamma$ degrees of freedom and non-centrality parameter,

$$\Delta_l = \sum_{j=1}^{r} \left\{ \frac{1}{\sigma_j^2(1-\bar{\rho}_j)} \sum_{k=1}^{p_j} (\theta_{kj} - \bar{\theta}_j)^2 \right\} = \sum_{j=1}^{r} \Delta_L^{(j)} . \tag{6.12}$$

Then the A.R.E. of the W_N test defined in (3.8) with respect to the L_{1m} test defined in (6.8) is given by

$$e(W_N, L_{1m}) = \frac{\sum_{j=1}^{r} \Delta_W^{(j)}}{\sum_{j=1}^{r} \Delta_L^{(j)}} , \tag{6.13}$$

if we let

$$e^{(j)}(W_N, L_{1m}) = \frac{\Delta_W^{(j)}}{\Delta_L^{(j)}} , \quad j = 1,\dots,r , \tag{6.14}$$

then from Sen (1967*b*) it can be shown that when using normal scores with normally distributed data, $e^{(j)}$ is close to unity. Then we can argue that our $e(W_N, L_{1m})$ should be close to unity for the same reasons. Hence, rank order tests are as efficient as parametric tests when using normal scores. For the non-normal data case, the parametric test is not applicable and is less powerful than the nonparametric one for obvious reasons.

7. Example

In the following example we apply the nonparametric test statistic (using normal scores) that is given in (3.10). The results of this procedure are compared with the results from the parametric one that is given in Section 6.

Blood samples from 199 patients (101 females and 98 males) were taken at two different visits, with approximately a one month period between the visits. The patients were not told their cholesterol and triglycerides values until after both measurements had been made. Thus, unless the patients changed their diet or other behavior, one would expect the two cholesterol measurements to be interchangeable. The following are the blood fat levels that were measured at visits one and two for each patient:

Observation	SEX	$X_{i1}^{(1)}$	$X_{i1}^{(2)}$	$X_{i2}^{(1)}$	$X_{i2}^{(2)}$
1	2	182	180	53	57
2	1	173	173	115	136
3	2	220	224	186	216
⋮	⋮	⋮	⋮	⋮	⋮
199	2	215	236	100	118

1) $X_{i1}^{(1)}$ = CHOL1 = Cholesterol level at visit 1,
2) $X_{i1}^{(2)}$ = CHOL2 = Cholesterol level at visit 2,
3) $X_{i2}^{(1)}$ = TG1 = Triglycerides level at visit 1,
4) $X_{i2}^{(2)}$ = TG2 = Triglycerides level at visit 2.

To be consistent with the previous notation we denote

$$\boldsymbol{X}_i = (X_{i1}^{(1)}, X_{i1}^{(2)}, X_{i2}^{(1)}, X_{i2}^{(2)}) .$$

Then our purpose is to study the interchangeability within the Cholesterol set $(X_1^{(1)}, X_1^{(2)})$ and within the Triglycerides set $(X_2^{(1)}, X_2^{(2)})$ simultaneously. Using (2.2), (2.3), (3.3), (3.4), (3.5) and the normal scores, we show the data summary of these equations in Table 1. Note that for SEX = 2 in the cholesterol set, $(T_{N_1,1}^{(1)} - T_{N_1,2}^{(1)})^2 > \sigma_{N_1}^2$, and from the definition of $T_{N_j,k}^{(j)}$ that $T_{N_j,1}^{(j)} = -T_{N_j,2}^{(j)}$ for $j = 1, 2$.

Table 2 shows the parametric test, using (6.7), (6.9), (6.10) and (6.11).

Table 3 contains test statistics, degrees of freedom and p-values of both the parametric and the nonparametric procedures. As we notice, the

Table 1. Data summary of the nonparametric statistics by SEX.

SEX	CHOL				TG			
	$T_{N_1,1}^{(1)}$	$T_{N_1,2}^{(1)}$	$\bar{E}_{N_1}^{(1)}$	$\sigma_{N_1}^2$	$T_{N_2,1}^{(2)}$	$T_{N_2,2}^{(2)}$	$\bar{E}_{N_2}^{(2)}$	$\sigma_{N_2}^2$
1	− .0305	.0305	.0000	.1487	− .0099	.0099	.0000	.2310
2	− .0922	.0922	.0000	.1731	− .0387	.0387	.0000	.2400
1 & 2	− .0612	.0612	.0000	.1666	− .0208	.0208	.0000	.2261

$n = 199$, $N_1 = N_2 = 398$, $r = 2$, $p_1 = p_2 = 2$.

Table 2. Data summary of the parametric statistics by SEX.

SEX	CHOL				TG			
	$\bar{X}_{\cdot 1}^{(1)}$	$\bar{X}_{\cdot 1}^{(2)}$	$\bar{X}_{\cdot 1}^{(\cdot)}$	B_1/n	$\bar{X}_{\cdot 2}^{(2)}$	$\bar{X}_{\cdot 2}^{(2)}$	$\bar{X}_{\cdot 2}^{(\cdot)}$	B_2/n
1	199.49	201.89	200.69	129.16	134.63	127.29	130.96	3014.4
2	198.10	204.11	201.10	133.52	95.30	99.62	97.46	486.86
1 & 2	198.78	203.01	200.90	264.31	114.64	113.25	113.94	3518.4

Table 3. Parametric and nonparametric test statistics and their p-values for each (and combined) set(s) of fat levels for the whole sample and for each SEX.

	SEX	CHOL			TG			CHOL, TG		
		1	2	1 & 2	1	2	1 & 2	1	2	1 & 2
Nonpar	W_N	1.227	9.920	9.012	0.083	1.251	0.758	1.310	11.17	9.770
	d.f.	1	1	1	1	1	1	2	2	2
	p-val	0.268	0.002	0.003	0.773	0.263	0.384	0.519	0.004	0.007
Param	$\log L^{\dagger}$	1.084	6.680	6.678	0.393	1.015	0.060	1.481	7.656	6.739
	d.f.	1	1	1	1	1	1	2	2	2
	p-val	0.298	0.010	0.010	0.531	0.314	0.810	0.477	0.022	0.030

$^{\dagger}\log L$ is the parametric test $-n \log (L_{1m})$.

results match with little variation in the p-values. Both procedures reject the null hypothesis that the two cholesterol levels and the two triglycerides levels are simultaneously interchangeable. Looking at the p-values within each set in Table 3, we can notice that this significance is due to the difference in the cholesterol levels for the females.

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