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ESTIMATION OF THE RATIO OF THE SCALE PARAMETERS OF TWO EXPONENTIAL DISTRIBUTIONS WITH UNKNOWN LOCATION PARAMETERS

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Abstract. We consider the estimation of the ratio of the scale parameters of two independent two-parameter exponential distributions with unknown location parameters. It is shown that the best affine equivariant estimator (BAEE) is inadmissible under any loss function from a large class of bowl-shaped loss functions. Two new classes of improved estimators are obtained. Some values of the risk functions of the BAEE and two improved estimators are evaluated for two particular loss functions. Our results are parallel to those of Zidek (1973, *Ann. Statist.*, **1**, 264–278), who derived a class of estimators that dominate the BAEE of the scale parameter of a two-parameter exponential distribution.

Key words and phrases: Two-parameter exponential distribution, equivariant estimator, inadmissible, risk reduction, scale parameter.

1. Introduction

Several researchers have improved on the best affine equivariant estimator (BAEE) of the scale parameter of a distribution with unknown location parameter. Stein (1964) proved that the BAEE for the variance of a normal population is inadmissible under squared error loss if the mean is unknown, by showing that there is a scale equivariant estimator which is better than the BAEE for the variance of a normal population. Brown (1968), using a different method of proof, extended this result to a wider class of distributions and loss functions. Brewster and Zidek (1974) described two techniques for improving on equivariant estimators. When estimating the variance of a normal distribution, the first technique produces the estimator of Stein (1964), and the second technique produces a "smooth" improved estimator which is also generalized Bayes and admissible within the class of scale equivariant estimators for the variance of a normal distribution. Zidek (1973) used the first technique to obtain an estimator that dominates the BAEE of the scale parameter for the two-parameter exponential distribution. This same dominating procedure was obtained independently by Arnold (1970), who proved its superiority by evaluating its risk function and comparing it with that of the BAEE. Using the second technique of Brewster and Zidek (1974), Brewster (1974) presented a smooth, improved estimator of the scale parameter for the two-parameter exponential distribution. Gelfand and Dey (1988) extended the results of Stein (1964) and Brown (1968) to estimation under the squared error loss function of the ratio of the variances of two independent normal random variables when the means are unknown.

Section 2 of this paper demonstrates the inadmissibility of the BAEE of the ratio of the scale parameters of two independent exponential distributions with unknown location parameters. Using a technique similar to that of Zidek (1973), we obtain an estimator of the ratio dominating the BAEE. Although the estimator obtained has a relatively simple form, it is not smooth. Consequently, we derive a smooth estimator dominating the BAEE. Section 3 presents examples of dominating rules under two particular loss functions. Numerical results are also presented.

2. Inadmissibility of the best equivariant estimator

2.1 *Definitions and problem statement*

DEFINITION 2.1. A real-valued function f is said to be strictly bowl-shaped in a domain D of the real line, if there exists x_0 in D such that f is strictly decreasing for $x \leq x_0$ and strictly increasing for $x \geq x_0$.

Let $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$ be two independent random samples from exponential distributions with unknown (location, scale) parameters (μ_1, σ_1) and (μ_2, σ_2), respectively. We consider the problem of estimation of the ratio θ of the two scale parameters σ_1 and σ_2. Let $X_{(1)}$ be the minimum of the X_i's, $Y_{(1)}$ be the minimum of the Y_j's, $\overline{X}$ be the sample mean of the X_i's and $\overline{Y}$ be the sample mean of the Y_i's. Furthermore, let $T_1 = \overline{X} - X_{(1)}$, $Z_1 = X_{(1)}/(\overline{X} - X_{(1)})$, $T_2 = \overline{Y} - Y_{(1)}$, $Z_2 = Y_{(1)}/(\overline{Y} - Y_{(1)})$ and $T = T_1/T_2$. Denote $\boldsymbol{T} = (T_1, T_2)$ and $\boldsymbol{Z} = (Z_1, Z_2)$. With this notation, mT_1/σ_1 follows a chi-squared distribution with $m-1$ degrees of freedom, $[\theta n(m-1)]^{-1} \cdot m(n-1)T$ has an F-distribution with $2(m-1)$ and $2(n-1)$ degrees of freedom, and the conditional density of T_1 given $Z_1 = z_1$ is equal to $K(z_1)t_1^{m-1}e^{-mt_1(1+z_1)}I_{(t_1>0,\, z_1>\mu_1 t_1^{-1})}$ for some positive value $K(z_1)$. Furthermore, $(\boldsymbol{T}, \boldsymbol{Z})$ is a sufficient statistic.

For an estimator $\delta = \delta(\boldsymbol{T}, \boldsymbol{Z})$ of $\theta = \sigma_1/\sigma_2$, we assume that loss is measured by $L(\delta, \theta) = W(\delta/\theta)$, where W satisfies the following condition:

CONDITION 1.

(i) W is differentiable with a derivative denoted by $W'(\cdot)$.

(ii) W is strictly bowl-shaped, assuming its minimum at 1.

(iii) If $\sigma_i = 1$ for $i = 1, 2$, then $E(|W'(cT)|) < \infty$ for all $c > 0$.

Let $\boldsymbol{\mu} = (\mu_1, \mu_2)$ and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)$. The risk function of an estimator $\delta(\boldsymbol{T}, \boldsymbol{Z})$ of θ is $R(\delta, \boldsymbol{\mu}, \boldsymbol{\sigma}) = E\{W(\delta(\boldsymbol{T}, \boldsymbol{Z})|\theta)\}$. Let $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$. Consider the affine group G of transformations on the original sample space: $(x, y) \rightarrow (ax + b,\ cy + d)$, where $a > 0$, $c > 0$ and b and d are real numbers. Our problem is invariant with respect to G with the induced transformations on the parameter space and the action space given by:

$$(\mu_1, \mu_2) \rightarrow (a\mu_1 + b,\ c\mu_2 + d)\ ,$$

$$(\sigma_1, \sigma_2) \rightarrow (a\sigma_1, a\sigma_2)\ ,$$

$$\delta \rightarrow (a/c)\delta\ .$$

For a given loss function $W(\cdot)$ satisfying Condition 1, a G-equivariant estimator of θ has the form

$$\delta(\boldsymbol{T}, \boldsymbol{Z}) = cT \quad \text{for some} \quad c > 0\ .$$

The best G-equivariant (affine equivariant) estimator of θ is denoted by $\delta_0(\boldsymbol{T}, \boldsymbol{Z}) = c_0 T$. Note that c_0 depends on $W(\cdot)$. Now consider the subgroup H of G, constructed from G by taking $b = d = 0$. Then any H-equivariant estimator of θ is of the form $\delta(\boldsymbol{T}, \boldsymbol{Z}) = TF(\boldsymbol{Z})$ for some measurable function F. The risk function of an H-equivariant (scale equivariant) estimator depends on $\boldsymbol{\mu} = (\mu_1, \mu_2)$ and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)$ only through $\mu_i \sigma_i^{-1}$, $i = 1, 2$; thus, without loss of generality, we assume that $\sigma_1 = 1$, $i = 1, 2$ and hence $\theta = 1$. Clearly, the class of H-equivariant estimators contains the class of G-equivariant estimators. We show in the next two subsections that some H-equivariant estimators are better than the BAEE δ_0.

2.2 *An improved estimator*

This section derives a relatively simple estimator (given in (2.4) below) that improves on the BAEE.

LEMMA 2.1. *The functions $h_1(c) = E(W(cT))$ and $h_2(c) = E_{\mu_1}(W(cT)|Z_1 = z_1)$ for $c > 0$, and for each possible μ_1 and z_1, are strictly bowl-shaped and differentiable with derivatives $E(TW'(cT))$ and $E_{\mu_1}(TW'(cT)|Z_1 = z_1)$, respectively.*

PROOF. Apply Lemma 2.1 in Brewster and Zidek (1974).

Now denote the unique minimizers of $E(W(cT))$ and $E[W(cT)|Z_1 = z_1]$ as $c = c_0$ and $c = c(z_1, \mu_1)$, respectively. In particular, c_0 is the minimizer of the function

$$g_0(c) = \int_0^\infty [W(ct)t^{m-2}]/(1 + mt/n)^{m+n-2} dt \,, \tag{2.1}$$

which is an unstandardized expectation of $W(cT)$.

THEOREM 2.1. *The best affine equivariant estimator*

$$\delta_0(\boldsymbol{T}, \boldsymbol{Z}) = c_0 T \,, \tag{2.2}$$

is inadmissible, when loss is measured by any member of the class of loss functions satisfying Condition 1.

PROOF. Let c_1 be the minimizer of

$$g_1(c) = \int_0^\infty [W(ct)t^{m-1}]/(1 + mt/n)^{m+n-1} dt \,. \tag{2.3}$$

Recall that $W(c_0 t)$ is minimized at $t = c_0^{-1}$ and hence $W'(c_0 t) \leq 0$ for $t \leq c_0^{-1}$ and $W'(c_0 t) \geq 0$ for $t \geq c_0^{-1}$. We have,

$$\begin{aligned} g_1'(c_0) &= \int_0^{c_0^{-1}} W'(c_0 t)t^m/(1 + mt/n)^{m+n-1} dt \\ &\quad + \int_{c_0^{-1}}^\infty W'(c_0 t)t^m/(1 + mt/n)^{m+n-1} dt \\ &> c_0^{-1}(1 + mc_0^{-1}/n)^{-1} g_0'(c_0) = 0 \,, \end{aligned}$$

since $t/(1 + mt/n)$ is strictly increasing in t. Thus $c_1 < c_0$, as $g_1(c)$ is strictly bowl-shaped. Now $c(z_1, 0)$ is the minimizer of

$$G_0(c) = \int_0^\infty W(ct)t^{m-1}/(1 + mt\xi/n)^{m+n-1} dt \,,$$

where $\xi = 1 + z_1$, hence

$$\int_0^\infty W'[c(z_1, 0)x/\xi]x^m/(1 + mx/n)^{m+n-1} dx = 0 \,,$$

which implies that $c_1 = c(z_1, 0)/\xi$, or equivalently, $c(z_1, 0) = c_1(1 + z_1)$. Let $\Gamma(a, b)$ denote the incomplete gamma function (Abramowitz and Stegun (1972), p. 262). We next note that when $\mu_1 > 0$ (and hence $z_1 > 0$) $c(z_1, 0) \geq$

$c(z_1, \mu_1)$, since the unstandardized conditional expectation of $W(cT)$,

$$G_{\mu_1}(c) = \int_0^\infty W(ct)t^{m-1}\Gamma\{m + n - 1, \mu_1[m(1 + z_1)t + n]/(z_1 t)\}$$
$$/[mt(1 + z_1) + n]^{m+n-1}dt$$

is bowl-shaped (Lemma 2.1), and the function $\zeta(x) = \Gamma\{m + n - 1, \mu_1[m(1 + z_1)x + n]/(z_1 x)\}$ is increasing in x. When $\mu_1 \le 0$, $c(z_1, 0) = c(z_1, \mu_1)$ for $z_1 > 0$, since the conditional density function of T given $Z_1 = z_1$, is the same for all $\mu_1 \le 0$. Now define $F_0(z_1) = \min\{c_0, c_1(1 + z_1)\}$ for $z_1 > 0$ and $F_0(z_1) = c_0$ for $z_1 \le 0$. Then the estimator

$$\delta(\boldsymbol{T}, \boldsymbol{Z}) = F_0(Z_1)T \tag{2.4}$$

satisfies

$$E_{\mu_1}\{W[TF_0(Z_1)] | Z_1 = z_1\} \le E_{\mu_1}\{W(c_0 T) | Z_1 = z_1\}$$

with strict inequality for all μ_1, and all $z_1 > 0$ such that $c(z_1, 0) = c_1(1 + z_1) < c_0$, which holds with positive probability since $c_1 < c_0$. Therefore, $\delta(\boldsymbol{T}, \boldsymbol{Z})$ is better than $\delta_0(\boldsymbol{T}, \boldsymbol{Z})$. That is, $\delta_0(\boldsymbol{T}, \boldsymbol{Z})$ is inadmissible.

COROLLARY 2.1. *Let* c_1 *be as defined in* (2.3). *Suppose the loss function* $W(\cdot)$ *satisfies Condition* 1. *For any scale equivariant estimator* $TF(\boldsymbol{Z})$, *let* $F^*(\boldsymbol{Z}) = \min\{F(\boldsymbol{Z}), c_1(1 + Z_1)\}$ *for* $Z_1 > 0$ *and* $F^*(\boldsymbol{Z}) = F(\boldsymbol{Z})$ *for* $Z_1 \le 0$. *Then* $E_\mu\{W[TF^*(\boldsymbol{Z})]\} \le E_\mu\{W[TF(Z)]\}$, *with strict inequality if* $P(F^*(\boldsymbol{Z}) \ne F(\boldsymbol{Z})) > 0$.

2.3 *Smooth improved estimator*

Although the improved estimator in (2.4) is relatively simple, it is not smooth. In this subsection, we develop a smooth improved estimator.

LEMMA 2.2. (1) *The function* $h_3(c) = E_{\mu_1}[W(cT) | Z_1 \in (0, r)]$ *is strictly bowl-shaped in* c *for any* $r > 0$. (2) *Let* $c_{\mu_1}(r)$ *be the minimizer of* $h_3(c)$ *in* (1). *Then* $c_{\mu_1}(r)$ *is an increasing function of* r. (3) $c_{\mu_1}(r) \le c_0(r)$ *for all real* μ_1. *Consequently*, $c_0 = c_0(\infty)$, *and* $c_0(r)$ *is increasing in* r.

The proof of Lemma 2.2 is given in the Appendix.

LEMMA 2.3. $\Phi_r(Z_1)T$ *has uniformly smaller risk than* $c_0 T$, *where*

$$\Phi_r(z_1) = \begin{cases} c_0(r) & 0 < z_1 < r\,, \\ c_0 & \textit{otherwise}\,. \end{cases}$$

PROOF.

$$\begin{aligned} E\{W[\Phi_r(Z_1)T]\} &= E\{W[c_0(r)T] \mid Z_1 \in (0,r)\}P(0 < Z_1 < r) \\ &\quad + E\{W(c_0T) \mid Z_1 \notin (0,r)\}P(Z_1 \notin (0,r)) \\ &< E[W(c_0T)]\,. \end{aligned}$$

LEMMA 2.4. *Let $0 < r' < r$. $\Phi_{r,r'}(Z_1)T$ has uniformly smaller risk than $\Phi_r(Z_1)T$ where*

$$\Phi_{r,r'}(z_1) = \begin{cases} c_0(r') & 0 < z_1 \le r'\,, \\ c_0(r) & r' < z_1 \le r\,, \\ c_0 & otherwise\,. \end{cases}$$

PROOF. The result follows from decomposing the risk as in the proof of Lemma 2.3 and noting that

$$E\{W[c_0(r')T] \mid Z_1 \in (0,r')\} < E\{W[c_0(r)T] \mid Z_1 \in (0,r')\}.$$

THEOREM 2.2. *The risk function of $\psi(Z_1)T$ is nowhere larger than that of c_0T, where $\psi(z_1) = c_0(z_1)$ if $z_1 > 0$ and $\psi(z_1) = c_0$ if $z_1 \le 0$.*

PROOF. For each $i = 1, 2, \ldots$, let $0 = r_{i_0} < r_{i_1} < r_{i_2} < \cdots < r_{i_{n_i}} < \infty$ be such that

(i) $\lim_{i\to\infty} r_{i_{n_i}} = \infty$,

(ii) $\max_{1<j<n_i} |r_{i_j} - r_{i_{j-1}}| \to 0$ as i goes to infinity. Define

$$\Phi^{(i)}(z_1) = \begin{cases} c_0(r_{i_j}) & r_{i_{j-1}} < z_1 < r_{i_j} \quad j = 1, 2, \ldots, n_i\,, \\ c_0 & \text{otherwise}\,. \end{cases}$$

By Lemma 2.4, $\Phi^{(i)}(Z_1)T$ has smaller risk than that of c_0T. Now, as i tends to infinity $\Phi^{(i)}(z_1) \to \psi(z_1)$, where $\psi(z_1)$ is $c_0(z_1)$ or c_0 according to $z_1 > 0$ or $z_1 \le 0$, respectively. Hence

$$\begin{aligned} E\{W[\psi(Z_1)T]\} &= E\Big\{\lim_{i\to\infty} W[\Phi^{(i)}(Z_1)T]\Big\} \\ &< \lim_{i\to\infty} E\{W[\Phi^{(i)}(Z_1)T]\} < \lim_{i\to\infty} E[W(c_0T)] = E[W(c_0T)]\,. \end{aligned}$$

2.4 *Other versions*

Instead of conditioning on the first component of the maximal invariant under the scale group in the construction proofs in Subsections 2.2 and 2.3

we can derive dominating estimators by doing similar calculations conditioning on the second component of the maximal invariant. The results are described as follows. Let c_2 be the minimizer of $h_2(c) = \int_0^\infty W(ct)t^{m-2}/(1 + mt/n)^{m+n-1}dt$, where $W(\cdot)$ satisfies Condition 1. Then the estimator defined by $\delta(\boldsymbol{X}, \boldsymbol{Y}) = (\overline{X} - X_{(1)})(\overline{Y} - Y_{(1)})^{-1} \max\{c_0, c_2(1 - Y_{(1)}/\overline{Y})\}$ for $Y_{(1)} > 0$ and $\delta(\boldsymbol{X}, \boldsymbol{Y}) = \delta_0 = c_0(\overline{X} - X_{(1)})(\overline{Y} - Y_{(1)})^{-1}$ for $Y_{(1)} \le 0$, dominates δ_0 under $W(\cdot)$. Let $c = c^*_{\mu_2}(r)$ be the minimizer of $E[W(cT) | Z_2 \in (0, r)]$, then $c_0^*(r) = c^*_{\mu_2=0}(r) \le c^*_{\mu_2}(r)$ for any $r > 0$ and $-\infty < \mu_2 < \infty$. A smooth estimator that dominates δ_0 under $W(\cdot)$ is defined by $\delta(\boldsymbol{X}, \boldsymbol{Y}) = (\overline{X} - X_{(1)})(\overline{Y} - Y_{(1)})^{-1}c_0^*(Y_{(1)}/(\overline{Y} - Y_{(1)}))$ for $Y_{(1)} > 0$ and $\delta(\boldsymbol{X}, \boldsymbol{Y}) = \delta_0$ for $Y_{(1)} \le 0$.

Notice that the risk functions of the estimators derived by conditioning on one component of the maximal invariant depend only on the corresponding location parameter.

3. Magnitudes of risk reduction

In this section, we use numerical integration to calculate the risk functions of two improved estimators of the ratio $\theta = \sigma_1/\sigma_2$ when the loss function $W(\delta/\theta)$ is the squared error loss $L_s(\delta, \theta) = (1 - \delta/\theta)^2$, and when $W(\delta/\theta) = L_u(\delta, \theta) = (\delta/\theta) - \ln(\delta/\theta) - 1$. $L_s(\cdot)$ heavily penalizes overestimation, and $L_u(\cdot)$ penalizes underestimation. The risk functions of the improved estimators and the BAEE are computed for specific sample sizes m of the X_i's and n of the Y_j's, and specific values of the location parameter μ_1.

Let c_0 and c_1 be as defined in (2.1) and (2.3). Then

$$
(3.1) \qquad \begin{aligned} &c_0 = 1 - 3n^{-1}, && c_1 = m(n-3)/[n(m+1)], && \text{under loss } L_s(\delta, \theta), \\ &c_0 = m(n-2)/[n(m-1)], && c_1 = 1 - 2n^{-1}, && \text{under loss } L_u(\delta, \theta). \end{aligned}
$$

In terms of the original samples $\boldsymbol{X} = (X_1, \ldots, X_m)$ and $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$, the first improved estimator of θ is

$$
(3.2) \qquad \delta^1(\boldsymbol{X}, \boldsymbol{Y}) = \begin{cases} (\overline{X} - X_{(1)})(\overline{Y} - Y_{(1)})^{-1} \min\{c_0, c_1\overline{X}/(\overline{X} - X_{(1)})\}, & \\ & \text{if } X_{(1)} > 0, \\ c_0(\overline{X} - X_{(1)})(\overline{Y} - Y_{(1)})^{-1}, & \text{if } X_{(1)} \le 0. \end{cases}
$$

The estimator $\delta^1(\boldsymbol{X}, \boldsymbol{Y})$ is not smooth. To derive a smoother improved estimator of θ, note that the value of c that minimizes $E_{\mu_1}[W(cT) | Z_1 \in (0, r)]$, where T and Z_1 are as defined in Section 2, can be expressed as

$$
c_{\mu_1}(r) = \int_0^\infty t^{j-1}\beta(\mu_1; t, r)dt \Big/ \int_0^\infty t^j\beta(\mu_1; t, r)dt ,
$$

with

$$\begin{aligned}(3.3)\quad \beta(\mu_1; t, r) = e^{-m\mu_1}\Gamma[m+n-2, \mu_1(mt+n)/(rt)]/(mt+n)^{m+n-2} \\ -\Gamma[m+n-2, \mu_1(m(r+1)+n)/(rt)] \\ /[mt(r+1)+n]^{m+n-2},\end{aligned}$$

for $j = m$ if the loss function $W(\cdot)$ is L_s and $j = m-1$ if the loss function $W(\cdot)$ is L_u. $c_{\mu_1}(r) \le c_{\mu_1=0}(r) \equiv c_0(r)$ for $-\infty < \mu_1 < \infty$. In terms of the X_i's and Y_j's, the second improved estimator of $\theta = \sigma_1/\sigma_2$ is

$$(3.4)\quad \delta^2(\boldsymbol{X}, \boldsymbol{Y}) = \begin{cases} \dfrac{c_0\overline{X}(\overline{X}-X_{(1)})[\overline{X}^j - (\overline{X}-X_{(1)})^j]}{(\overline{Y}-Y_{(1)})[\overline{X}^{j+1} - (\overline{X}-X_{(1)})^{j+1}]}, & \text{if} \quad X_{(1)} > 0, \\ c_0(\overline{X}-X_{(1)})(\overline{Y}-Y_{(1)})^{-1}, & \text{if} \quad X_{(1)} \le 0, \end{cases}$$

where $j = m$ if the loss function is L_s and $j = m-1$ if the loss function is L_u.

Recall that the risk functions of $\delta^1(\boldsymbol{X}, \boldsymbol{Y})$ and $\delta^2(\boldsymbol{X}, \boldsymbol{Y})$ depend on the parameters (μ_1, σ_1, μ_2, σ_2) only through (μ_1/σ_1, μ_2/σ_2). Hence, in the calculation of the risk functions of δ^1 and δ^2 below, we assume $\theta = 1$.

The risk function of $\delta^1(\boldsymbol{X}, \boldsymbol{Y})$ has the following integral form

$$\begin{aligned}(3.5)\quad &\int_0^{m^{-1}} dz_1 \int_0^\infty [1 - c_1(1+z_1)t]^2 f(t, z_1)dt + \int_{m^{-1}}^\infty dz_1 \int_0^\infty [1 - c_0t]^2 f(t, z_1)dt \\ &+ \int_{-\infty}^0 dz_1 \int_0^\infty [1 - c_0t]^2 f(t, z_1)dt,\end{aligned}$$

where $f(t, z_1)$ is a probability density proportional to

$$\Gamma[m+n-1, \mu_1(m(1+z_1)t+n)/(z_1t)]t^{m-1}/[mt(1+z_1)+n]^{m+n-1}$$

$$t > 0, \quad z_1 > 0.$$

The risk function of $\delta^2(\boldsymbol{X}, \boldsymbol{Y})$ has the form

$$(3.6)\quad \int_0^\infty \int_0^\infty [1 - c_0(z_1)t]^2 f(t, z_1)dtdz_1 + \int_{-\infty}^0 dz_1 \int_0^\infty (1 - c_0t)^2 f(t, z_1)dt,$$

where $f(t, z_1)$ was defined in the last paragraph and

$$c_0(z_1) = m(m+n-j-3)[nj]^{-1}[1 - (z_1+1)^{-j}]/[1 - (z_1+1)^{-j-1}]$$

for $j = m$ if the loss function is L_s and $j = m-1$ if the loss function is L_u. We evaluated numerically the risk functions (3.5), (3.6) of $\delta^1(\boldsymbol{X}, \boldsymbol{Y})$ and

$\delta^2(X, Y)$ for $m = 4$, $n = 7$. Tables 1 and 2 compare these risk functions to the risk function of the BAEE, for several values of the location μ_1. The maximum risk reduction upon the BAEE is about 3.5% when loss is measured by L_s, and about 8% when loss L_u is used.

Table 1. Percentage of risk reduction in using estimators δ^1 and δ^2 instead of the BAEE under loss function L_s.

	Value of μ_1								
	0	.1	.15	.2	.3	.4	.6	.8	1
Risk reduction for δ^1	1.55	3.13	2.80	2.13	1.00	0.40	0.07	0.03	0.00
Risk reduction for δ^2	0.03	1.78	2.30	2.68	3.00	3.40	3.25	2.88	2.30

Risk of BAEE = .4.

Table 2. Percentage of risk reduction in using estimators δ^1 and δ^2 instead of the BAEE under loss function L_u.

	Value of μ_1							
	.05	.1	.2	.3	.4	.5	.6	.8
Risk reduction for δ^1	4.81	8.23	6.43	4.22	2.83	2.17	2.1	1.77
Risk reduction for δ^2	3.41	4.96	6.61	7.78	7.93	8.11	7.89	7.27

Risk of BAEE = .2725.

Appendix

The proofs of the following two lemmas, which are used in proving Lemma 2.2, are modifications of the proof of Lemma 2(i) of Lehmann ((1986), p. 85). The lemmas should be of independent interest.

LEMMA A.1. *Let $\{p_\eta(x)\colon \eta > 0\}$ be a monotone likelihood ratio (MLR) decreasing family of densities on the real line. If $\Psi(x)$ is a decreasing (increasing) function of x, then $E_\eta \Psi(X)$ is an increasing (decreasing) function of η.*

LEMMA A.2. (1) *Let $p_\eta(x)$ be an MLR-increasing density function. If $\Psi(x; \eta)$ is increasing (decreasing) in x and η, then $E_\eta \Psi(X; \eta)$ is increasing (decreasing) in η.* (2) *Let $p_\eta(x)$ be an MLR-decreasing density function. If $\Psi(x; \eta)$ is increasing (decreasing) in x and decreasing (increasing) in η, then $E_\eta \Psi(X; \eta)$ is decreasing (increasing) in η.*

PROOF OF LEMMA 2.2.

Part (1). First note that $E[W(cT) \mid Z_1 \in (0, r)] \propto \int_0^\infty W(ct) h_{\mu_1}(t; r) dt$

where $h(t) = h_{\mu_1}(t;r)$ is a density function proportional to $\int_0^\infty yh_{1\mu_1}(ty;r)h_2(y)dy$, with $h_1(t) = h_{1\mu_1}(t;r) \propto t^{m-2}e^{-mt}(e^{-m\mu_1} - e^{-mtr})I_{(t>\mu_1 r^{-1})}$ and $h_2(t) \propto t^{n-2}e^{-nt}I_{(t>0)}$. We need to show that $\{h(t\tau^{-1}), \tau > 0\}$ is MLR-increasing i.e.,

(A.1) $$h(t\eta_1)/h(t\eta_2) \text{ is increasing in } t\,,$$
$$\text{for any } \eta_1, \eta_2 \text{ such that } \quad 0 < \eta_1 < \eta_2\,.$$

After putting $x = ty$, and differentiating $h(t\eta_1)/h(t\eta_2)$ with respect to t, we see that for (A.1) to hold it suffices to show that

(A.2) $$\Phi(\eta) = \int_0^\infty x^2 h_1(\eta x)h_2'(xt^{-1})dx \Big/ \int_0^\infty xh_1(\eta x)h_2(xt^{-1})dx$$
is increasing in η.

Rewrite $\Phi(\eta) = E_\eta[\Psi(X)]$ where $\Psi(x) = xh_2'(xt^{-1})/h_2(xt^{-1})$ and the expectation is taken with respect to the density

$$p_\eta(x) = xh_1(\eta x)h_2(xt^{-1}) \Big/ \int_0^\infty xh_1(\eta x)h_2(xt^{-1})dx\,.$$

Since $p_\eta(x)$ is MLR-decreasing $\Psi(x)$ is decreasing in x, $\Phi(\eta)$ is then increasing in η, by Lemma A.1.

Part (2). To show that $c_{\mu_1}(r)$ is increasing in r, let $r < r^*$ and put $F(c) = E[W(cT) | Z_1 \in (0, r^*)]$. It is then enough to show that $F'[c_{\mu_1}(r)] < 0$. Noting that $F'[c_{\mu_1}(r)] \propto \text{cov}\,[W'(c_{\mu_1}(r)T),\ h(T;r^*)/h(T;r)]$ with respect to $h(t;r)$, and that $W'[c_{\mu_1}(r)t]t$ has only one sign change, it is then sufficient to show that

(A.3) $$\Delta(t) = h(t;r^*)/h(t;r) \quad \text{is decreasing in } t\,.$$

Making the same change of variable as in Part (1), and differentiating $\Delta(t)$ with respect to t, it is not difficult to see that (A.1) is implied by the statement that $H(r)$ increases in r, where $H(r) = E_r\Lambda(x)$, $\Lambda(x) = x \cdot h_2'(xt^{-1})/h_2(xt^{-1})$, and E_r denotes the expectation taken with respect to the density $p_r(x)$ proportional to $xh_1(x;r)h_2(xr^{-1})$. To complete the proof of (A.3) using Lemma A.1, we must verify that

(A.4) $$\xi(x) = h_1(x;r^*)/h_1(x;r) \quad \text{decreases in } x \text{ for } r < r^*\,.$$

For (A.4) to hold it is enough to prove that $\zeta(r) = E_r[\gamma(Z;r)]$ increases in r where $\gamma(z;r) = mz + [m\mu_1 x^{-1}e^{-m\mu_1}]/(e^{mxr} - e^{-m\mu_1})$, and where the expectation is taken with respect to the density $h_r(z) \propto e^{-mxz}I_{(\mu_1 x^{-1} \le z \le r)}$. With $h_r(z)$ MLR-increasing, $\gamma(z;r)$ increasing in z and r, we conclude by Lemma A.2(1) that

$\zeta(r)$ increases in r. Hence (A.4) holds.

Part (3). To show that $c_{\mu_1}(r) \le c_{\mu_1=0}(r) \equiv c_0(r)$, we first assume that $\mu_1 > 0$. Since the function $h_3(c)$ is strictly bowl-shaped, it suffices to prove that $h_3'(c_0(r)) > 0$. For this latter claim, and following the same argument used to show part (2) of this lemma, we need to verify that $H(y) = h_{\mu_1}(y;r)/h_0(y;r)$ is increasing in y. Observe that

$$H(y) = \int_0^\infty uh_{1\mu_1}(uy;r)h_2(u)du \Big/ \int_0^\infty uh_{10}(uy;r)h_2(u)du$$

$$= \int_0^\infty xh_{1\mu_1}(x;r)h_2(xy^{-1})dx \Big/ \int_0^\infty xh_{10}(x;r)h_2(xy^{-1})dx$$

can be rewritten as $H(y) = E_y[M(X)]$, with $M(x) = h_{1\mu_1}(x;r)/h_{10}(x;r)$ and with the expectation taken with respect to the density function $g_y(x) \propto xh_{10}(x;r)h_2(xy^{-1})$. But for $0 < y_1 < y_2$, the ratio $g_{y_2}(x)/g_{y_1}(x) \propto e^{-x(y_2^{-1}-y_1^{-1})}$, is increasing in x. Hence $g_y(x)$ is MLR-increasing.

Furthermore, it can be checked that $M(x) = (e^{-m\mu_1} - e^{-mxr})(1 - e^{-mxr})^{-1} \cdot I_{(x>\mu_1 r^{-1})}$ is increasing in x. Thus, by Lemma 2.1 of Lehmann (1986), $H(y)$ is increasing in y.

For the case $\mu_1 < 0$, observe that $E_{\mu_1}[W(cT) \mid Z_1 \in (0,r)] = E_{\mu_1=0}[W(cT) \mid Z_1 \in (0,r)]$. This follows from the fact that the function $h(t;r)$ defined in the proof of Part (1) in this appendix is same for all $\mu_1 \le 0$. Hence $c_{\mu_1}(r) = c_{\mu_1=0}(r) \equiv c_0(r)$.

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